8.022 (E&M) – Lecture 13

Topics:

- B's role in Maxwell's equations
- Vector potential
- Biot-Savart law and its applications

What we learned about magnetism so far…

- Magnetic Field B
  - Experiments: currents in wires generate forces on charges in motion
    - Force exerted on charge q with velocity v: $\vec{F} = q\,\dfrac{\vec{v}}{c}\times\vec{B}$
    - Explanation: there must exist a magnetic field B
  - Special Relativity: B is just E seen from another reference frame…
- Ampere's Law: $\oint_C \vec{B}\bullet d\vec{s} = \dfrac{4\pi}{c} I_{encl}$
  - Application: B generated by current in a wire: $\vec{B} = \dfrac{2I}{cr}\hat{\varphi}$

Divergence of B

- Consider the B produced by a wire of current: $\vec{B} = \frac{2I}{cr}\hat{\varphi}$
- Calculate its divergence in Cartesian coordinates:

$$\text{Given } r = \sqrt{x^2+y^2} \text{ and } \hat{\varphi} = \hat{y}\cos\varphi - \hat{x}\sin\varphi = \frac{x\hat{y}}{\sqrt{x^2+y^2}} - \frac{y\hat{x}}{\sqrt{x^2+y^2}} \Rightarrow$$

$$\vec{B} = \frac{2I}{cr}\left(\frac{x\hat{y}}{x^2+y^2} - \frac{y\hat{x}}{x^2+y^2}\right) \Rightarrow \vec{\nabla}\bullet\vec{B} = \frac{2I}{cr}\left(\frac{2yx}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2}\right) = 0$$

- This is a general property of the magnetic field: $\boxed{\vec{\nabla}\bullet\vec{B} = 0}$
- Similar equation for E: $\vec{\nabla}\bullet\vec{E} = 4\pi\rho$
  - The divergence of E is related to the density of electric charges
  - The divergence of B must be related to the density of magnetic charges
    → Magnetic monopole don't exist

(There may be magnetic monopoles leftover from the Early Universe, but never observed experimentally so far)

Ampere's law in differential form

- Apply Stoke's theorem to Ampere's law:

$$\oint_C \vec{B}\bullet d\vec{s} = \frac{4\pi}{c} I_{encl}$$

$$\oint_C \vec{B}\bullet d\vec{s} = \int_S \vec{\nabla}\times\vec{B}\bullet d\vec{S} = \frac{4\pi}{c}\int_S \vec{J}\bullet d\vec{S}$$

$$\int_S \left(\vec{\nabla}\times\vec{B} - \frac{4\pi}{c}\vec{J}\right)\bullet d\vec{S} = 0 \text{ for any surface}$$

→ Ampere's law in differential form: $\boxed{\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}}$

Toward Maxwell's equations

- Let's collect all the equations in differential form that we found so far:

$$\begin{cases} \vec{\nabla} \bullet \vec{E} = 4\pi\rho \\ \vec{\nabla} \bullet \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = 0 \\ \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} \end{cases}$$

← Relates E and charge density (ρ) - Gauss

← No magnetic monopoles!

← E is a conservative field

← Relates B and its sources (J) - Ampere

- Not complete Maxwell's equations yet, but we are getting closer...

Vector potential A

- Definition of potential for electric field:
  - ϕ(P) = work needed to move a unit charge from reference to P
  - Relationship between ϕ and E: $\vec{E} = -\vec{\nabla}\phi$
  - Hidden advantage:

    If $\vec{E} = -\nabla\phi \Rightarrow \nabla \times \vec{E} \equiv 0$ because $\nabla \times (\nabla\phi) = 0 \;\; \forall\phi$
- Can we introduce something similar for B?
  - Goal: enforce div B=0
  - Since $\vec{\nabla} \bullet \vec{\nabla} \times \vec{f} = 0$ for any $\vec{f}$, we define

$$\vec{B} \equiv \vec{\nabla} \times \vec{A}$$

  - A is called "vector potential" in analogy with ϕ
  - A is not connected to work or energy (but to angular momentum)

Non Uniqueness

- Electrostatics: given a charge distribution and boundary conditions → potential ϕ is uniquely identified
- Magnetism: does it work the same for A? No, there are infinite number of A corresponding to a single B
  - Example: $\vec{B} = B_0\hat{z}$. Find $\vec{A}$ that creates this $\vec{B}$ field.

Requirements:

Q: what current creates this B?

$$\begin{cases} B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 0 \\ B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = 0 \\ B_z = \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} = B_0 \end{cases} \Rightarrow \text{Possible solutions: } \begin{cases} \vec{A} = -yB_0\hat{x} \\ \vec{A} = xB_0\hat{y} \\ \vec{A} = \frac{B_0}{2}(-y\hat{x} + x\hat{y}) \\ \vec{A} = \text{...infinite others!} \end{cases}$$

- We are given one "coupon" to simplify equations when needed

Poisson's equation for A

- Electrostatics:

$$\begin{cases} \vec{E} = -\vec{\nabla}\phi \\ \vec{\nabla}\bullet\vec{E} = 4\pi\rho \end{cases} \Rightarrow \boxed{\nabla^2\phi = -4\pi\rho} \text{ Poisson's equation}$$

- Magnetism:

$$\begin{cases} \vec{B} = \vec{\nabla}\times\vec{A} \\ \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} \end{cases} \Rightarrow \vec{\nabla}\times\vec{\nabla}\times\vec{A} = \frac{4\pi}{c}\vec{J} \Rightarrow \boxed{\vec{\nabla}(\vec{\nabla}\bullet\vec{A}) - \nabla^2\vec{A} = \frac{4\pi}{c}\vec{J}}$$

We used the identity: $\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \vec{\nabla}(\vec{\nabla}\bullet\vec{A}) - \nabla^2\vec{A}$ (Pset#7) ☹

- Use your coupon now!

Choosing $\boxed{\vec{\nabla}\bullet\vec{A}=0} \Rightarrow \boxed{\nabla^2\vec{A} = -\frac{4\pi}{c}\vec{J}}$

Solving Poisson's equation for A

How do you solve $\nabla^2 \vec{A} = -\frac{4\pi}{c}\vec{J}$?

Think of it in cartesian coordinates: $\begin{cases} \nabla^2 A_x = -\frac{4\pi}{c} J_x \\ \nabla^2 A_Y = -\frac{4\pi}{c} J_Y \\ \nabla^2 A_Z = -\frac{4\pi}{c} J_Z \end{cases}$

Remember Poisson's equation $\nabla^2\phi = -4\pi\rho$ and its solution $\phi = \int_V \frac{\rho}{r} dV$

Same as our new equation if replace $\phi \rightarrow \vec{A}$ and $\rho \rightarrow \frac{\vec{J}}{c} \Rightarrow \boxed{\vec{A} = \frac{1}{c}\int_V \frac{\vec{J}}{r} dV}$

For current flowing in a wire: $\boxed{\vec{A} = \frac{I}{c}\int_{wire} \frac{d\vec{l}}{r}}$

Biot-Savart Law

Find $\vec{B}$ produced from current knowing that $\vec{A} = \frac{I}{c}\int_{wire} \frac{d\vec{l}}{r}$.

$$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \frac{I}{c}\int_{wire} \frac{d\vec{l}}{r} = \frac{I}{c}\int_{wire} \vec{\nabla} \times \frac{d\vec{l}}{r}$$

Using the fact that $\nabla \times (a\vec{b}) = a(\nabla \times \vec{b}) + (\vec{\nabla} a) \times \vec{b}$:

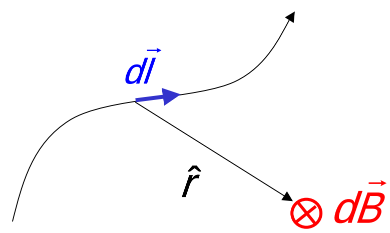
$$= \frac{I}{c}\left[\int_{wire} \frac{1}{r}(\vec{\nabla} \times d\vec{l}) + \vec{\nabla}\frac{1}{r} \times d\vec{l}\right] = \frac{I}{c}\int_{wire} \frac{1}{r}(\vec{\nabla} \times d\vec{l}) + \vec{\nabla}\frac{1}{r} \times d\vec{l}$$

Since $\vec{\nabla} \times d\vec{l} = 0$ and $\vec{\nabla}\frac{1}{r} = -\frac{\hat{r}}{r^2}$:

$$\Rightarrow \boxed{\vec{B} = \frac{I}{c}\int_{wire} d\vec{l} \times \frac{\hat{r}}{r^2}}$$

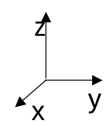
Biot-Savart Law: illustration

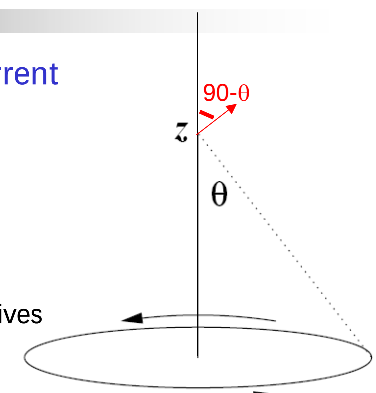
- Biot-Savart: $d\vec{B} = \frac{I}{c} d\vec{l} \times \frac{\hat{r}}{r^2}$



- dB is perpendicular to current and to radial direction
- E.g.: if you have dl // x, r // y → B//z

Application of Biot-Savart:
B from loop of current





- Calculate B created by a loop of current
  - Radius: R
  - Distance from center of the loop: z
- Solution on axis
  - Apply Biot-Savart
  - Determine direction of dB
  - Symmetry → only component // z survives

$$B = \int_{wire} \left(d\vec{B}\right)_z = \int_{wire} \frac{I}{cr^2} \left|d\vec{l} \times \hat{r}\right| \sin\theta$$

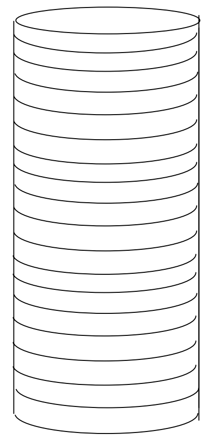
$$\left|d\vec{l} \times \hat{r}\right| = \left|d\vec{l}\right| = R\,d\varphi; \quad \sin\theta = R / r; \quad r = \sqrt{R^2 + h^2}$$

$$\vec{B} = \frac{I}{cr^2} R \sin\theta \int_0^{2\pi} d\varphi\,\hat{z} = \frac{2\pi I R^2}{c\left(R^2 + z^2\right)^{3/2}}\hat{z} \quad \Rightarrow \quad \boxed{\vec{B}_{loop\ center} = \frac{2\pi I}{cR}\hat{z}}$$

Application of Biot-Savart:

B from solenoid

- What if we stack a N rings over a length L?
- Use result of single loop + superposition:



Single ring: $d\vec{B} = \frac{2\pi R^2}{c(R^2+z^2)^{3/2}} dI$

Integrate on all rings (in the middle of the solenoid)

$$\vec{B} = \int_{-L/2}^{L/2} \frac{2\pi R^2}{c(R^2+z^2)^{3/2}} nIdz = \frac{2\pi nI}{c} \int_{-L/2}^{L/2} \frac{R^2 dz}{(R^2+z^2)^{3/2}}$$

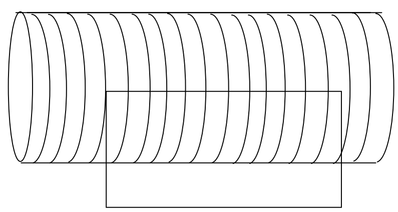
$$= \frac{2\pi nI}{c} \frac{2L}{\sqrt{L^2+4R^2}}$$

With n=N/L

- For L>>R: $\vec{B} = \frac{4\pi nI}{c}$

Solenoid and Ampere's law

- One can prove that B outside the solenoid is =0
- Ampere can be used to simply prove that B does not depend on r:



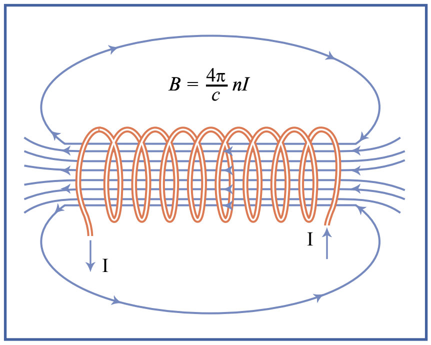
$$\oint_{rectangle} \vec{B} \bullet d\vec{l} = \frac{4\pi}{c} I_{encl}$$

Since $\vec{B}$ is //z and present only inside the solenoid:

$$B(r)L = \frac{4\pi}{c} NI \Rightarrow B(r) = \frac{4\pi}{c}\frac{N}{L} I = \frac{4\pi}{c} nI$$ no dependence on R

Solenoid's magnetic field: demos

- Expected:



- Can we test this experimentally?
  - G12: B from a single wire using iron filings
  - G13: B from 2 wires
  - G16: B inside solenoid

More demos on magnetic fields

- More demos:
  - G14: map B around a wire using a compass
  - G9a: collapsing solenoid
    - Can you explain what's happening?
  - G18: Long solenoid
    - Long solenoid with N_{turn}=2760, I=4.5 mA, length = 46 cm
      - (R=10 Ω, L=128 mH)
    - What is B?

$$B = \frac{4\pi}{c} nI = \frac{4\pi}{3 \cdot 10^{10}} \frac{2760}{50} 4.5 = 230 \cdot 10^{-8} \text{ Gauss ???}$$

    - Verify with Hall probe

Thompson's experiment: variation

- Variation on a theme: instead of canceling effects of E and B, one could tune the fields and measure the radius of curvature of the electron beam.
- Parameters of the problem:
  - V= 300 V
  - I= 1.4 A
  - R= 5 cm
- Solution:
  - e/m= 2.02 x 10^{11} C/Kg (cfr: 1.76 x 10^{11} C/Kg)

Summary and outlook

- Today:
  - Toward Maxwell's equations: $\vec{\nabla} \bullet \vec{B} = 0$ and $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}$
  - Vector Potential: $\vec{B} \equiv \vec{\nabla} \times \vec{A}$
  - Biot-Savart Law: $d\vec{B} = \frac{I}{c} d\vec{l} \times \frac{\hat{r}}{r^2}$
- Next time:
  - What happens when B varies in time?
    - Faraday's and Lenz's laws and their applications