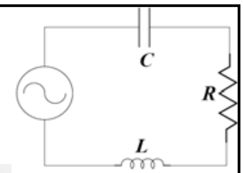
8.022 (E&M) – Lecture 18

Topics:

- RCL circuits: the hardest of the easiest part of the course?
  - More on complex impedance
  - Power and energy
  - Filters

Last time: AC driven RCLs



Simple solution when introducing following rules:

- Work with complex V and I
  - Real currents and voltages are just the real part of the $\tilde{V}$ and $\tilde{I}$.
- Generalization of Ohm's law to complex V and I:

$$\boxed{\tilde{V}(t) = \tilde{I}(t) Z_X}$$

where Z_X is the impedance of component X: $\begin{cases} Z_R = R \\ Z_C = \dfrac{1}{i\omega C} \\ Z_L = i\omega L \end{cases}$

- Analyze circuit as if it were DC with only resistors
- Take the real part of I(t) and V(t)
- The End.

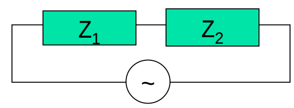
“Analyze as DC with only resistors”

What do I mean with this statement?

- Impedances in series
  - Same current flowing in each element

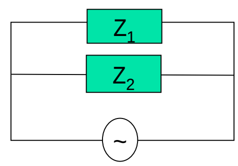
    $I_1Z_1=V_1$; $I_2Z_2=V_2$; $V_1+V_2=V$; $V=ZI$

    → $Z_{eq}=Z_1+Z_2$



- Impedances in parallel
  - Same voltage drop across each element
  - $V_1/Z_1=V_2/Z_2=V/Zeq$; $V_1=V_2=V$

    → $1/Z_{eq}=1/Z_1+1/Z_2$

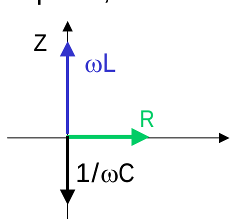


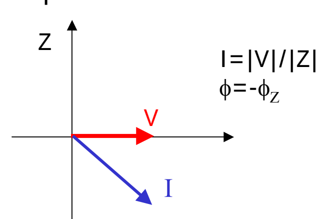
→ Same rules as resistors in series and parallel!

Is the current leading or lagging?

Instead of thinking of the problems in terms of complex currents, think in terms of complex impedance!

- Generalized Ohm’s law: $\tilde{V}(t)=\tilde{I}(t)Z_C$
  - All what we really care about is amplitude of I and relative phase between I and V
  - Trick: let’s choose V real (no law against it!) and draw the complex I, V and Z in the complex plane





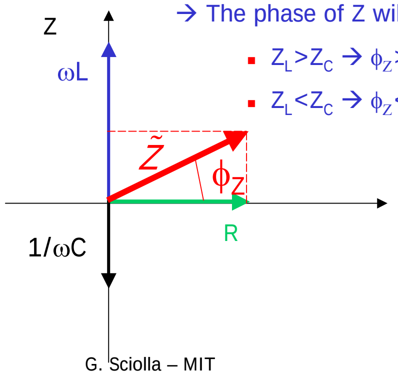
Is current leading or lagging? (2)

Consider the complex impedance:

- Real part: only R contributes
- Imaginary part: Z_L "pulls up" by ωL and Z_C pulls down by $1/\omega C$

→ The phase of Z will depend on who prevails:

- $Z_L > Z_C$ → $\phi_Z > 0$
- $Z_L < Z_C$ → $\phi_Z < 0$;



$$|Z| = \sqrt{\mathrm{Re}^2(Z) + \mathrm{Im}^2(Z)} = $$

$$= \sqrt{\mathrm{R}^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$$tg\phi_Z = \frac{\mathrm{Im}(Z)}{\mathrm{Re}(Z)} = \frac{\omega L - \frac{1}{\omega C}}{\mathrm{R}}$$

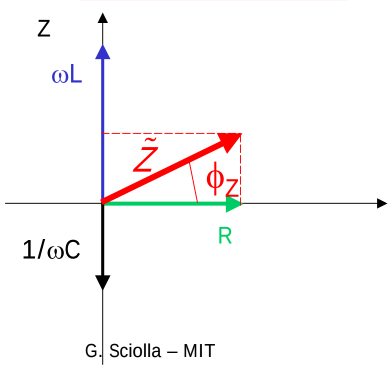
Is current leading or lagging? (3)

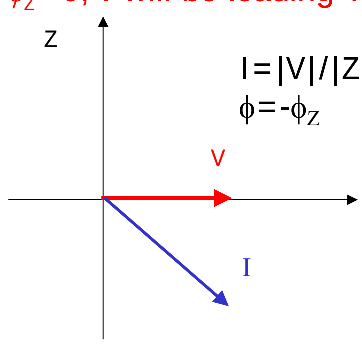
Now remember that $\tilde{V}(t) = \tilde{I}(t)\tilde{Z}_c$ and that we chose a real V:

$$\tilde{I}(t) = \frac{V(t)}{\tilde{Z}_c} = \frac{V(t)}{|\tilde{Z}_c|} e^{-i\phi_Z}$$

⇒ if $\phi_Z > 0$, I will be lagging V

if $\phi_Z < 0$, I will be leading V



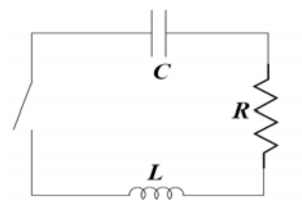


$I = |V|/|Z|$

$\phi = -\phi_Z$

Power in RCL circuits

- Power delivered in a circuit is
$$P(t) = V(t)I(t)$$
- Given $\begin{cases} V(t) = V_0 \cos \omega t \\ I(t) = I_0 \cos(\omega t - \phi) \end{cases}$



- The average power over a period T will be
$$\langle P \rangle = \frac{1}{T}\int_T V(t)I(t)dt = \frac{\omega}{2\pi}\int_T V_0 \cos \omega t I_0 \cos(\omega t - \phi)dt =$$
$$= \frac{\omega}{2\pi}\frac{V^2{}_0}{|Z|}\int_T \cos \omega t \cos(\omega t - \phi)dt$$
  - NB: when we say light bulb has a P of 100W we are referring to <P>
- Using the identity: cos(α–β)=cosαcosβ+sinαsinβ we obtain:
$$\langle P \rangle = \frac{\omega}{2\pi}\frac{V_0^2}{|Z|}\left[\int_0^{\frac{2\pi}{\omega}} \cos \omega t \cos \omega t \cos \phi dt + \int_0^{\frac{2\pi}{\omega}} \cos \omega t \sin \omega t \sin \phi dt\right]$$

Power in RCL circuits (2)

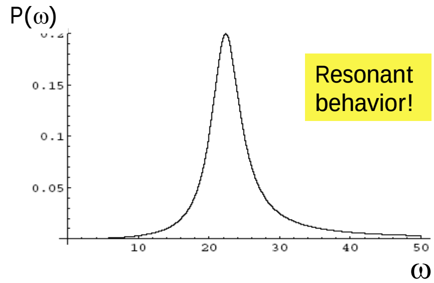
- Since: $\begin{cases} \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \cos^2 \omega t dt = \frac{1}{2} \\ \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \cos \omega t \sin \omega t dt = 0 \end{cases}$ $\Rightarrow \langle P \rangle = \frac{1}{2}\frac{V_0^2}{|Z(\omega)|}\cos \phi$
- NB: Power depends on relative phase between I and V
  - cosϕ=0 → no power dissipated in the circuit → no work done!
  - cosϕ=0 when ϕ= 90º → when Z is purely imaginary: R needed!
- Introducing: RMS (root mean squared) voltage and currents:
$$V_{RMS} = \frac{V_0}{\sqrt{2}} \text{ and } I_{RMS} = \frac{I_0}{\sqrt{2}}$$
  - NB: in the US: outlet voltage is 120 V. This is the RMS voltage: V_{max}=170

→ $\boxed{\langle P \rangle = \frac{V_{RMS}{}^2}{|Z(\omega)|}\cos \phi = RI_{RMS}(\omega)^2}$ remembering that $\cos \phi = \frac{R}{|Z(\omega)|}$

Power vs. frequency

NB: Z depends on ω → power dissipated depends on driving frequency!

$$\langle P \rangle = \frac{{V_{RMS}}^2}{|Z(\omega)|^2} R = \frac{{V_{RMS}}^2}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} R$$



Resonant behavior!

- At what ω is P is max?
  - $\omega L - \frac{1}{\omega C} = 0 \Rightarrow \omega = \frac{1}{\sqrt{LC}} = \omega_0$
- What ω is the max P?
  - $P_{max} = \frac{{V_{RMS}}^2}{R}$
- What is the corresponding phase?
  - Zero: the imaginary part due to C and L exactly cancel out!

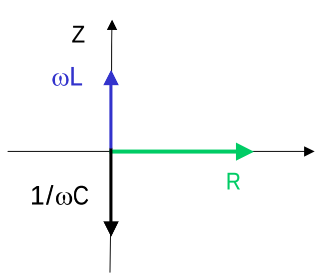
ω_0 in term of L and C

What does $\omega=\omega_0$ mean in terms of L and C?

- Remember:

$$\omega_0 = \frac{1}{\sqrt{LC}} \Leftrightarrow \omega L = \frac{1}{\omega C}$$

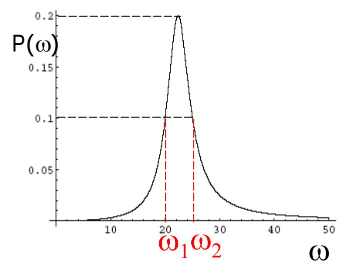
- Back to the phasor representation for Z



The imaginary part due to C exactly compensates the one due to L
→ Z is purely real!

How good is the resonant system?

- Definition: width of resonance wrt the height
- Width: $\Delta\omega$ between the points where the power goes to $P_{max}/2$: ω_1 and ω_2



$$\frac{V_{RMS}^2}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} R = \frac{V_{RMS}^2}{2R} \Rightarrow \left|\omega L - \frac{1}{\omega C}\right| = \pm R$$

$$\begin{cases} \omega_1 L - \frac{1}{\omega_1 C} = -R \\ \omega_2 L - \frac{1}{\omega_2 C} = R \end{cases} \Rightarrow \begin{cases} \omega_1^2 LC + RC\,\omega_1 - 1 = 0 \\ \omega_2^2 LC - RC\,\omega_2 - 1 = 0 \end{cases}$$

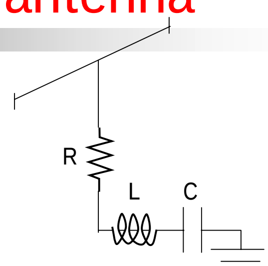
$$\begin{cases} \omega_1 = \frac{-RC \pm \sqrt{R^2C^2 + 4LC}}{2LC} = \\ \omega_2 = \frac{RC \pm \sqrt{R^2C^2 + 4LC}}{2LC} \end{cases} \Rightarrow \Delta\omega = \omega_2 - \omega_1 = \frac{R}{L} \Rightarrow \boxed{Q = \frac{\omega_{res}}{\Delta\omega} = \frac{L\omega_0}{R}}$$

○ unphysical

Application: FM antenna

Consider the following circuit:

- L=8.22 μH
- C=0.27 pF=0.27×10^{-12} F
- R=75 Ω



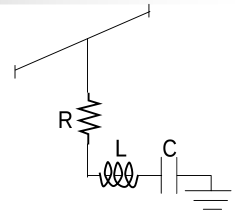
The radio signal in the air induces an alternated emf in the antenna: V_{RMS}=9.13 μV

- Find frequency of incoming wave for which antenna is in tune

$$\text{Resonance frequency: } \omega_0 = \frac{1}{\sqrt{LC}} = 6.7x10^8$$

$$\omega_0 = 2\pi\nu \Rightarrow \nu_0 = \frac{\omega_0}{2\pi} = 106 MHz \text{ YES, FM radio!}$$

Application: FM antenna (cont)



- $L=8.22\ \mu H$
- $C=0.27\ pF=0.27x10^{-12}\ F$
- $R=75\ \Omega$
- $V_{RMS}=9.13\ \mu V$

- Calculate I_{RMS}

$$I_{RMS} = = \frac{I_0}{\sqrt{2}} = \frac{V_{RMS}}{|Z_0|} = \frac{V_{RMS}}{R} \text{ (NB : at resonance } |Z_0|=R)$$

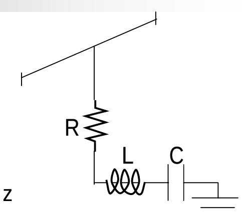
- ΔV_{RMS} across C

$$V_C = I_{RMS} Z_C = \frac{1}{\omega C}\frac{V_{RMS}}{R} = 0.66 mV$$

Question: V_c=0.66 mV while V_{RMS}=9 μV. How can this happen?

L and C cancel almost perfectly ⇒ Z can be small while C and L are large and Z~real. NB: all circuits with good Q value have this feature!

Application: FM antenna (cont)



- Calculate width of resonance

$$\Delta\omega = \frac{R}{L} = 9 \cdot 10^6 \Rightarrow \Delta\nu = \frac{\Delta\omega}{2\pi} = 1.4 MHz$$

Q: is this a good antenna?

No, since separation between stations is ~ 0.2 MHz

- Q factor

$$Q = \frac{\omega_{res}}{\Delta\omega} = \frac{L\omega_0}{R} = \text{ 73 good but not enough for a radio.}$$

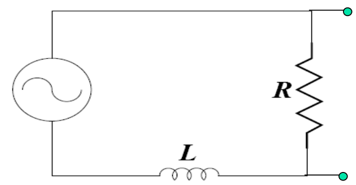
How can this be improved?

Can we increase L? No, it would change frequency

⇒ decreasing R is the solution

Low pass RL filter

- RCL circuits have a frequency dependent response: they can act as filters (select only certain frequencies)
- Example: RL circuit
  - Calculate the complex current

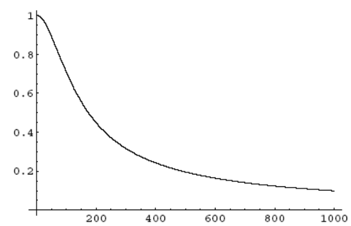


$$\tilde{I} = \frac{\tilde{V}}{\tilde{Z}} = \frac{\tilde{V}}{R + i\omega L} \quad \Rightarrow$$

$$|I| = \frac{|\tilde{V}|}{|R + i\omega L|} = \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}}$$

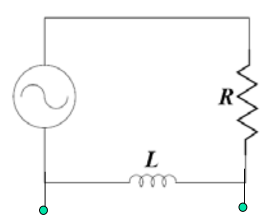
$$V_R = |I| R = \frac{V_0 R}{\sqrt{R^2 + \omega^2 L^2}}$$

$$\Rightarrow \begin{cases} \omega \to 0 : \mathrm{V_R} \to \mathrm{V_0} \\ \omega \to \infty : \mathrm{V_R} \to 0 \end{cases} \Rightarrow \text{low pass filter}$$



High pass RL filter

- What if we take the voltage V_L across the inductor?
  - Same complex current



$$\tilde{I} = \frac{\tilde{V}}{\tilde{Z}} = \frac{\tilde{V}}{R + i\omega L} \quad \Rightarrow$$

$$|I| = \frac{|\tilde{V}|}{|R + i\omega L|} = \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}}$$

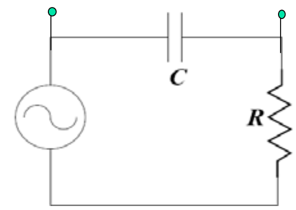
$$|V_L| = \omega L |I| = \frac{\omega L V_0}{\sqrt{R^2 + \omega^2 L^2}}$$

$$\Rightarrow \begin{cases} \omega \to 0 : \mathrm{V_L} \to \dfrac{\omega L V_0}{\sqrt{R^2 + \omega^2 L^2}} \to \dfrac{\omega L V_0}{R} \to 0 \\ \omega \to \infty : \mathrm{V_L} \to \dfrac{\omega L V_0}{\omega\sqrt{\dfrac{R^2}{\omega^2} + L^2}} \to \dfrac{L V_0}{L} \to V_0 \end{cases} \Rightarrow \text{high pass filter}$$

Demo on filters

Low pass RC filter

- Let's now study the voltage across a capacitor of a driven RC circuit
  - The complex current is now:



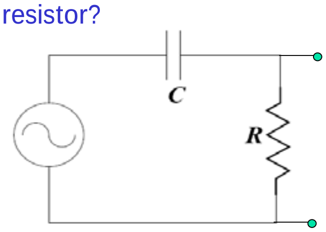
$$\tilde{I} = \frac{\tilde{V}}{\tilde{Z}} = \frac{\tilde{V}}{R - \dfrac{i}{\omega C}} \Rightarrow$$

$$|I| = \frac{|\tilde{V}|}{\left|R - \dfrac{i}{\omega C}\right|} = \frac{V_0}{\sqrt{R^2 + \dfrac{1}{\omega^2 C^2}}}$$

$$V_C = \frac{|I|}{\omega C} = \frac{\dfrac{V_0}{\omega C}}{\sqrt{R^2 + \dfrac{1}{\omega^2 C^2}}} = \frac{V_0}{\sqrt{\omega^2 C^2 R^2 + 1}} \Rightarrow \begin{cases} \omega \to 0 : V_R \to V_0 \\ \omega \to \infty : V_R \to 0 \end{cases} \Rightarrow \text{low pass filter}$$

High pass RC filter

- What if we take the voltage V_R across the resistor?
  - Same complex current



$$\tilde{I} = \frac{\tilde{V}}{\tilde{Z}} = \frac{\tilde{V}}{R - \dfrac{i}{\omega C}} \Rightarrow$$

$$|I| = \frac{|\tilde{V}|}{\left|R - \dfrac{i}{\omega C}\right|} = \frac{V_0}{\sqrt{R^2 + \dfrac{1}{\omega^2 C^2}}}$$

$$V_R = R|I| = \frac{V_0}{\sqrt{R^2 + \dfrac{1}{\omega^2 C^2}}} = \frac{\omega C R V_0}{\sqrt{\omega^2 C^2 R^2 + 1}} \Rightarrow \begin{cases} \omega \to 0 : V_R \to 0 \\ \omega \to \infty : V_R \to V_0 \end{cases} \Rightarrow \text{high pass filter}$$

Summary and outlook

- Today:
  - End of RCL circuits
    - Some tricks to make RCL calculations easier
    - Power dissipated in RCL circuits
    - Antennas and high and low pass filters
- Next time:
  - Back to Maxwell's equation:
    - The missing ingredient!