8.022 (E&M) – Lecture 14

Topics:

- Electromagnetic Inductance
  - Faraday's and Lentz's laws

Last time

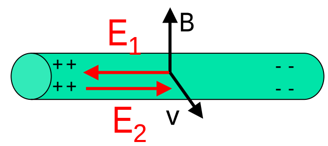
- Parallel between Electric and Magnetic Fields
  - Toward Maxwell's equations:

$$\begin{cases} \vec{\nabla}\bullet\vec{E} = 4\pi\rho & \Leftrightarrow \quad \vec{\nabla}\bullet\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = 0 & \Leftrightarrow \quad \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} \end{cases}$$

  - Vector Potential: $\vec{E} = -\vec{\nabla}\phi \quad \Leftrightarrow \quad \vec{B} \equiv \vec{\nabla}\times\vec{A}$
  - Biot-Savart: $\vec{B} = \frac{I}{c}\int_{wire} d\vec{l} \times \frac{\hat{r}}{r^2}$

Moving rod in uniform B

- Let's move a conducting rod in a uniform B
  - Charges move with velocity v//x axis
  - B//y axis



- What happens?

  1) Lorentz force: $\vec{F}_{Lorentz} = q\frac{\vec{v}}{c} \times \vec{B} = q\vec{E}_1$

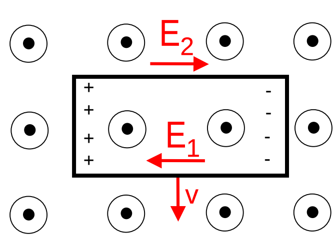
  2) Electric field E_1 causes separation of charges on the wire

  3) Separation of charges creates an opposite electric field E_2 that exactly compensates E_1 and equilibrium is established:

$$\vec{E}_2 = -\frac{\vec{v}}{c} \times \vec{B}$$

Moving a loop in uniform B

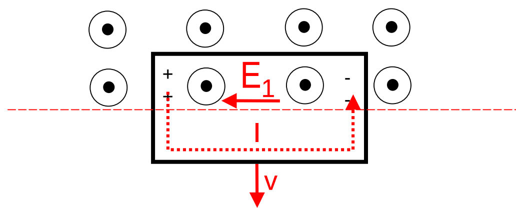
- Now move a rectangular loop of wire in B
  - Same velocity
  - Same B



- What happens?
  - Lorentz force → E_1
  - E_1 → separation of charges on the wire
  - Separation of charges creates opposite electric field E_2= $-E_1$:

What if B is non uniform?

- Now move the rectangular loop of wire in non uniform B
  - Velocity v
  - $B = B_0$ above - - -
  - B = 0 below - - -



- What happens?
  - Lorentz force → E_1
  - E_1 → separation of charges on the wire
  - Separation of charges creates charges to flow in the loop (no opposing force in the bottom part!)
- This phenomenon is called electromagnetic induction

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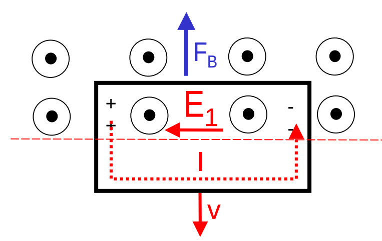
Comments on induction

Please notice the following:



- End of electrostatics!

$$\oint_{loop} \vec{E} \bullet d\vec{l} \neq 0 \ or \ \nabla \times \vec{E} \neq 0$$

- The current flowing in top leg of the loop will feel a force F_B from B pointing up
  - Lentz's law

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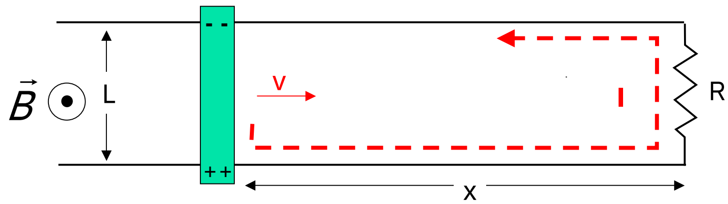
Induced emf

- Consider a sliding conducting bar on rails closed on a resistor R in a region of constant magnetic field B



- Charge separation in the bar will induce current → e.m.f.

$$e.m.f. = \frac{1}{q} W(- \to +) = \frac{1}{q}\int_{-}^{+} \vec{F} \bullet d\vec{s} = \frac{1}{c}\int_{-}^{+} \left(\vec{v} \times \vec{B}\right) \bullet d\vec{s} = \frac{vBL}{c}$$

  - Current flowing in the loop: $I = \frac{vBL}{cR}$

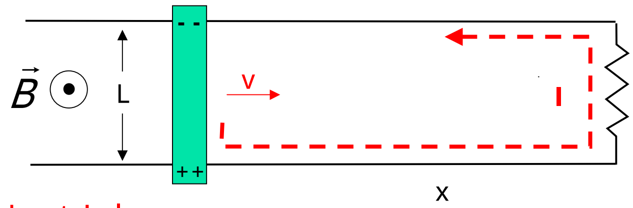
Faraday's law

- EMF in the loop: $e.m.f. = \frac{vBL}{c} = \frac{BL}{c}\frac{dx}{dt}$
- Magnetic flux in the rectangle is defined as: $\Phi_B = Blx$
- Combine the two keeping in mind that given the direction of v, flux decreases with time:

→ Faraday's law: $$e.m.f. = -\frac{1}{c}\frac{\partial \Phi_B}{\partial t}$$

- The minus sign is important: Lentz's law
  - It indicates that the direction of the current is such to oppose the changes in flux of B: ~"electromagnetic inertia"

Thoughts on Lentz's law



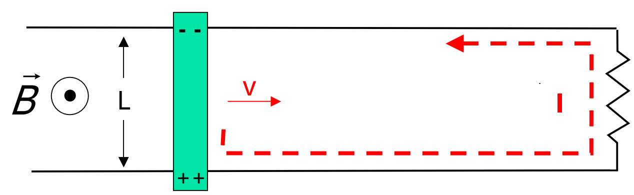
$$e.m.f. = -\frac{1}{c}\frac{\partial \Phi_B}{\partial t}$$

Lentz's law:

The current generated in wire opposes changes in flux of B

- v is L→R:
  - Flux of B decreases over time → e.m.f. is created with direction that compensates this change: counterclockwise
- v is R→L:
  - Flux of B increases over time → e.m.f. is created with direction that compensates this change: clockwise

Another way of looking at Lentz



$$e.m.f. = -\frac{1}{c}\frac{\partial \Phi_B}{\partial t}$$

When current flows in magnetic field it feels a force

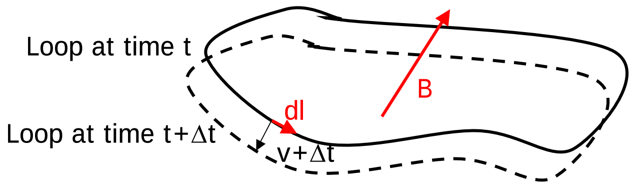
Lentz's law: the force will be will try to slow down the bar

- If I clockwise:
  - It creates a B pointing into the board → **I** x **B** points to the left
- If I counterclockwise:
  - It creates a B pointing out of the board → **I** x **B** points to the right

NB: the – sign in Lentz's law is what allows conservation of energy

General proof of Faraday's law

- Consider a loop of arbitrary shape moving with velocity v through a static magnetic field B



- At time t, the flux through the loop is: $\Phi_B = \int_S \vec{B} \bullet d\vec{a}$
- How does it change when t → t+Δt?

$$\Delta\Phi_B = \Phi_B(t+\Delta t) - \Phi_B(t) = \Phi_{ribbon} = \int_{ribbon} \vec{B} \bullet d\vec{a}$$

- On the ribbon:

$$d\vec{a} = (\vec{v}\Delta t) \times d\vec{l}$$

Proof of Faraday's law(2)

- This means that:

$$\Delta\Phi_B = \int_{ribbon} \vec{B} \bullet d\vec{a} = \int_{ribbon} \vec{B} \bullet (\vec{v}\Delta t) \times d\vec{l} = \int_{ribbon} \Delta t \; \vec{B} \bullet (\vec{v} \times d\vec{l})$$

- Using the identity $\vec{a} \bullet (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \bullet \vec{c}$ we obtain:

$$\Delta\Phi_B = \int_{loop} \Delta t \; \vec{B} \bullet (\vec{v} \times d\vec{l}) = \Delta t \int_{loop} \vec{B} \times \vec{v} \bullet d\vec{l} = -\Delta t loop \int_{ribbon} \vec{v} \times \vec{B} \bullet d\vec{l}$$

For $\Delta t \to 0$: $\frac{\partial \Phi_B}{\partial t} = -c \int_{loop} \left(\frac{\vec{v}}{c} \times \vec{B} \right) \bullet d\vec{l}$

Since **v**/c x **B** is the magnetic force for unit charge

→ its line integral on the loop is the work necessary to move a unit charge around the wire: e.m.f!

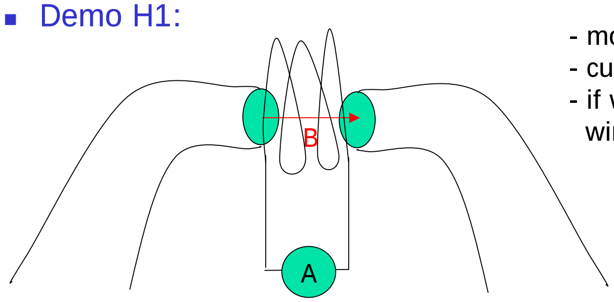
$$\rightarrow \; e.m.f. = -\frac{1}{c} \frac{\partial \Phi_B}{\partial t}$$

Work from B???

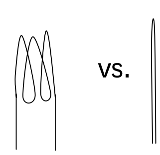
- Faraday's law: $e.m.f. = -\frac{1}{c}\frac{\partial \Phi_B}{\partial t}$
- This means that $\vec{v}/c \times \vec{B}$ integrated over the loop is the work that we have to do to move a unit charge around the loop
- But last time we proved that B cannot do work
- Are these 2 statements inconsistent???
  - No, the work done to move the charges is not done by B
  - It's done by whoever is moving the loop in B

Verification of Faradya's law

- Faraday's law: $e.m.f. = -\frac{1}{c}\frac{\partial \Phi_B}{\partial t}$
- What does it mean?
  - E.m.f. Is produced when the <u>flux</u> of B changes over time
  - → area of the loop cannot be null!
- Demo H1:



- move loop in B
- current flows in wire
- if we used instead a wire with 0 area: no I



“Relativity”

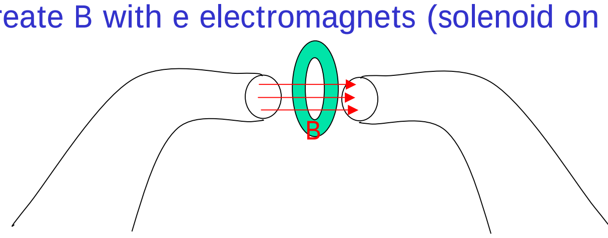
- What if loop is static and B changes?
  - Relativity tells me that we should get the same result
    - Same problem from another reference frame
- Does this make sense?
  - Charges do not move in the other reference frame
  - What causes the force? The induced electric field
- Since e.m.f. is the work necessary to move a unit charge around the loop:

$$e.m.f. = \oint_C \vec{E} \bullet d\vec{l}$$

- Demo H3: magnet bar moving in the loop

Application of Lentz’s law

- H5: disk falling in a magnetic field B
  - Create B with e electromagnets (solenoid on Fe core)



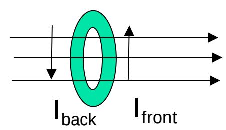
  - What happen if we drop a disk of conductor?
    - With and without B
  - What if we drop a full disk
  - What if we drop a disk with a cut?

Explanation

- Falling Loop:
  - B perpendicular to loop is limited in space → flux changes during fall → induced I

    → loop will levitate

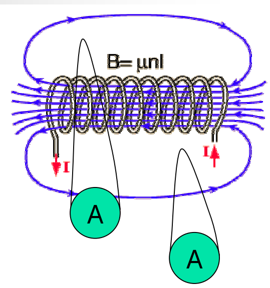
  (Eddie currents)



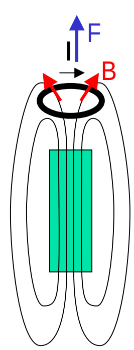
- Falling Disk
  - Will it slow down?
- Falling open ring
  - Will it levitate?

More demos on Faraday's law

- H8: current generated by a solenoid
  - Where to put the loop of wire to have current?
  - Remember: B of solenoid is 0 outside
  - Switch I on and off

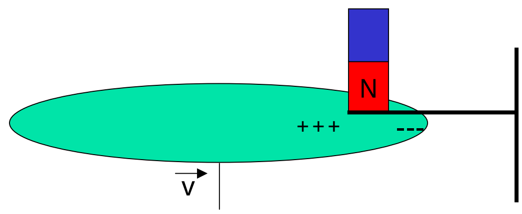


- H22 Levitating rings



More demos on Faraday's law

- H15a: current generated by a solenoid
  - Spinning disk of conductor
  - Magnet sitting on top separated by a plastic sheet
  - When disk starts spinning, magnet levitates
  - Why?



Faraday's law in differential form

- Faraday's law in integral form: $e.m.f. = -\frac{1}{c}\frac{\partial \Phi_B}{\partial t}$
- Right term (apply Stokes): $e.m.f. = \oint_C \vec{E} \bullet d\vec{l} = \int_S \vec{\nabla} \times \vec{E} \bullet d\vec{a}$
- Left term: $-\frac{1}{c}\frac{\partial \Phi_B}{\partial t} = -\frac{1}{c}\frac{\partial}{\partial t}\int_S \vec{B} \bullet d\vec{a}$

$$\rightarrow \int_S \left(\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} \right) \bullet d\vec{a} = 0$$

- Since this is valid for any surface: $\boxed{\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}}$
  - curl E is not longer zero: bye bye electrostatics!
  - Explicit link between E and B, as in relativity!

Another step toward Maxwell's equations…

- All the equations in differential form that we found so far:

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = 4\pi\rho & \leftarrow \text{Relates E and charge density } (\rho) \text{ - Gauss} \\ \vec{\nabla}\cdot\vec{B} = 0 & \leftarrow \text{Magnetic field lines are closed} \\ \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial B}{\partial t} & \leftarrow \text{Change in B creates E - Faraday} \\ \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} & \leftarrow \text{Relates B and its sources (J) - Ampere} \end{cases}$$

- Another step toward Maxwell's equations: one last missing ingredient… Can you guess what?
  - Symmetry will guide you… Hint:
  - Or vector calculus… Hint: take the divergence of Faraday's law…

Summary and outlook

- Today:
  - Faraday's (and Lentz's) law:
    - Integral form: $e.m.f. = -\frac{1}{c}\frac{\partial \Phi_B}{\partial t}$
    - Differential form: $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$
- Next time:
  - Mutual and self inductance