Massachusetts Institute of Technology

Physics 8.03 Fall 2016

Exam 1 Formula Sheet

Springs and masses:

$$m\frac{d^2}{dt^2}x(t) + b\frac{d}{dt}x(t) + kx(t) = F(t)$$

More general differential equation with harmonic driving force:

$$\frac{d^2}{dt^2}x(t) + \Gamma\frac{d}{dt}x(t) + \omega_0^2 x(t) = \frac{F_0}{m}\cos(\omega_d t)$$

Steady state solutions:

$$x_s(t) = A\cos(\omega_d t - \delta)$$

where

$$A = \frac{\frac{F_0}{m}}{\sqrt{(\omega_0^2 - \omega_d^2)^2 + \omega_d^2\Gamma^2}}$$

and

$$\tan\delta = \frac{\Gamma\omega_d}{\omega_0^2 - \omega_d^2}$$

General solutions:
For $\Gamma = 0$ (undamped system):

$$x(t) = R\cos(\omega_0 t + \theta) + x_s(t)$$

where R and θ are unknown coefficients.
For $\Gamma < 2\omega_0$ (under damped system):

$$x(t) = Re^{-\frac{\Gamma}{2}t}\cos\left(\sqrt{\omega_0^2 - \frac{\Gamma^2}{4}}\ t + \theta\right) + x_s(t)$$

where R and θ are unknown coefficients.
For $\Gamma = 2\omega_0$ (critically damped system):

$$x(t) = (R_1 + R_2 t)e^{-\frac{\Gamma}{2}t} + x_s(t)$$

where R_1 and R_2 are unknown coefficients.
For $\Gamma > 2\omega_0$ (over damped system):

$$x(t) = R_1 e^{-\left(\frac{\Gamma}{2} + \sqrt{\frac{\Gamma^2}{4} - \omega_0^2}\right)t} + R_2 e^{-\left(\frac{\Gamma}{2} - \sqrt{\frac{\Gamma^2}{4} - \omega_0^2}\right)t} + x_s(t)$$

where R_1 and R_2 are unknown coefficients.
Coupled oscillators

$$F_j = -\sum_{k=1}^{n} K_{jk} x_k$$

Examples for $n = 2$

$$\mathcal{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\mathcal{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

$$\mathcal{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

Matrix equation of motion, matrices $\mathcal{M}$, $\mathcal{K}$, $\mathcal{I}$ are $n \times n$, vectors $\mathcal{X}$, $\mathcal{Z}$ are $n \times 1$.

$$\frac{d^2}{dt^2}\mathcal{X}(t) = -\mathcal{M}^{-1}\mathcal{K}\mathcal{X}(t)$$

$$\mathcal{Z}(t) = \mathcal{A}e^{-i\omega t}$$

$$(\mathcal{M}^{-1}\mathcal{K} - \omega^2\mathcal{I})\mathcal{A} = 0$$

To obtain the frequencies of normal modes solve:

$$det(\mathcal{M}^{-1}\mathcal{K} - \omega^2\mathcal{I}) = 0$$

For $n = 2$

$$det \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = M_{11}M_{22} - M_{12}M_{21}$$

If the system is driven by force one can find the response amplitudes $\mathcal{C}(\omega_d)$

$$\mathcal{F}(t) = \mathcal{F}_0 e^{-i\omega_d t}$$

$$\mathcal{W}(t) = \mathcal{C}(\omega_d)e^{-i\omega_d t}$$

$$\mathcal{C}(\omega_d) = \begin{bmatrix} c_1(\omega_d) \\ c_2(\omega_d) \end{bmatrix}$$

$$(\mathcal{M}^{-1}\mathcal{K} - \omega_d^2\mathcal{I})\mathcal{C}(\omega_d) = \mathcal{F}_0$$

solving the equation above one can find the response amplitudies for the first ($c_1(\omega_d)$) and second ($c_2(\omega_d)$) objects in the system.

Reflection symmetry matrix:

$$\mathcal{S} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Eigenvalues (β) and eigenvectors ($\mathcal{A}$) of this 2×2 $\mathcal{S}$ matrix:

(1) $\beta = -1$, $\mathcal{A} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(2) $\beta = 1$, $\mathcal{A} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

1D infinite coupled system which satisfy space translation symmetry:

Given a eigenvalue β, the corresponding eigenvector is

$$A_j = \beta^j A_0$$

where

$$A_j(A_0)$$

is the normal amplitude of jth(0th) object in the system.

Consider an one dimentional system which consists infinite number of masses coupled by springs, β can be written as $\beta = e^{ika}$ where k is the wave number and a is the distance between the masses.

Kirchoff's Laws (be careful about the signs!)

$$\text{Node}: \sum_i I_i = 0 \ \text{Loop}: \sum_i \Delta V_i = 0$$

$$\text{Capacitors}: \Delta V = \frac{Q}{C} \quad \text{Inductors}: \Delta V = -L\frac{dI}{dt} \quad \text{Current}: I = \frac{dQ}{dt}$$

8.03SC Physics III: Vibrations and Waves
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