Yen-Jie Lee

Reminder : EXAM1 , Review section

This is what we have done :

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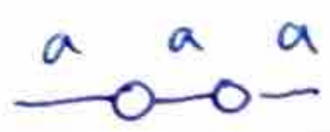
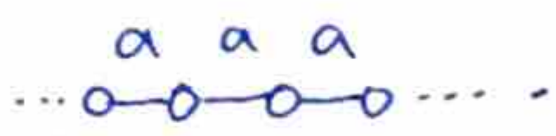
From  to 

N-coupled oscillator $\qquad$ ∞ coupled oscillator

$\Rightarrow$ N coupled equations of motion $\Rightarrow$ ∞ coupled equation of motion

Idea we got : make use of the property :

" Space Translation Invariance "

This symmetry can be translated into mathematics

$$A' = SA \quad \text{such that} \quad A_j' = A_{j+1}$$

If A is an eigenvector of S

$$\Rightarrow SA = \beta A$$

$$\Rightarrow A_j' = \beta A_j = A_{j+1}$$

$$\Rightarrow A_j = \beta^j A_0 \propto \beta^j$$

Consider $\beta = e^{ika}$ (don't want $A_j \to \infty$ when $j \to \infty$).

Need $|\beta| = 1$

$\beta = e^{i\theta}$

factorize the length scale a out

$$\Rightarrow A_j \propto e^{ijka}$$

(a : space between masses)

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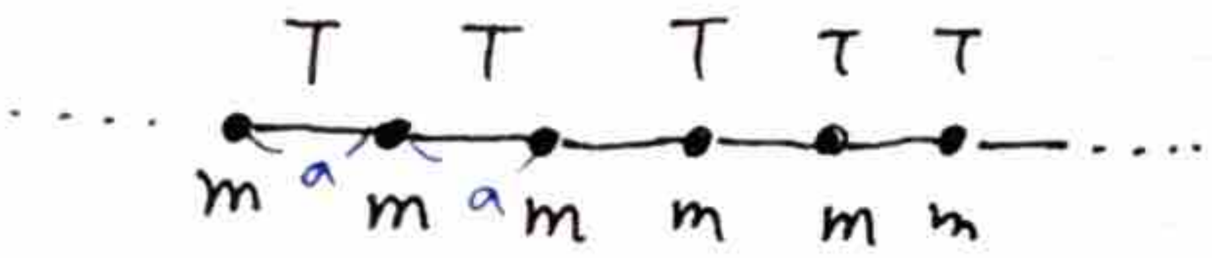
Let's consider this example.

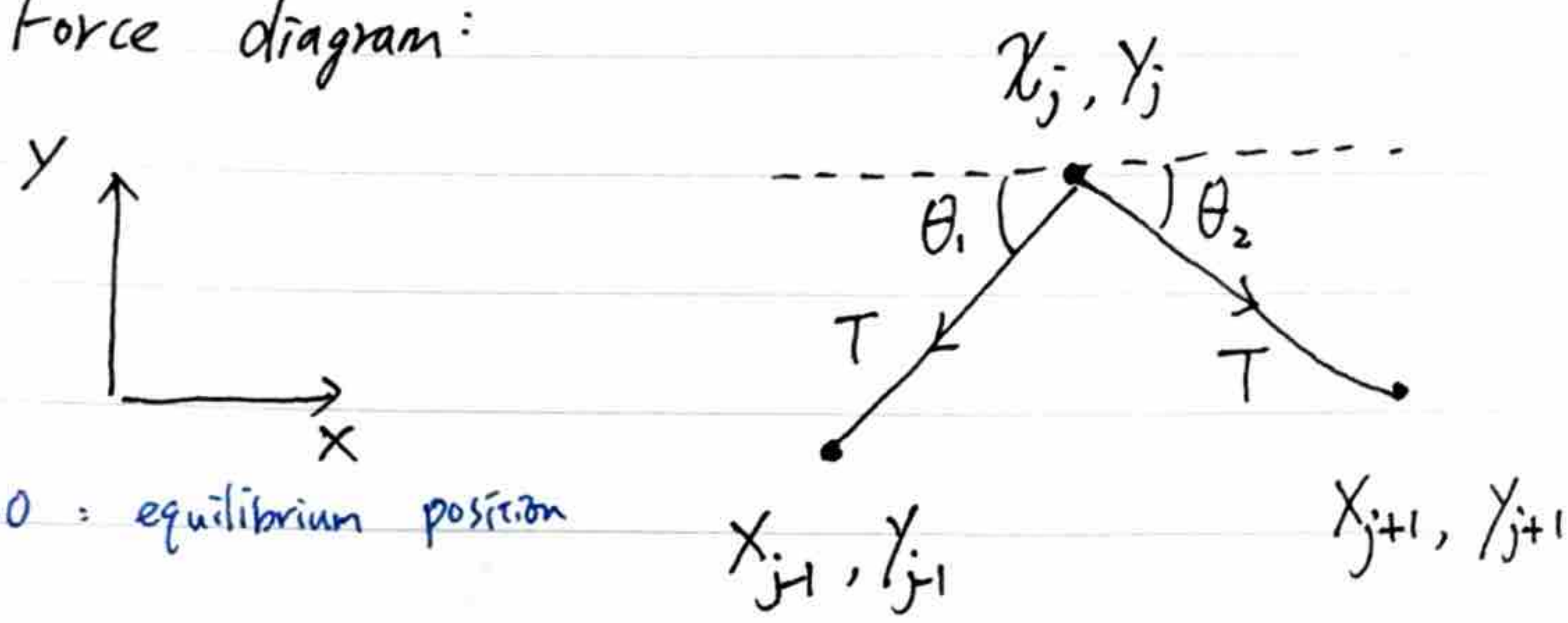


A lot of point like massive particles connected by massless strings.

These particles can only move up and down. We have constant tension T and small vibration. Distance between particles : a

Question: what will be the resulting motion of the system?

Force diagram:



$y=0$: equilibrium position

Assume $y_j \ll a \Rightarrow \theta_1 \ \& \ \theta_2 \ll 1$

Horizontal Direction : $m\ddot{x}_j = -T\cos\theta_1 + T\cos\theta_2$ ..... ①

Vertical Direction : $m\ddot{y}_j = -T\sin\theta_1 - T\sin\theta_2$ ..... ②

since $\theta_1 \ \& \ \theta_2$ are very small $\Rightarrow \cos\theta \sim 1 \ , \ \sin\theta \sim \theta$

① $\Rightarrow m\ddot{x}_j = -T + T = 0$ No motion in the horizontal direction

② $\Rightarrow$
$$m\ddot{y}_j = -T(\sin\theta_1 + \sin\theta_2)$$
$$\approx -T\left(\frac{y_j - y_{j-1}}{a} + \frac{y_j - y_{j+1}}{a}\right)$$
$$m\ddot{y}_j = +\frac{T}{a}\left(y_{j-1} - 2y_j + y_{j+1}\right)$$

Normal modes: $y_j = \mathrm{Re}\left(A_j e^{i(\omega t + \phi)} \right)$

From S matrix, the eigenvectors are $A = \begin{pmatrix} \vdots \\ A_j \\ A_{j+1} \\ A_{j+2} \\ \vdots \end{pmatrix}$

$$A_j \propto \beta^j = e^{ijka}$$

← giving β a fancy name.

* reminder: a : distance between particles in the x direction

To get $M^{-1}K$ matrix:

$$M = \begin{pmatrix} m & & & 0 \\ & m & & \\ & & m & \\ 0 & & & m \end{pmatrix} \qquad K = \begin{pmatrix} \cdots & -\frac{T}{a} & \frac{2T}{a} & -\frac{T}{a} & \cdots & 0 \\ \cdots & 0 & -\frac{T}{a} & \frac{2T}{a} & -\frac{T}{a} & \cdots \\ & 0 & 0 & \ddots & \ddots & \ddots \end{pmatrix}$$

$$M^{-1}K = \begin{pmatrix} \cdots & -\frac{T}{ma} & \frac{2T}{ma} & -\frac{T}{ma} & \cdots & \\ \cdots & 0 & -\frac{T}{ma} & \frac{2T}{ma} & -\frac{T}{ma} & \cdots \\ \cdots & & \ddots & \ddots & \ddots & \end{pmatrix}$$

To get ω:

Since $M^{-1}K$ and S share the same eigenvectors

Calalate $M^{-1}KA = \omega^2 A$

j_{th} term: $\omega^2 A_j = \frac{T}{ma}\left(-A_{j-1} + 2A_j - A_{j+1}\right)$

$$\omega^2 \cancel{A_j} = \frac{T}{ma} \cancel{A_j} \left(-e^{-ika} + 2 - e^{ika}\right)$$

$$\omega^2 = \underbrace{\frac{T}{ma}}_{= \omega_0^2}\left(2 - 2\cos ka\right)$$

$\omega_0^2 \equiv \frac{T}{ma}$

$$= 2\omega_0^2 \left(1 - \cos ka\right)$$

$$\omega^2 = 4\omega_0^2 \sin^2 \frac{ka}{2}$$

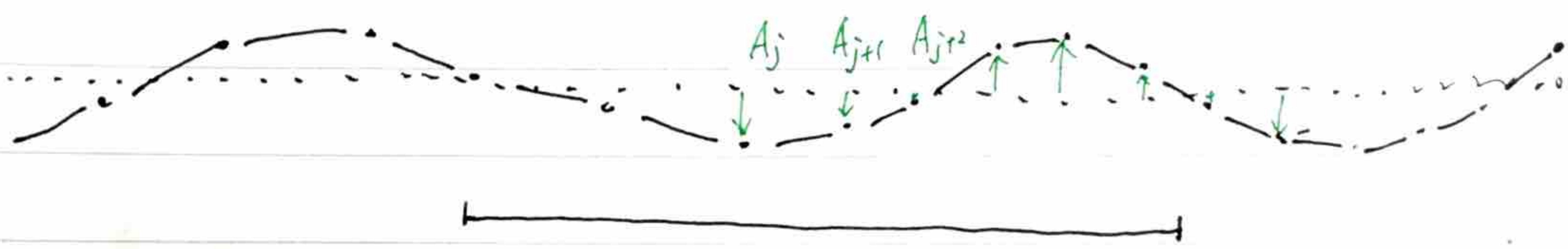
Almost the same as what we get in the last lecture!

$\omega = \omega(k)$ is a function of k

"Dispersion Relation"

k is given $\Rightarrow$ ω is determined
↳ wave number ↳ angular frequency
$k = \frac{2\pi}{\lambda}$

Normal modes: Standing waves!



$\lambda = \frac{2\pi}{k}$

Oscillating at frequency ω, determined by k

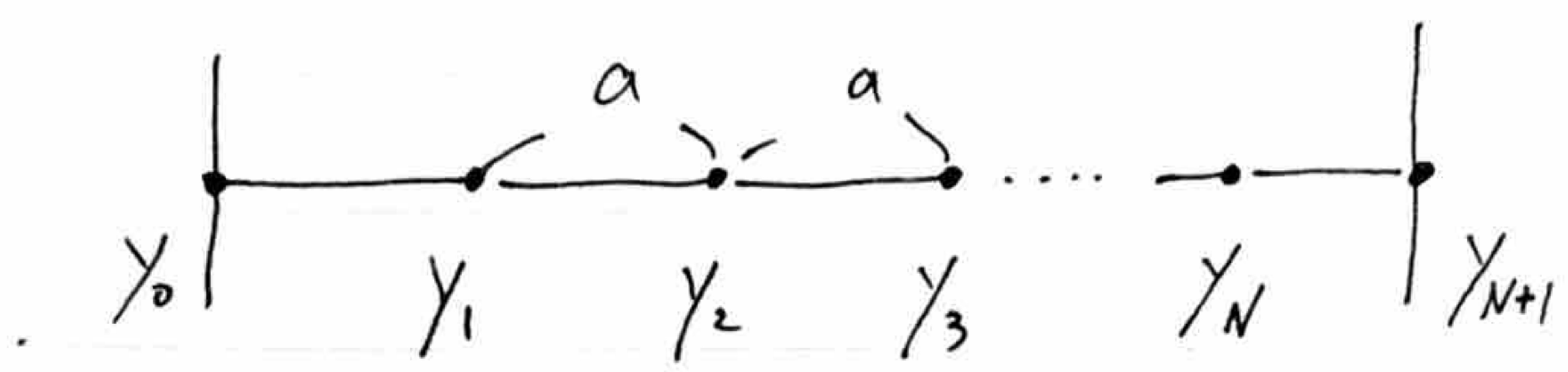
This system is infinitely long.

All possible k values (thus wavelength) are allowed. Each k value correspond to a different normal mode with angular frequency $\omega(k)$

Now we will try to solve a finite system using this infinitely long system.

Consider the following boundary conditions:

(1) Fixed end:



Boundary conditions: ① $y_0 = 0$, ② $y_{N+1} = 0$

What are the normal modes satisfying the boundary conditions?

There are two k values which give the same ω

$$\omega(k) = \omega(-k)$$

Therefore: linear combinations of e^{ijka} and e^{-ijka} are also normal modes

Guess: $$y_j = \mathrm{Re}\left[e^{i(\omega t+\phi)} \left(\alpha e^{ijka} + \beta e^{-ijka} \right) \right]$$

α, β are constants.

Use boundary conditions: ① $y_0 = 0 \Rightarrow \alpha + \beta = 0 \Rightarrow \alpha = -\beta$

② $y_{N+1} = 0 \Rightarrow \alpha \left(e^{i(N+1)ka} + e^{-i(N+1)ka} \right) = 0$

$$2i \sin (N+1)ka = 0$$

$\Rightarrow ka = \dfrac{n\pi}{N+1}$ $\quad n = 1, 2, 3, \dots N$

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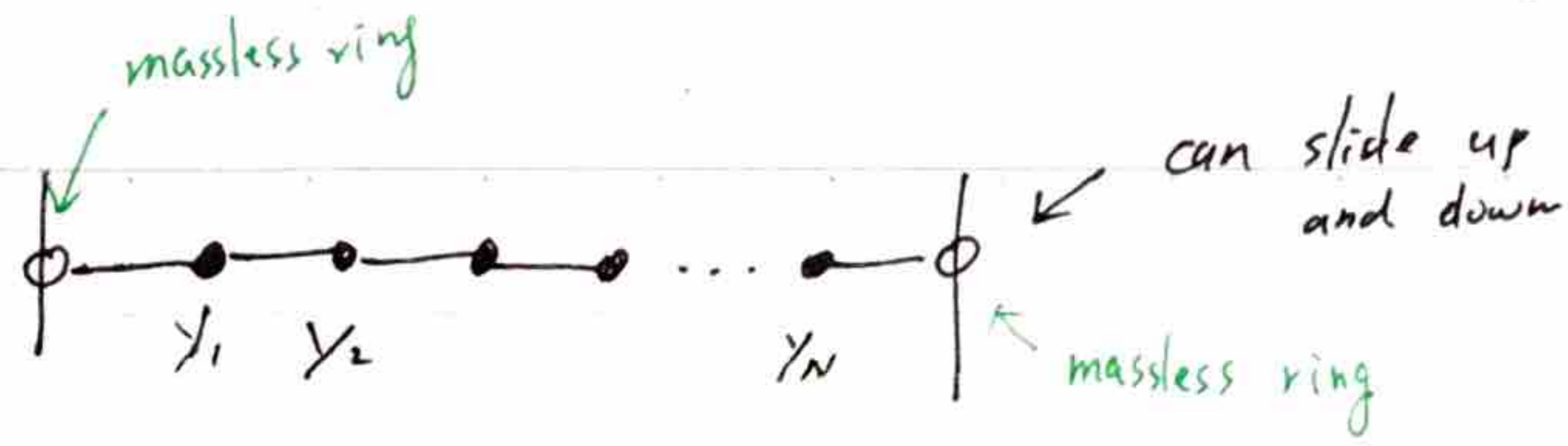
(More examples:)

(2) Open End:



Boundary Conditions

① $y_1 = y_0$ ② $y_N = y_{N+1}$

From first boundary condition

① $\Rightarrow \quad \alpha + \beta = \alpha e^{ika} + \beta e^{-ika}$

$\Rightarrow \quad \alpha(1 - e^{ika}) = \beta(e^{-ika} - 1)$

Second boundary condition

② $\Rightarrow \quad \alpha e^{iNka} + \beta e^{-iNka} = \alpha e^{i(N+1)ka} + \beta e^{-i(N+1)ka}$

$\Rightarrow \quad \alpha e^{iNka}(1 - e^{ika}) = \beta e^{-iNka}(e^{-ika} - 1)$

$\frac{①}{②} \quad e^{iNka} = e^{-iNka} \quad \Rightarrow e^{2iNka} = 1$

$\Rightarrow ka = \frac{2n\pi}{2N} = \frac{n\pi}{N}$

$n = 1, 2, 3, 4 \cdots N$

① $\Rightarrow \quad \beta = \alpha e^{ika} \quad \Rightarrow y_j = \alpha(e^{ijka} + e^{-i(j-1)ka})$

$= \alpha e^{-i\frac{ka}{2}}(e^{i(j-\frac{1}{2})ka} + e^{-i(j-\frac{1}{2})ka})$

$\propto \cos(ka(j - \tfrac{1}{2}))$

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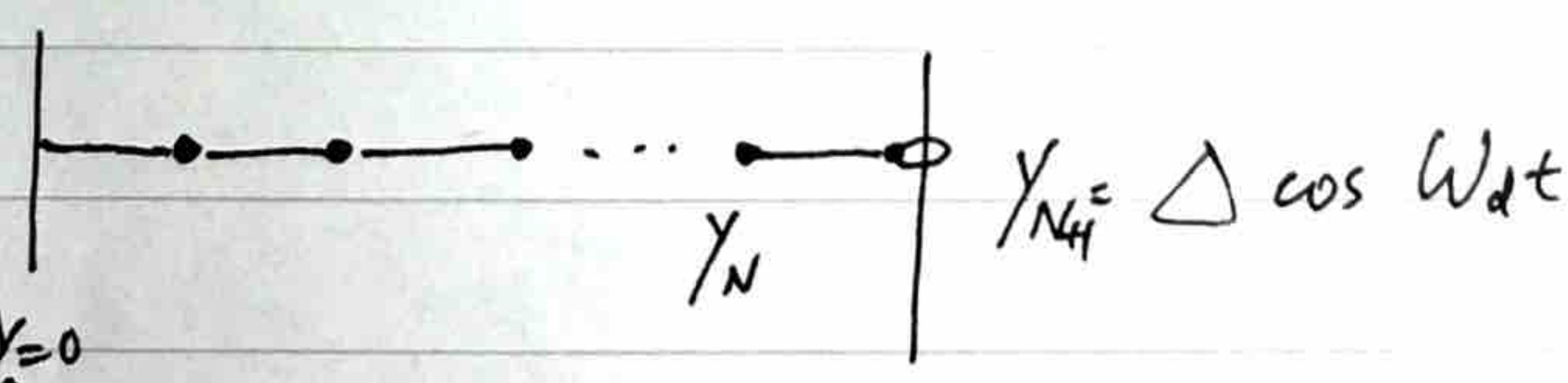
(3)



Boundary Conditions: ① $y_0 = 0$, ② $y_{N+1} = \Delta \cos W_d t$

Need to find the "particular solution"

y_j must be oscillating at a frequency W_d

What is the corresponding k_d which gives W_d ?

↳ Use $W(k)$

$$W_d^2 = 2W_0^2 (1 - \cos k_d a)$$

$\Rightarrow$ Solve to get $k_d a = \cos^{-1}\left(1 - \frac{W_d^2}{2W_0^2}\right)$

Guess $Y_j = Re\left[e^{iW_d t} \left(\alpha e^{ijk_d a} + \beta e^{-ijk_d a} \right) \right]$

↑ oscillating at W_d

Boundary condition at $j=0$:

$$Y_0 = 0 \Rightarrow \alpha + \beta = 0 \Rightarrow \beta = -\alpha$$

$$\Rightarrow Y_j = Re\left[2i e^{i(W_d t)} A \sin j k_d a \right]$$

Boundary condition at $j = N+1$

$$Y_{N+1} = Z \cos W_d t = Re\left[\Delta e^{iW_d t} \right]$$

$$\Rightarrow 2i A \sin (N+1) k_d a = \Delta$$

$$A = \frac{\Delta}{2i \sin (N+1) k_d a}$$

$$\Rightarrow Y_j = Re\left[\frac{\Delta \sin j k_d a}{\sin (N+1) k_d a} e^{i(W_d t)} \right]$$

$$= \frac{\Delta \sin j k_d a}{\sin (N+1) k_d a} \cos W_d t$$

↳ Explode when $k_d a = \frac{n\pi}{N+1}$!!

(Match with normal mode frequency)

Break?

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Summary:

① Symmetry + doesn't explode at the edge of the universe

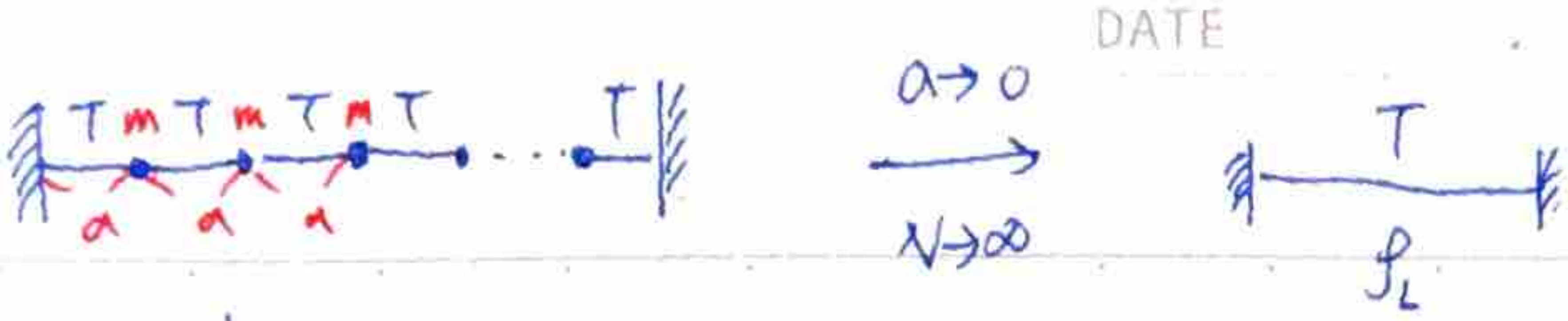
choose ⇒ $\beta = e^{ika}$

② E.O.M can be derived from Physical laws

③ Dispersion relation $\omega(k)$ can be derived from ① and ②

④ The allowed k value is determined by boundary condition. Full solution = linear combination of normal modes

⑤ Use initial condition to determine the unknowns.



Now make it continuous !!!

jth term of

$$M^{-1}KA \Rightarrow \omega^2 A_j = \frac{T}{ma}\left(-A_{j-1} + 2A_j - A_{j+1}\right)$$

$$\Downarrow$$

at x

$$M^{-1}KA \Rightarrow \omega^2 A(x) = \frac{T}{ma}\left(-A(x-a) + 2A(x) - A(x+a)\right)$$

Taylor Series: $f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{1}{2!}\Delta x^2 f''(x)$

$$A(x-a) = A(x) - aA'(x) + \tfrac{1}{2}a^2 A''(x)$$
$$A(x+a) = A(x) + aA'(x) + \tfrac{1}{2}a^2 A''(x)$$

$$\Rightarrow -A(x-a) + 2A(x) - A(x+a) = -\frac{\partial^2 A(x)}{\partial x^2}a^2 + \cdots$$

$$M^{-1}KA(x) = \frac{T}{ma}\frac{\partial^2 A(x)}{\partial x^2}a^2 + \cdots$$

In the limit $a \ll$ wave length

$\Rightarrow$ We can ignore the higher order terms

$$\rho_L \equiv \frac{m}{a}$$

$$\Rightarrow M^{-1}K \rightarrow -\frac{T}{\rho_L}\frac{\partial^2}{\partial x^2}$$

($M^{-1}K$ is now an "operator")

$$\Rightarrow \frac{\partial^2 \psi(x,t)}{\partial t^2} = \frac{T}{\rho_L}\frac{\partial^2 \psi(x,t)}{\partial x^2}$$

plug in normal mode $e^{ikx}e^{i\omega t}$

$\Rightarrow$ Dispersion relation: $\omega^2 = \frac{T}{\rho_L}k^2$

$$\frac{\omega}{k} = v_p = \sqrt{\frac{T}{\rho_L}}$$

v_p: phase velocity
ω: angular frequency
k: wave number