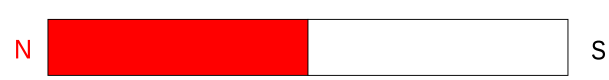
8.022 (E&M) – Lecture 10

Topics:

- Magnetic field B
- Magnetic force acting on charges in motion
- Ampere's law

The Origins of Magnetism

- Ancient Greeks noticed that a piece of a mineral magnetite (an oxide of iron) had very special properties:
  - Could attract a piece of iron, but no effect on Au, Ag, Cu, etc
  - Can attract or repel piece of magnetite depending on relative orientation
- By the 12th century people could build a magnetic compass
  - A small magnetic needle is suspended so it can pivot around vertical axis
  - The needle will always come to rest with one end pointing North
  - By definition we call that end "North" and the other "South"



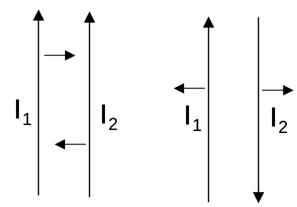
  - Like poles repel, unlike poles attract: demo
  - North and South cannot be separated in a magnet: demo
  - Magnetic forces can be pretty strong! Demo G3: nail on a sting

The big step forward

- In 1820 Oersted realized that current flowing in a wire made the needle of a compass swing
  - The direction depends on the direction of the current

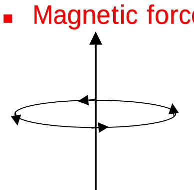
BIG discovery: proves that Electricity and Magnetism are related!

- Soon after, Ampere's experiment with parallel wires carrying current
  - If currents are parallel, wires attract
  - If anti-parallel, wires repel
  - No force on a stationary charge nearby...
  - NB: wires are overall neutral!
  - Demo



Magnetic force between currents

- More refined observations followed:
  - $F \sim I_1 I_2$ → F is proportional to velocity of charges in motion
  - Direction of F is perpendicular to velocity
- Interpretation
  - Some field (magnetic field B) is created by the charges in motion
  - Magnetic force is proportional to cross product **v** x **B**



$$\vec{F} = q \frac{\vec{v}}{c} \times \vec{B}$$

  - Direction of **B**: **B** curls around the current (right hand rule)
  - Iron fillings can be used to visualize B field lines: demo G2

NB: this is an empirical law so far

Lorentz force

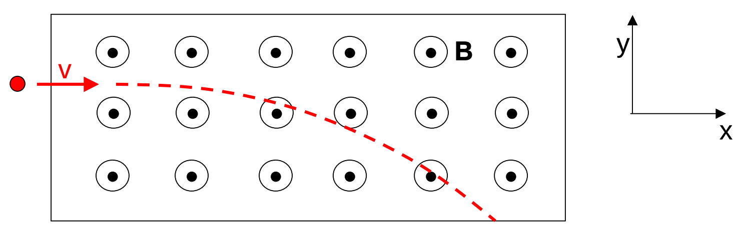
- When a charged particle moves in electric (E) and magnetic (B) fields it feels a force ($F_{Lorentz}$):

$$\vec{F}_{Lorentz} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

  - The above formula defines the magnetic field B
  - Units of B in cgs:
    - [B] = [F]/[q] = dyne/esu = Gauss (G)
    - NB: [B] = [E]
  - Units of B in SI: $\vec{F}_{Lorentz} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$
    - [B] = [F]/[q v] = N s /(m C) = Tesla (T)
  - Conversion: 1 T = 10^4 G

Trajectory in magnetic fields

- A particle of charge q and mass m moves with velocity **v** // +**x** axis in a magnetic field **B** // +**z** axis (out of the page):

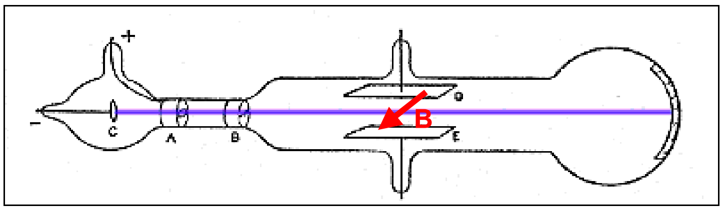


- What is the trajectory of q in the magnetic field? $\vec{F} = q\frac{\vec{v}}{c} \times \vec{B}$
  - v, B and F (a) are always perpendicular → circular motion!

$$F_{Lorentz} = F_{centripetal} \Rightarrow \frac{qvB}{c} = \frac{mv^2}{R} \Rightarrow \boxed{R = \frac{mvc}{qB}}$$

Deflection of electron beam by B

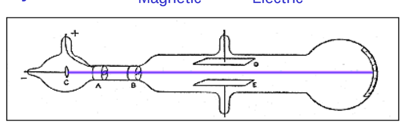
- An electron beam is produced by a cathode in a vacuum tube
  - Velocity of electrons: $\mathbf{v}_e$
- Magnetic field **B** perpendicular to $\mathbf{v}_e$ is produced by current in a wire or by permanent magnet
- What do we expect to happen?
  - Electrons curve according to Lorentz force (Demo G5, G6 TV)

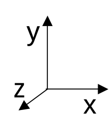


J.J. Thompson's experiment

- Discovery of electrons and measurement of e/m_e in 1897
- The idea:
  - A beam of "cathode rays" crosses a region with E and B present
  - Choosing v_e//x axis, B//z axis, E//y axis → $F_{Lorentz}$ // $F_{Electric}$
  - E and B can be adjusted so $F_{Magnetic} = -F_{Electric}$ so that e will go straight

$$\vec{F}_{Lorentz} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$$

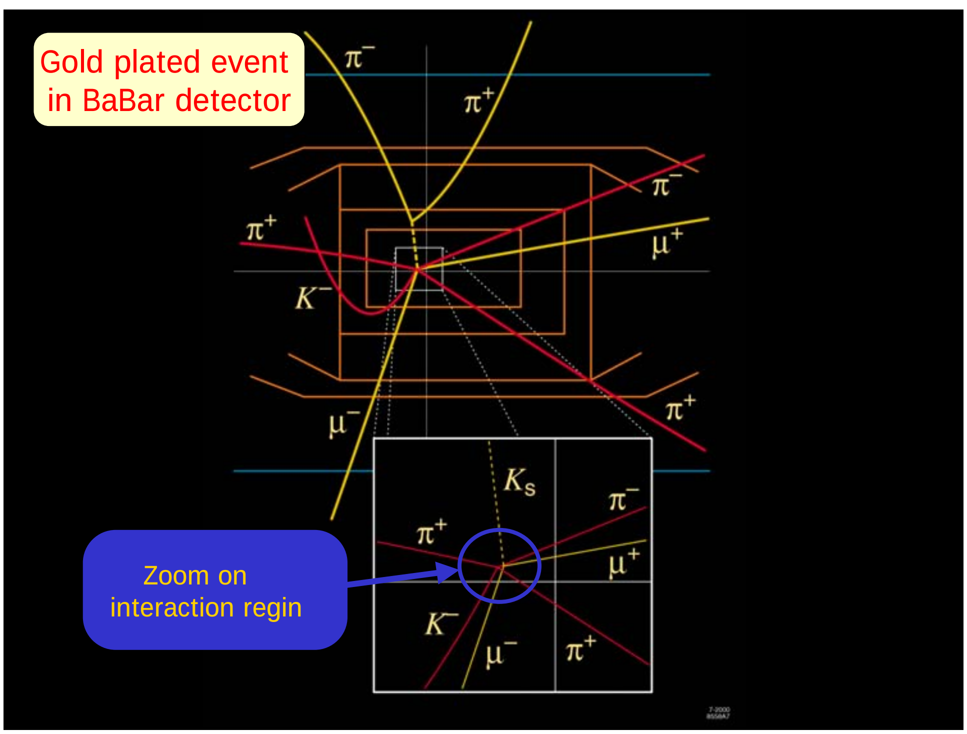




- Electric field alone causes a shift: $\Delta y = -\dfrac{qEL^2}{2mv^2}$
- Now turn on B and set it to cancel the shift due to E: $v = c\dfrac{E}{B}$
- Substituting this in the previous equation gives: $\dfrac{e}{m_e} = \dfrac{q}{m} = \dfrac{2\Delta y c^2 E}{B^2 L^2}$

Application in modern physics

- Tracking detectors in modern particle physics
- The problem
  - High energy collisions between elementary particles (such as e^+e^-) produce many particles (protons, electrons, pions, muons,…)
  - How can we “see” these particles?
    - Build detectors that can “visualize” the trajectory of charged particles using the fact that particles ionize the material they cross
  - How can I measure the properties of these particles?
    - E.g.: measure momentum, energy, mass, etc.
    - Immerse the detector in a very strong magnetic field B ~ 2 T
    - Charged particles will curve according to $R = \frac{mvc}{qB}$
      - Direction measures the charge
      - Radius of curvature measures momentum p=mv



Magnetic force and work

- Moving a charge in an electric field E requires work:

$$W_{12} = -q\int_1^2 \vec{E} \bullet d\vec{s}$$

- How much work does it take to move a charge in a magnetic field?

$$dW = \vec{F} \bullet d\vec{s} = \vec{F} \bullet \vec{v}dt = \frac{q}{c}(\vec{v} \times \vec{B}) \bullet \vec{v}dt = 0$$

→ No work is needed to move a particle in a magnetic field because v and F are always perpendicular!

Force on a current

- A magnetic field will excerpt a force on a current
  - Since a current is just a stream of moving charges!
- Current I flowing in a wire can be seen a density of charges λ moving with velocity v: I=λv
- The force dF exerted on the infinitesimal wire dl is:

$$d\vec{F} = (\lambda dl)\frac{\vec{v}}{c} \times \vec{B}$$

- Rewrite this in terms of the current: $d\vec{F} = \frac{I}{c} d\vec{l} \times \vec{B}$
- Total force F: $\vec{F} = \frac{I}{c} \int_{wire} d\vec{l} \times \vec{B}$
- For a long straight wire in a constant magnetic field:

$$\vec{F} = \frac{I}{c} L\hat{n} \times \vec{B}$$

Ampere's law

- In electrostatics, the electric field E and its sources (charges) are related by Gauss's law: $\int_{Surface} \vec{E} \bullet d\vec{A} = 4\pi Q_{encl}$
  - Why useful? When symmetry applies, E can be easily computed
- Similarly, in magnetism the magnetic field B and its sources (currents) are related by Ampere's law:

$$\oint_C \vec{B} \bullet d\vec{s} = \frac{4\pi}{c} I_{encl}$$

  - Why useful? When symmetry applies, E can be easily computed
  - NB: This is a line integral!

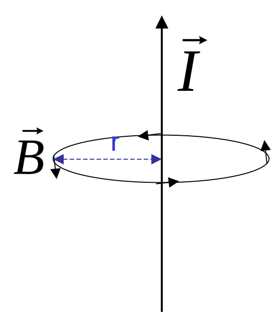
NB: no demonstration has been given so far for Ampere's law.

Application of Ampere's law:

B created by current in a wire

- Long, straight wire in which flows a current I
- Calculate magnetic field B created by I
- Solution:
  - Apply Ampere's law:

$$\oint_C \vec{B} \bullet d\vec{s} = B(r) 2\pi r = \frac{4\pi}{c} I_{encl} \quad \Rightarrow \quad \vec{B} = \frac{2I}{cr} \hat{\varphi}$$



  - Direction: right hand rule
- NB: B_{wire} ~ 1/r. Does this look familiar?
  - Remember E created by a line of charge: $E(r) = \frac{2\lambda}{r}$
  - Coincidence? Not at all...

Force between 2 wires

- Force on wire 1 due to magnetic field B created by wire 2:

$$\vec{F}_1 = \frac{I_1}{c} L\hat{n} \times \vec{B}_2$$

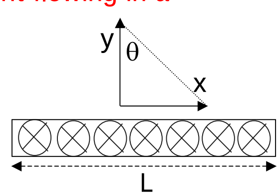
- Magnetic field created by wire 2: $\vec{B}_2 = \frac{2I_2}{cr}\hat{\varphi}$
- Total force F: $F = \frac{2I_2 I_1}{c^2 r} L$
- Usually we quote the force/unit length: $\frac{F}{L} = \frac{2I_2 I_1}{c^2 r}$
  - Direction? $\vec{F} \propto I_1 \times \hat{\varphi}_2$ Using right hand rule:
    - I_1 and I_2 parallel: attractive
    - I_1 and I_2 anti-parallel: repulsive

Can we test this experimentally? Demo G8, G9

Another application of Ampere's law:

B created by sheet of current

- Calculate the magnetic field B created by current flowing in a sheet of conductor
  - Current // -z axis (into the page)
  - Width of sheet of conductor: L
  - Current in a metal sheet ~ N parallel wires



- Solution:
  - B from a wire is know: $\vec{B} = \frac{2I}{cr}\hat{\varphi}$
  - Just apply superposition...
    - Direction: for y>0: B // +x; for y<0: B // -x
    - Magnitude: integrate dB = B field from each infinitesimal wire

$$B = \frac{2I}{Lc}(2\theta) \quad \text{When L>>y,} \quad \theta \to \pi/2 \quad \Rightarrow \quad B = \frac{2\pi I}{Lc}$$

NB: magnitude of B does not depend on y. As for E of sheet of charges

Another application of Ampere's law:

B created by plane of current

Calculation:

$B = \int dB_x$ (only component // $\hat{x}$ survives because of symmetry)

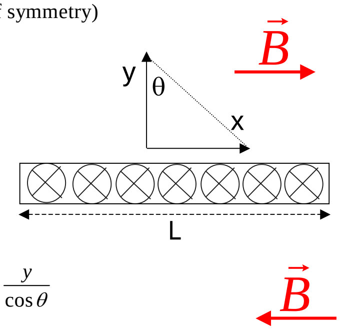
$= \int_{x=-L/2}^{x=L/2} \left(\frac{2dI}{cr}\right)\cos\theta$

$= \int_{x=-L/2}^{x=L/2} \frac{2\left(\frac{I}{L}dx\right)}{cr}\cos\theta$

$= \frac{2I}{Lc}\int_{x=-L/2}^{x=L/2} \frac{dx}{r}\cos\theta$

$= \frac{2I}{Lc}\int_{-\theta}^{\theta} \frac{\frac{yd\theta}{\cos^2\theta}}{\frac{y}{\cos\theta}}\cos\theta$ $\quad x = ytg\theta \Rightarrow dx = \frac{yd\theta}{\cos\theta}; \quad r = \frac{y}{\cos\theta}$

$\Rightarrow \boxed{\vec{B} = \pm\frac{4I}{Lc}\theta\hat{x}}$ + for y>0; - for y<0



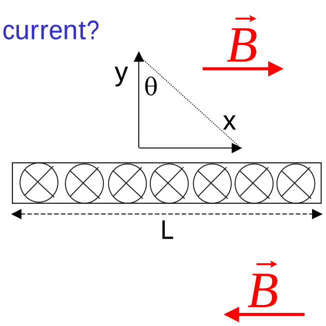
More on B from sheet of current

- If we define current per unit length K=I/L: $\vec{B} = \pm\frac{2\pi K}{c}\hat{x}$
- What is the change of B across the sheet of current?

$$\Delta B = \frac{4\pi K}{c}$$

- Does it ring a bell?
  - Yes, ΔE across a plane of charge!

$$\Delta E = 4\pi\sigma$$



- Another similarity between electric and magnetic fields.
  ...This must be more than a pure coincidence...

Ampere's law in SI

- In SI Ampere's law takes the form: $\oint_C \vec{B} \bullet d\vec{s} = \mu_0 I_{encl}$
  - where μ_0=4 10^{-7} N/A^2 is the magnetic permeability of free space
- Be careful not to mix cgs and SI formulae!
  - To convert cgs → SI: multiply by $\mu_0 c/(4\pi)$
  - Examples:
    - Magnetic field created by a wire: $\vec{B} = \frac{2I}{cr}\hat{\varphi} \quad \Rightarrow \quad \vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\varphi}$
    - Force between 2 wires:
      - NB: factor 1/c missing in $F_{Lorentz}$ in SI

$$\frac{F}{L} = \frac{2 I_2 I_1}{c^2 r} \quad \Rightarrow \quad \frac{F}{L} = \frac{\mu_0 I_2 I_1}{2\pi r}$$

Divergence of B

- Consider the B produced by a wire of current: $\vec{B} = \frac{2I}{cr}\hat{\varphi}$
- Calculate its divergence in Cartesian coordinates:

$$\text{Given } r = \sqrt{x^2+y^2} \text{ and } \hat{\varphi} = \hat{y}\cos\varphi - \hat{x}\sin\varphi = \frac{x\hat{y}}{\sqrt{x^2+y^2}} - \frac{y\hat{x}}{\sqrt{x^2+y^2}} \Rightarrow$$

$$\vec{B} = \frac{2I}{cr}\left(\frac{x\hat{y}}{x^2+y^2} - \frac{y\hat{x}}{x^2+y^2}\right) \Rightarrow \vec{\nabla} \bullet \vec{B} = \frac{2I}{cr}\left(\frac{2yx}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2}\right) = 0$$

- This is a general property of the magnetic field: $\boxed{\vec{\nabla} \bullet \vec{B} = 0}$
- Similar equation for E: $\vec{\nabla} \bullet \vec{E} = 4\pi\rho$
  - The divergence of E is related to the density of electric charges
  - The divergence of B must be related to the density of magnetic charges
    → Magnetic monopole don't exist

(There may be magnetic monopoles leftover from the Early Universe, but never observed experimentally so far)

Thoughts on B

- What exactly is a magnetic field B?
  - Why does it have so much in common with electric field E?
  - Why should there be a field that acts only on moving charges?
- Answer: Special Relativity
  - Relativity: the physics must be the same in all reference frames
  - A charge at rest for observer 1 appears in motion to observer 2 that moves with a certain velocity w.r.t. observer 1:
    - Observer 1 will measure an electric field
    - Observer 2 will measure a magnetic field
    - Calculating attractive or repulsive force acting on a test charge in the 2 reference frames will lead to the same conclusion

Summary and outlook

- Today:
  - Magnetic Field B
  - Magnetic Force acting on charges in motion
  - Ampere's Law
- Next time:
  - Quick Introduction to Special Relativity
  - Goals:
    - Understand how and why Magnetism and Electricity are related
    - Finally play with some really cool physics!