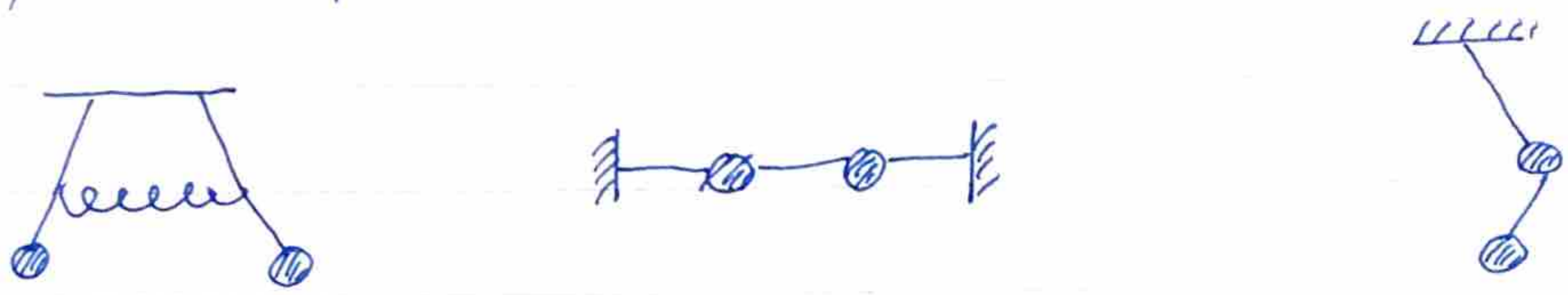
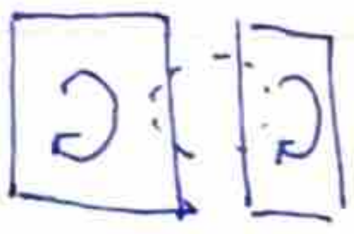
Reminder:

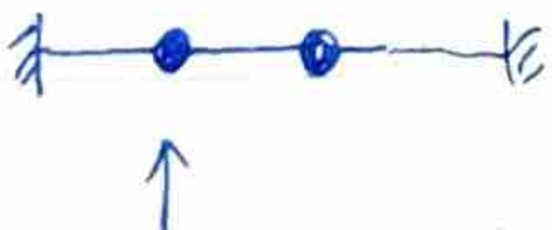
Summary of coupled oscillators:



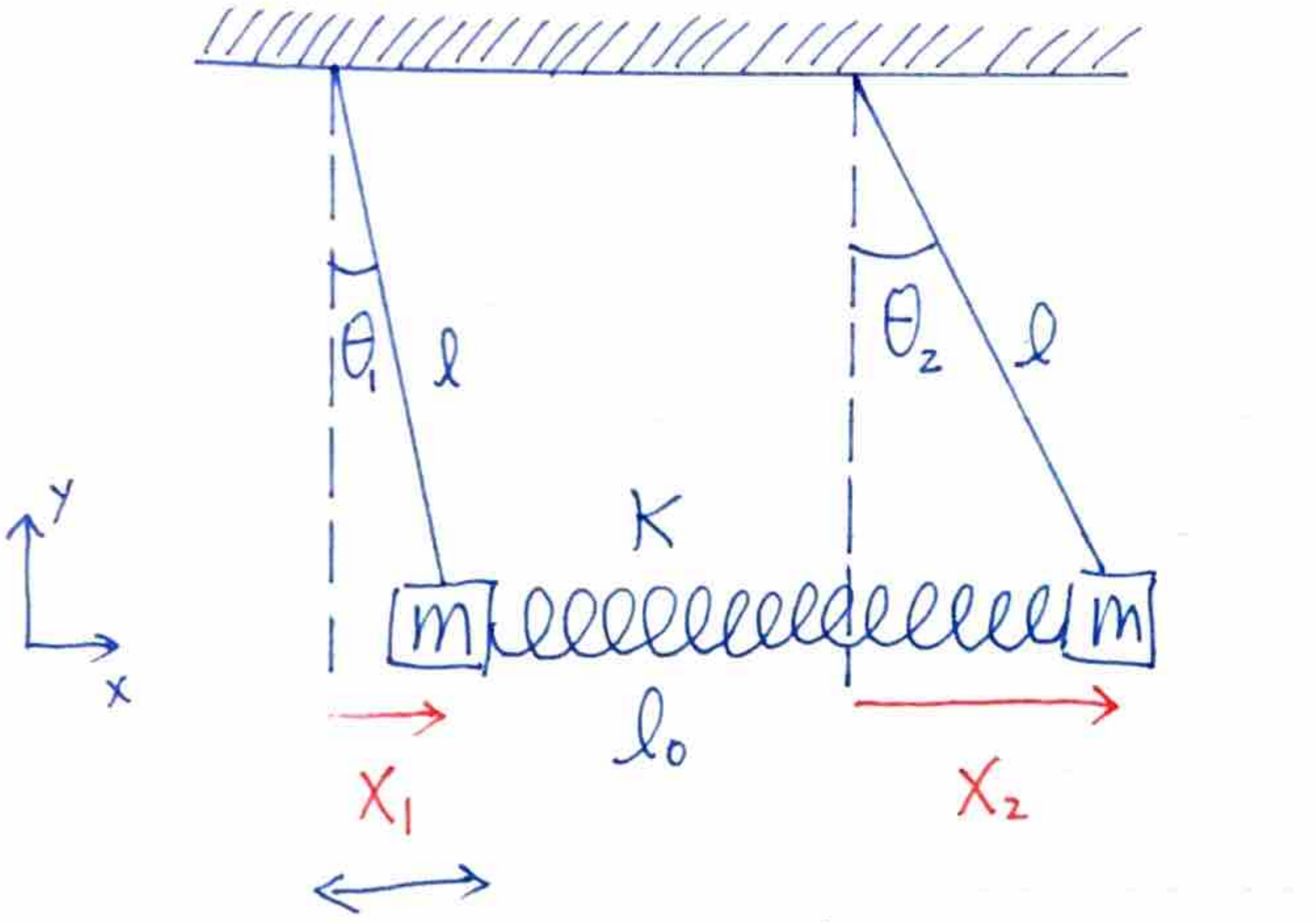


....	Arbitray Excitation	Normal Mode Excitation
Motion	Not harmonic	harmonic
Amplitude Ratio	Varies	constant
Energy	Migrates	stays

Next: Driven Coupled oscillators



add external force !



Driving force $\vec{F_d} = F_0 \cos(\omega_d t)\hat{x}$

Last time:

We solved the normal mode of this system

Now we would like to add a driving force into the game.

Equation of motion:

$$\begin{cases} m\ddot{x}_1 = -\left(k+\frac{mg}{l}\right)x_1 + kx_2 + F_0\cos(\omega_d t) \\ m\ddot{x}_2 = kx_1 - \left(k+\frac{mg}{l}\right)x_2 \end{cases}$$

Matrix form:

$$M\ddot{X} = -KX + F\cos(\omega_d t)$$

where

$$M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \qquad K = \begin{pmatrix} k+\frac{mg}{l} & -k \\ -k & k+\frac{mg}{l} \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad M^{-1} = \begin{pmatrix} \frac{1}{m} & 0 \\ 0 & \frac{1}{m} \end{pmatrix}$$

$(\times M^{-1})$

$$\rightarrow \quad \ddot{X} = -M^{-1}KX + M^{-1}F\cos(\omega_d t)$$

$$M^{-1}K = \begin{pmatrix} \frac{k}{m}+\frac{g}{l} & \frac{-k}{m} \\ \frac{-k}{m} & \frac{k}{m}+\frac{g}{l} \end{pmatrix} \qquad M^{-1}F = \begin{pmatrix} \frac{F_0}{m} \\ 0 \end{pmatrix}$$

Last time we have solved the homogeneous solution:

$$\det\left(M^{-1}K - \omega^2 I\right) = 0$$

Solutions $\Rightarrow$ ① $\omega_1^2 = \frac{g}{l}$ $\quad A^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

② $\omega_2^2 = \frac{g}{l} + \frac{2k}{m}$ $\quad A^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\det\left(M^{-1}K - \omega^2 I\right) = \left(\omega^2 - \omega_1^2\right)\left(\omega^2 - \omega_2^2\right) = 0$$

Homogeneous solution:

$$X = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_1 t + \phi_1) + \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_2 t + \phi_2)$$

Now we have an additional driving force:

$$\ddot{X} + M^{-1}K X = M^{-1}F \cos(\omega_d t)$$

Similar to driven oscillator problem we want to eliminate $\cos(\omega_d t)$ term ...

Go to complex notation $\quad X = \mathrm{Re}(Z) \quad \ddot{Z} + M^{-1}kZ = M^{-1}Fe^{i\omega_d t}$

Guess:

$$Z = B\, e^{i\omega_d t}$$

No δ ! (because it's undamped)

Amplitude of the driving force induced oscillation

$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

Plug into the equation:

$$\Rightarrow \left(-\omega_d^2 I + M^{-1}K\right) \underset{\substack{\downarrow \\ B\cancel{e^{i\omega_d t}}}}{Z} = M^{-1}F \cancel{e^{i\omega_d t}}$$

$$\Rightarrow \left(M^{-1}K - \omega_d^2 I\right) B = M^{-1}F$$

These are just two simultaneous equations:

$$\underbrace{\begin{pmatrix} \frac{k}{m}+\frac{g}{l}-\omega_d^2 & \frac{-k}{m} \\ \frac{-k}{m} & \frac{k}{m}+\frac{g}{l}-\omega_d^2 \end{pmatrix}}_{\overleftrightarrow{E}} \underbrace{\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}}_{\vec{B}} = \underbrace{\begin{pmatrix} \frac{F_0}{m} \\ 0 \end{pmatrix}}_{\vec{D}}$$

$$\Rightarrow \begin{cases} \left(\frac{k}{m}+\frac{g}{l}-\omega_d^2\right) B_1 - \frac{k}{m} B_2 = \frac{F_0}{m} \\ \frac{-k}{m} B_1 + \left(\frac{k}{m}+\frac{g}{l}-\omega_d^2\right) B_2 = 0 \end{cases}$$

We can go ahead and solve it directly to get B_1, B_2

Or, we use "Cramer's Rule"

Useful for large # of coupled oscillators.

$$\overleftrightarrow{E} = \begin{pmatrix} \frac{k}{m}+\frac{g}{\ell}-\omega_d^2 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m}+\frac{g}{\ell}-\omega_d^2 \end{pmatrix}$$

$$\vec{D} = \begin{pmatrix} \frac{F_0}{m} \\ 0 \end{pmatrix}$$

Replace the first column by $\vec{D}$

$$B_1 = \frac{\left| \begin{pmatrix} \vec{D} \end{pmatrix} \begin{pmatrix} \ \end{pmatrix} \right|}{\det \overleftrightarrow{E}}$$

$$= \frac{\begin{vmatrix} \frac{F_0}{m} & -\frac{k}{m} \\ 0 & \frac{k}{m}+\frac{g}{\ell}-\omega_d^2 \end{vmatrix}}{(\omega_d^2-\omega_1^2)(\omega_d^2-\omega_2^2)}$$

Yen-Jie: We have evaluated this before when we solve the normal mode frequency (replace ω by ω_d)

$$= \frac{\frac{F_0}{m}\left(\frac{k}{m}+\frac{g}{\ell}-\omega_d^2\right)}{(\omega_d^2-\omega_1^2)(\omega_d^2-\omega_2^2)}$$

explode when $\omega_d = \omega_1$ or $\omega_d = \omega_2$!!!

$\uparrow$ Normal mode 1 $\qquad$ $\uparrow$ Normal mode 2

Similarly:

$$B_2 = \frac{\left|\underset{\text{original content from } \overleftrightarrow{E}}{\left(\quad\right)}\underset{\text{Replace the second column by } \vec{D}}{\left(\vec{D}\right)}\right|}{\det \overleftrightarrow{E}}$$

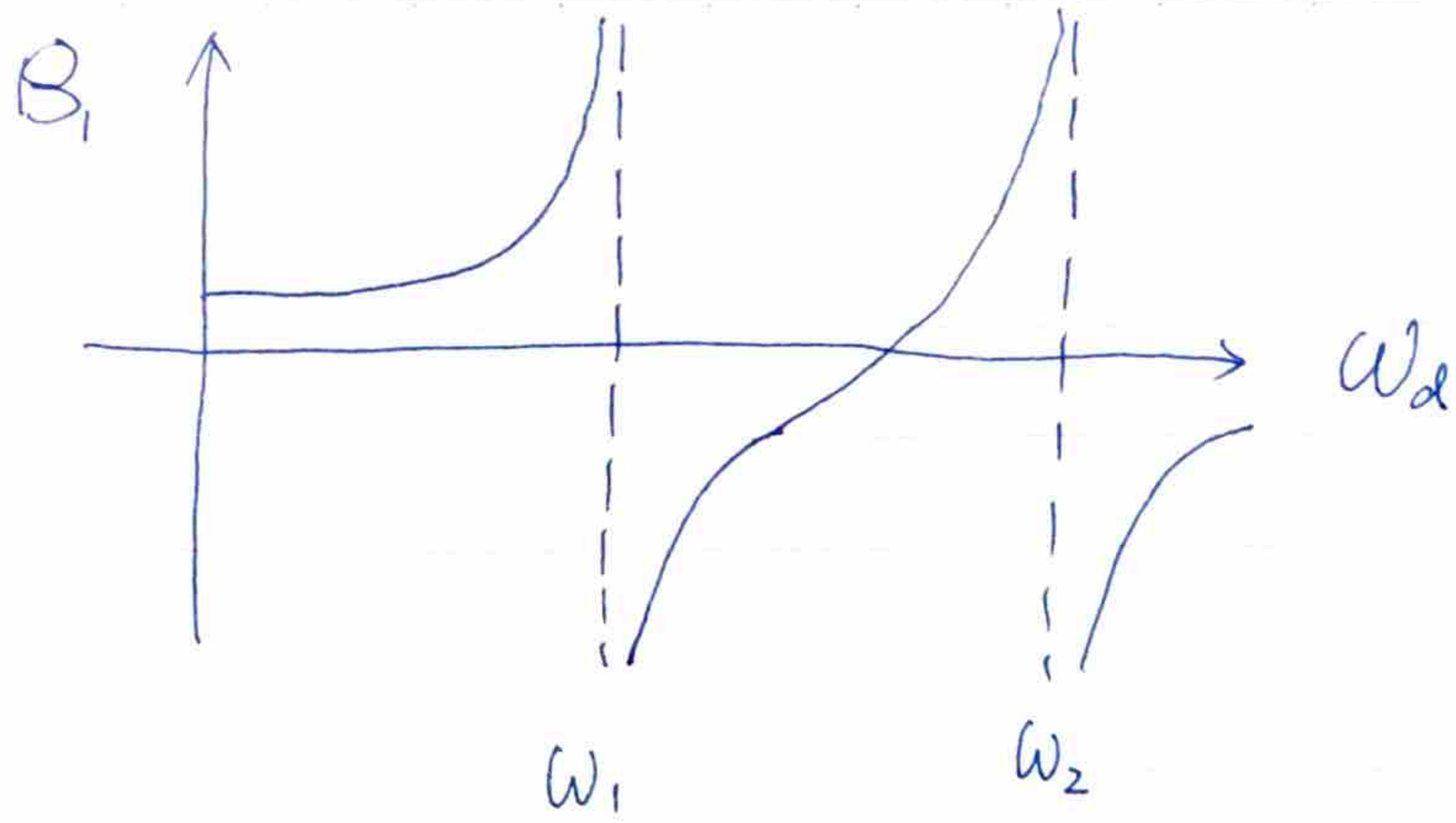
$$= \frac{\begin{pmatrix} \frac{k}{m}+\frac{g}{l}-W_d^2 & \frac{F_0}{m} \\ -\frac{k}{m} & 0 \end{pmatrix}}{(W_d^2-W_1^2)(W_d^2-W_2^2)}$$

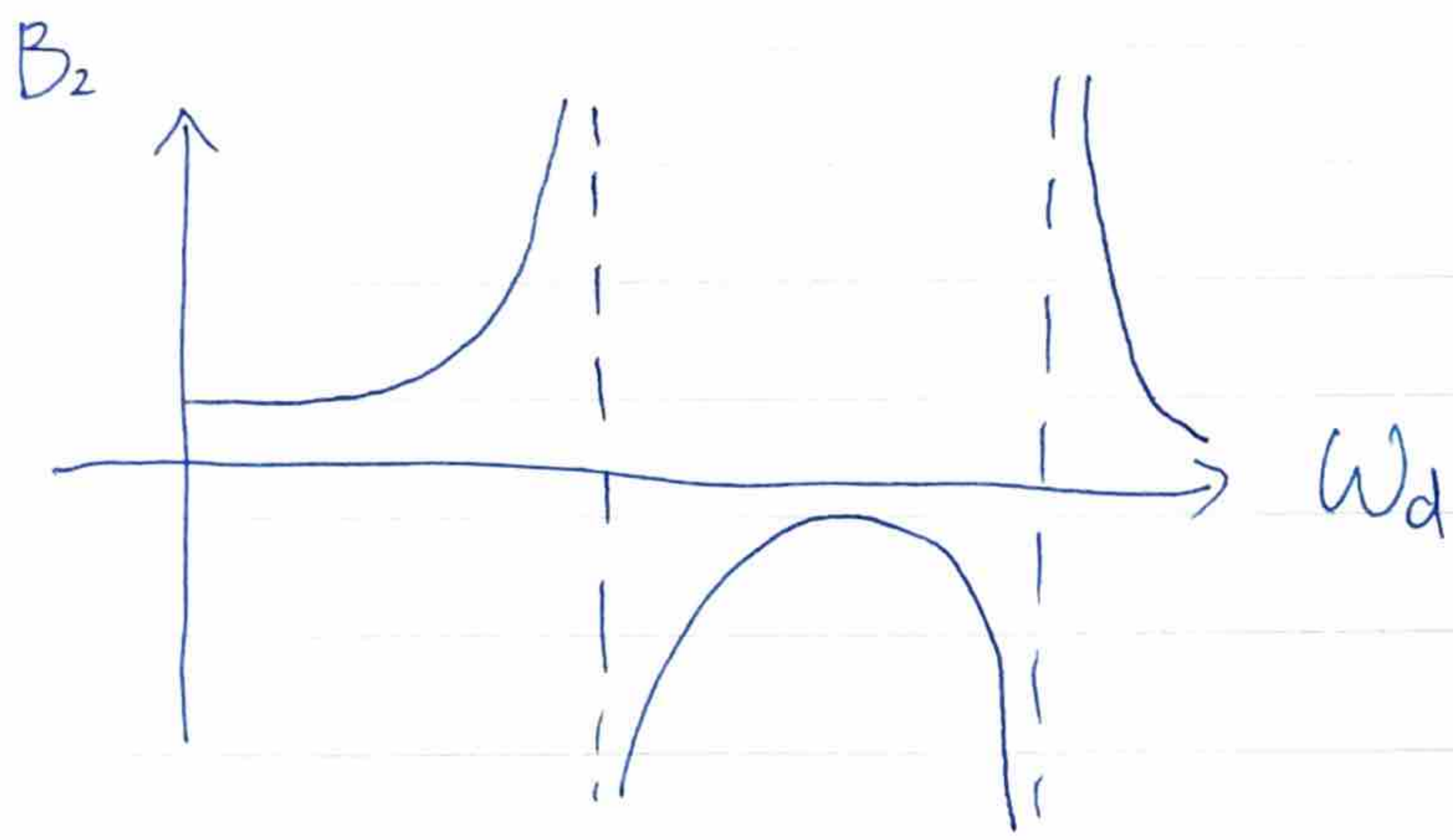
$$= \frac{\frac{F_0}{m}\left(\frac{k}{m}\right)}{(W_d^2-W_1^2)(W_d^2-W_2^2)}$$

$$\frac{B_1}{B_2} = \frac{\left(\frac{k}{m}+\frac{g}{l}-W_d^2\right)}{\frac{k}{m}}$$

(1) $W_d^2 = W_1^2 = \frac{g}{l} \Rightarrow \frac{B_1}{B_2} = 1$

(2) $W_d^2 = W_2^2 = \frac{g}{l}+\frac{2k}{m} \Rightarrow \frac{B_1}{B_2} = -1$





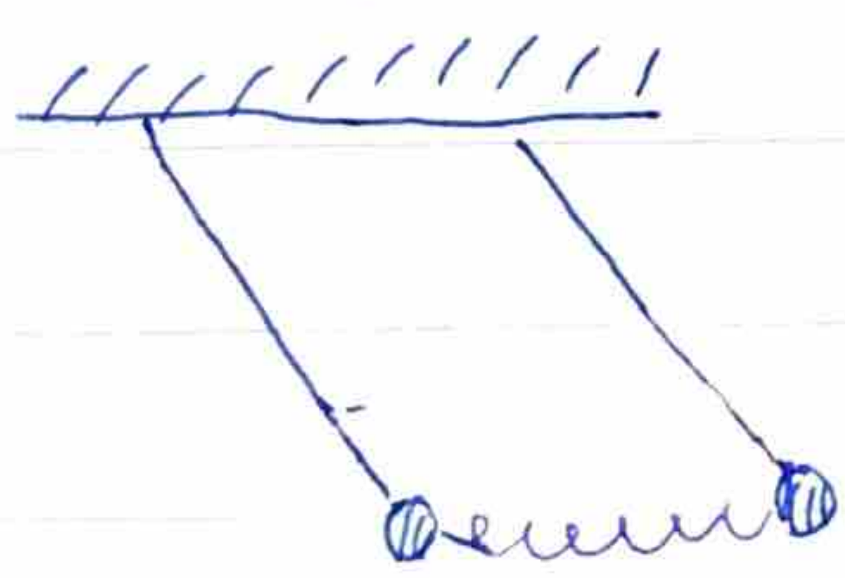
$\Updownarrow$ $\qquad$ $\Updownarrow$

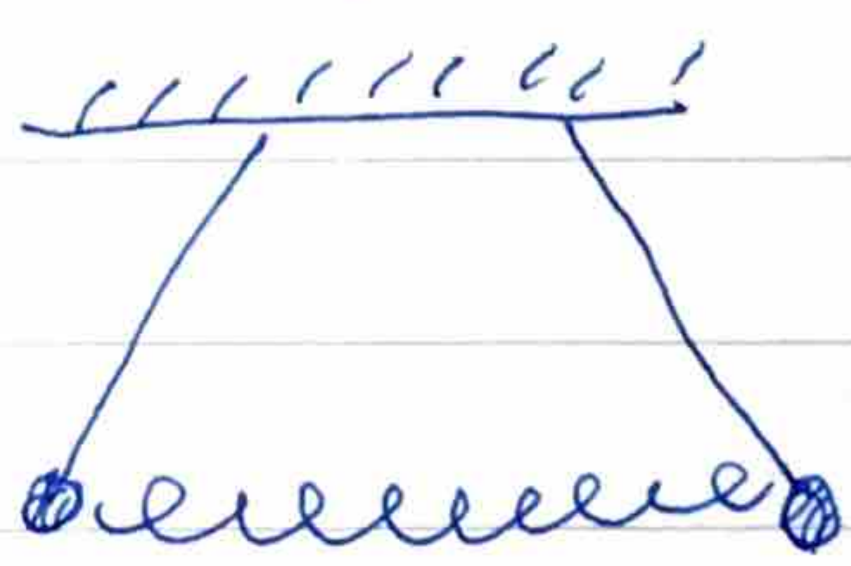
$B_1 \approx B_2$ $\qquad$ $B_1 \approx -B_2$

$\uparrow$ $\qquad$ $\uparrow$

Excite Mode 1 (Near ω_1) $\qquad$ Excite Mode 2 (Near ω_2)

$\Downarrow$ $\qquad$ $\Downarrow$





DEMO

Full solution:

$$x_1 = \alpha \cos(\omega_1 t + \phi_1) + \beta \cos(\omega_2 t + \phi_2) + B_1(\omega_d)\cos(\omega_d t)$$

$$x_2 = \underbrace{\alpha \cos(\omega_1 t + \phi_1) + (-\beta)\cos(\omega_2 t + \phi_2)}_{\text{Homogeneous Solution}} + \underbrace{B_2(\omega_d)\cos(\omega_d t)}_{\text{Particular Solution}}$$