Physics 8.03
Vibrations and Waves

Lecture 11

Fourier Analysis with traveling waves

Dispersion

Last time:

- Arbitrary motion ➔ Superposition of ALL possible normal modes

$$y(x,t) = \sum_{m=1}^{\infty} A_m \sin(k_m x)\cos(\omega_m t + \beta_m) + \sum_{n=0}^{\infty} B_n \cos(k_n x)\cos(\omega_n t + \beta_n)$$

- Orthogonal functions ➔ Fourier coefficients

$$A_m = \frac{2}{L}\int_0^L y(x,t=0)\sin(k_m x)\,dx$$

$$B_n = \frac{2}{L}\int_0^L y(x,t=0)\cos(k_n x)\,dx$$

Fourier expansion recipé

- Start with superposition of all possible modes
- Determine the simplest basis functions using
  - Boundary conditions → $[0, L]$ or $[-L/2, L/2]$ or $[-L, L]$
  - Symmetry → $f(-x, 0) = f(x, 0)$ or $f(-x, 0) = -f(x, 0)$
  - Initial condition → $y(x, 0) = 0$ *or* $v_y(x, 0) = 0$
- Determine the Fourier coefficients -- A_n and/or B_n
  Use orthogonality relations with
  - Initial deformation $y(x, t = 0)$ or
  - Initial velocity $v_y(x, t = 0)$
- Add the time-dependence

- Fourier expansions for traveling waves
- What happens if the Fourier components all travel at slightly different speeds?
  - $\omega_n \neq v\, k_n$ → DISPERSION
- Wave equation in dispersive media
  - Phase velocity → velocity of a single crest of the wave with average wave vector, $\bar{k}$ →

$$v_p = \frac{\omega}{k}$$

  - Group velocity → velocity of the slow envelope velocity of energy transport →

$$v_g = \frac{d\omega}{dk}$$

Corrections/comments on today's lecture

- Formula for approximation of ω_m was written incorrectly on the board; the correct version is

$$\omega^2 = c^2 k_m^2 (1 + \alpha k_m^2) \Rightarrow \omega = c k_m \sqrt{1 + \alpha k_m^2}$$

$$\omega \approx c k_m \left(1 + \frac{1}{2} \alpha k_m^2\right)$$

- Where does the equation for a stiff string come from?
  - For a derivation, see for example, Fetter and Walecka, "Theoretical mechanics of Particles and Continua," page 221