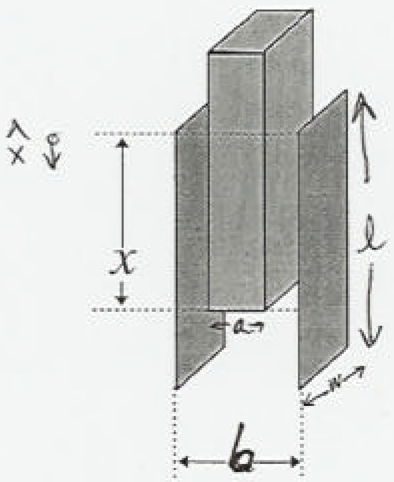
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS

Physics 8.022 | *Spring 2003*

Final Exam

Problem 1 (Capacitance and Energy) (20 pts)



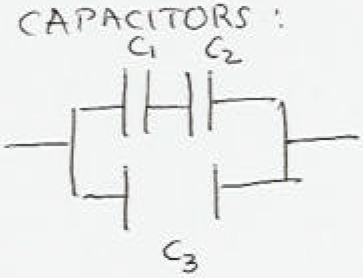
Consider a parallel plate capacitor with plate separation b and a sheet of metal of thickness a, as shown in the figure. Here, $b < a$. For this problem, consider the plates of the capacitor to be large enough so that fringing fields can be neglected.

a) Does the capacitance of the capacitor rise or fall after the metal sheet is inserted between the plates of the capacitor?

IT RISES (TOTAL CHARGE REMAINS SAME, VOLTAGE DROPS).

b) If the capacitance of the capacitor before the metal sheet was inserted was given by C_0, what is the capacitance after insertion of the metal sheet?

AT POSITION x THERE ARE EFFECTIVELY THREE CAPACITORS:



	Area	Length	C
C_1	$x \cdot w$	$\frac{b-a}{2}$	$\frac{xw}{4\pi(b-a)/2}$
C_2	xw	$\frac{b-a}{2}$	$\frac{xw}{4\pi(b-a)/2}$
C_3	$(\ell-x)\cdot w$	b	$\frac{(\ell-x)w}{4\pi b}$

$$C_{TOTAL} = C_3 + \frac{1}{\frac{1}{C_1}+\frac{1}{C_2}} = \frac{(\ell-x)w}{4\pi b} + \frac{xw}{4\pi(b-a)} = \frac{\ell w}{4\pi b} + \frac{xwa}{4\pi b(b-a)}$$

$$= C_0 + \frac{xwa}{4\pi b(b-a)} = C_0\left[1 + \left(\frac{x}{\ell}\right)\left(\frac{a}{b-a}\right)\right]$$

ACCEPTABLE IF ASSUMED TOTALLY INSIDE, I.E $x=\ell \Rightarrow C_{TOTAL} = C_0 \frac{b}{b-a}$

c) Consider a charge an amount of charge $+Q$ and $-Q$ placed on the plates of the capacitor. What is the energy stored in the capacitor before and after the metal plate is inserted between the plates?

BEFORE $\quad U_{BEFORE} = \frac{1}{2}\frac{Q^2}{C_{T0}}$.

AFTER $\quad U_{AFTER} = \frac{1}{2}\frac{Q^2}{C_{TOTAL}} = \frac{1}{2}\frac{Q^2}{C_0}\frac{1}{1+\frac{a}{\ell(b-a)}x}$

↳ function of $\underline{x}$ Note: $1+\frac{ax}{\ell(b-a)} > 1 \Rightarrow U_{AFTER} < U_{BEFORE}$

d) Given the equation $F = -\frac{\partial U}{\partial x}$, where x is the length of the metal plate that is inserted between the capacitor plates, what is the force on the metal? Which direction does it tend to move the metal plate?

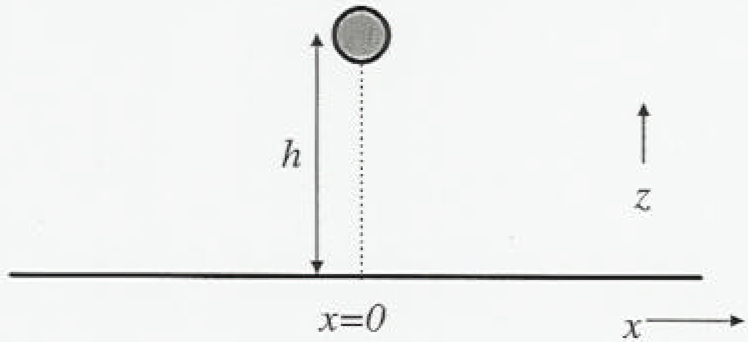
$$F = -\frac{\partial U}{\partial x} = \frac{1}{2}\frac{Q^2}{C_0}\frac{1}{\left[1+\frac{ax}{\ell(b-a)}\right]^2}\cdot\frac{a}{\ell(b-a)}$$

The force is along $+\hat{x}$ (positive F resulted from derivative).

It pulls the metal sheet inside the capacitor

Problem 2(Conducting plane and a line charge) **(22 pts)**

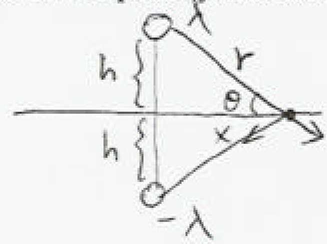
Consider an infinitely long line charge parallel to the y axis at a height h above a grounded conducting plane. "Grounded" means that charge is free to flow into the plane from a large reservoir of charge, and the potential at the plane is always zero (the potential at infinity is also zero). The line charge has a charge per unit length of λ.



a) What is the electric field as a function of x for all points just above the plane?

Use the imagine

x components cancel out, $E_x = 0$



$\sin\theta = \frac{h}{r}$

$$E_z = -2 \cdot \frac{2\lambda}{r} \sin\theta = -\frac{4\lambda h}{r^2}$$

$$= -\frac{4\lambda h}{h^2 + x^2}$$

b) What is the surface charge density as a function of x?

$\because$ $E = 4\pi\sigma$ for a conducting plane

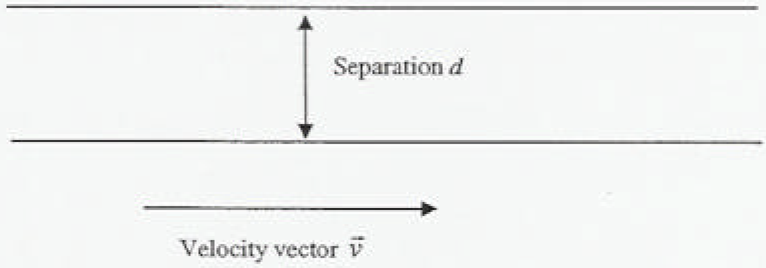
$$\therefore \sigma = \frac{E}{4\pi} = \frac{\lambda h}{\pi(x^2+h^2)}$$

c) What is the electrostatic potential at the position $z=h/2$, $x=0$?

$$\phi(z = h/2) = \int_0^{\frac{h}{2}} \left(\frac{2\lambda}{h-z} + \frac{2\lambda}{h+z} \right) dz = 2\lambda \ln \frac{h+z}{h-z} \Big|_0^{\frac{h}{2}}$$

$$= 2\lambda \ln 3$$

Problem 3 (Moving charged lines) (22 pts)



Two infinite lines of charge with charge per unit length λ_0 in their rest frame are separated by a distance d. These charges are moving in a direction parallel to their length with a velocity $\vec{v}$. This speed could be close to the speed of light!

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(a) First consider the case where the wires are not moving. What is the electric force per unit length that the top line feels due to the bottom line? Give the direction and magnitude.

$|\vec{E}| = \frac{2\lambda}{d}$, direction is away from bottom wire

$$\rightarrow \boxed{\frac{|\vec{F}|}{L} = \frac{2\lambda^2}{d} \quad \text{pointing up (repelled)}}$$

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(b) Now consider the case where the wires are moving (with respect to the lab frame) as described in the figure above. In the lab frame, what is the electric force per unit length that the top line feels due to the bottom line? Give direction and magnitude.

Lorentz contraction increases the density of the line charge:

$$\lambda_{moving} = \frac{\lambda_{not\ moving}}{\sqrt{1 - v^2/c^2}}$$

$$\rightarrow \boxed{\frac{|\vec{F}|}{L} = \frac{1}{(1 - v^2/c^2)} \frac{2\lambda^2}{d}}$$

same direction.

(c) In the lab frame, what is the magnetic force per unit length that the top line feels due to the bottom line? Give direction and magnitude.

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Moving wire makes a current

$$I = (\lambda_{moving})\, v = \frac{\lambda v}{\sqrt{1 - v^2/c^2}}$$

$$\frac{|\vec{F}|}{L} = \frac{IB}{c}, \quad \text{where} \quad B = \frac{2I}{dc}$$

$$\rightarrow \boxed{\frac{|\vec{F}|}{L} = \frac{v^2/c^2}{(1 - v^2/c^2)} \frac{2\lambda^2}{d} \quad \underline{\text{down}}: \text{force is attractive}}$$

(d) Write down an expression for the total force on the top wire. Is there a speed at which the force goes to zero?

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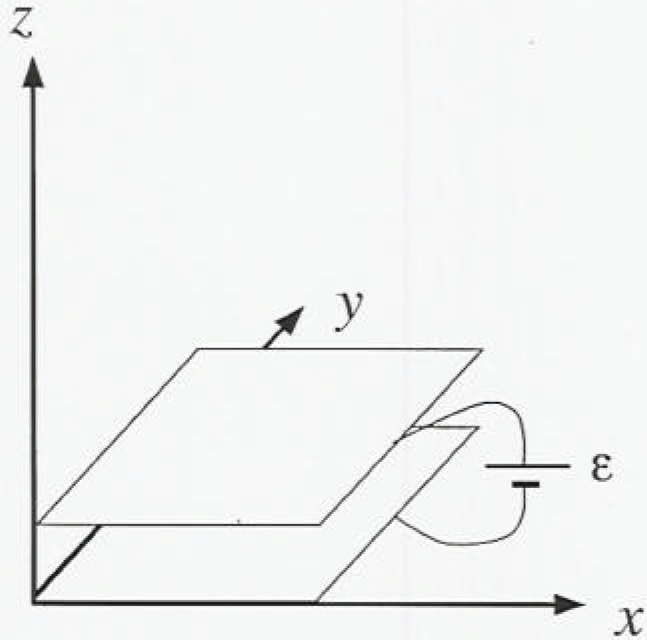
Total force per unit length is electric force minus magnetic (minus since they point in opposite directions):

$$\frac{|\vec{F}_{TOT}|}{L} = \frac{2\lambda^2}{d}\left[\frac{1 - v^2/c^2}{1 - v^2/c^2}\right] = \frac{2\lambda^2}{d}$$

Problem #4 (E and B fields in a parallel plate capacitor) 25 points

A parallel plate capacitor consists of two square plates of side L and separation d, with $d << L$, so you may ignore all fringing fields. The lower plate has one corner at the origin, and lies in the x-y plane with its edges aligned with the x-y axes; the upper plate is parallel to the lower at $z = d$. The plates are originally uncharged, but a source of EMF starts moving charge from the lower plate to the upper at $t = 0$. This battery maintains a steady current I, and the charge is drawn from the lower plate and put onto the upper plate evenly along their boundaries at $x = L$. Assume that the plates have a large conductivity so that at any instant the charge is uniformly distributed.



(a) What is the surface charge density $\sigma(t)$, and what is the electric field $E(t)$?

$$Q = \int I\,dt = It \qquad \sigma = \frac{It}{L^2}$$

$$E = 4\pi\sigma(-\hat{z}) \qquad \vec{E} = -\frac{4\pi It}{L^2}\hat{z}$$

(b) By symmetry, and the fact that the charge at $x = 0$ gets there by being injected at $x = L$, there must be surface currents $\boldsymbol{K}(x)$ (current per unit y-length) in the two plates. What are the surface currents $\boldsymbol{K}_b(x)$ and $\boldsymbol{K}_t(x)$ in the bottom and top plates? (Hint: the elegant way to solve this uses the continuity equation and $\sigma(t)$ from part a.)

$$L\left(K(x+\Delta x) - K(x)\right) = -\frac{d\sigma}{dt}\Delta x\, L$$

$$\frac{dK}{dx} = -\frac{d\sigma}{dt} = \frac{-I}{L^2}$$

$$\therefore \quad K_t = -\frac{I}{L^2} x\, \hat{x}$$

$$K_b = \frac{I}{L^2} x\, \hat{x}$$

(c) What is the magnetic field $\boldsymbol{B}(x)$ between the plates?

$$B = \frac{4\pi K}{c}(-\hat{y}) \text{ (from Ampère's law)}$$

$$\vec{B}(x) = -\frac{4\pi}{c}\cdot\frac{I}{L^2} x\, \hat{y}$$

$$= -\alpha x\, \hat{y} \qquad \text{let } \alpha = \frac{4\pi}{c}\frac{I}{L^2}$$

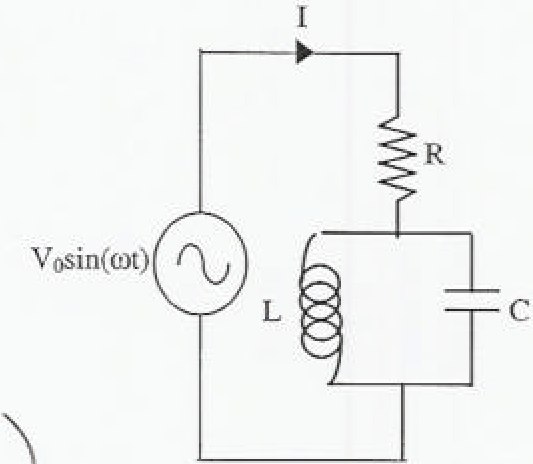
(d) Compute $\vec{\nabla} \times \vec{B}$ and compare it to $\frac{\partial \vec{E}}{\partial t}$.

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -\alpha x & 0 \end{vmatrix} = -\alpha \hat{z} = -\frac{4\pi}{c}\frac{I}{L^2}\hat{z}$$

$$\frac{\partial E}{\partial t} = -\frac{4\pi I t}{L^2}\hat{z} \qquad \therefore \quad \vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

Problem 5 (Impedance and Resonance) (21 pts)

a) What is the complex impedance of the combination of circuit elements, R, L, and C shown on the right? Please rationalize the expression into separate real and imaginary parts.



$$Z = R + \frac{1}{\frac{1}{i\omega L} + i\omega C}$$

$$= R + i\left(\frac{\omega L}{1-\omega^2 LC}\right)$$

b) What is the current, I (the actual and not the complex current,) flowing through the circuit? Please give an expression for the phase angle.

$$\tilde{I} = \frac{\tilde{V}}{Z} = \frac{V_0 e^{i\omega t}}{R + i\left(\frac{\omega L}{1-\omega^2 LC}\right)} = \frac{V_0 e^{i\omega t}}{\sqrt{R^2 + \left(\frac{\omega L}{1-\omega^2 LC}\right)^2}} \cdot e^{i\phi}$$

$$\phi = \arctan \frac{-\omega L/R}{1-\omega^2 LC} \qquad I = I_0 \sin(\omega t + \phi) \qquad I_0 = \frac{V_0}{\sqrt{R^2 + \left(\frac{\omega L}{1-\omega^2 LC}\right)^2}}$$

c) Explain the low and high frequency behavior of the phase shift of the current in terms of the currents through each of the circuit elements.

Low frequency: Most of the current goes through the inductor

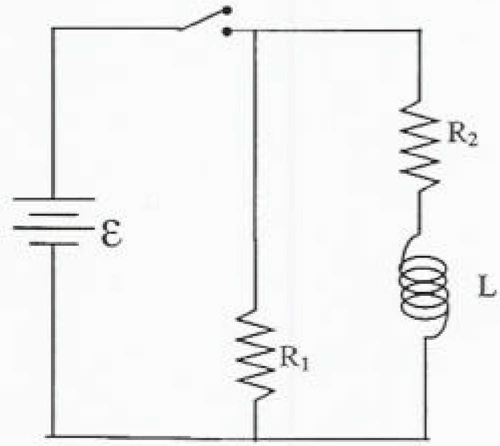
$\tan\phi \simeq \frac{-\omega L}{R} \rightarrow$ Phase Lags

High frequency: Most of the current goes through the capacitor

$\tan\phi \simeq \frac{1}{\omega C R} \rightarrow$ phase Leads

Problem #6 (Power and inductors) (20 pts)

In the schematic on the left, the switch S has been closed for a very long time, and then it is opened at time t=0.



(a) Calculate the power dissipated in the resistors R_1 and R_2 before the switch is opened.

$$\boxed{\begin{aligned} P_1 &= \mathcal{E}^2/R_1 \\ P_2 &= \mathcal{E}^2/R_2 \end{aligned}}$$

(b) Find the energy in the inductor immediately after the switch is opened.

Current through inductor was $I = \mathcal{E}/R_2$, doesn't want to change dammit:

$$\boxed{U = \frac{L\mathcal{E}^2}{2R_2^2}}$$

(c) Calculate the power dissipated in the resistors R_1 and R_2 after the switch is opened and as a function of time.

Current starts to decay:

$$I(t) = \frac{\mathcal{E}}{R_2} \exp\left[-(R_1+R_2)t/L\right]$$

$$P = I^2(t)\,[R_1+R_2]$$

$$\boxed{P = \frac{\mathcal{E}^2}{R_2^2}[R_1+R_2]\exp\left[-2(R_1+R_2)t/L\right]}$$

(d) Show that the total energy dissipated equals the result in (b).

$$U_{diss} = \int_0^\infty P \, dt$$

$$= \frac{\mathcal{E}^2}{R_2^2}[R_1+R_2] \int_0^\infty \exp\left[-2(R_1+R_2)t/L\right] dt$$

$$= \frac{-\mathcal{E}^2 L}{2R_2^2} e^{-2(R_1+R_2)t/L} \Big|_0^\infty$$

$$= \frac{-\mathcal{E}^2 L}{2R_2^2}(0-1)$$

$$\boxed{U_{diss} = \frac{\mathcal{E}^2 L}{2R_2^2}}$$

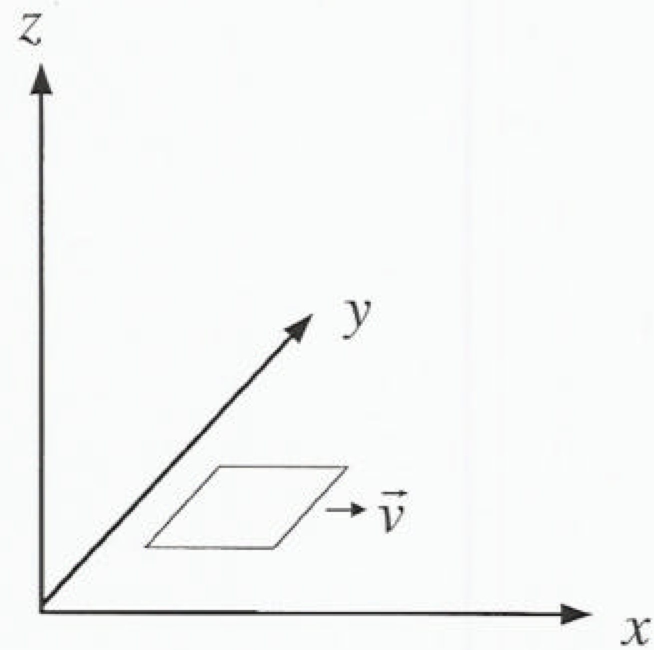
Same as (b)

Problem 7 – Square wire loop in a non-uniform magnetic field (20 pts)

In the figure on the right, there exists a static magnetic field of the form:

$$\vec{B} = B_0 x\hat{z}$$

That is to say, that the B-field points in the z-direction and increases in strength as you move in the x-direction. A rigid, square, wire frame of side l lies in the x-y plane with its center at the point $(x_0, y_0, z=0)$.



a) Compute the magnetic flux that points that resulting from magnetic field threading through the loop in the $\hat{z}$ direction.

$$\phi = \int B\, da = \ell \int_{x_0-\ell/2}^{x_0+\ell/2} B_0 x \cdot dx = \frac{B_0 \ell}{2} x^2 \Big|_{x_0-\ell/2}^{x_0+\ell/2}$$

$$= B_0 \ell^2 x_0$$

b) Next, suppose that the frame, still lying in the x-y plane, moves with a constant velocity $\vec{v} = v_0\hat{x}$. Compute the magnitude ε of the EMF generated around the frame.

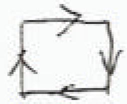
$$\varepsilon = -\frac{1}{c}\frac{d\phi}{dt} = -\frac{B_0 \ell^2}{c}\frac{dx_0}{dt} = -\frac{B_0 \ell^2 v_0}{c}$$

c) Suppose that the frame has a resistance R and a self-inductance L. Find the current in the loop after it has been moving for a long time (long compared with L/R) with constant velocity $\vec{v} = v_0\hat{x}$.

Sine past long time, the current is steady,

L has no effect.

$$I = \frac{\mathcal{E}}{R} = \frac{\ell^2 B_0 v_0}{cR}$$ along the directi



d) Find the magnetic force on the frame after it has been moving for a long time with constant velocity $\vec{v} = v_0\hat{x}$.

B doesn't vary along y directi, from

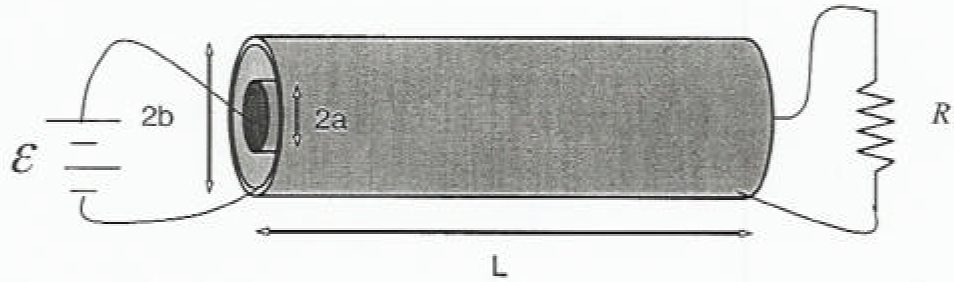
$d\vec{F} = \frac{I}{c} d\vec{\ell} \times \vec{B}$, the two forces along y directi cancel out.

Assume the locati of the center of the square is x:

$$F_x = -\frac{I}{c}\ell B(x+\ell) + \frac{I}{c}\ell B(x) = -\frac{I}{c}\ell B_0(x+\ell) + \frac{I}{c}\ell B$$

$$= -\frac{I}{c}\ell^2 B_0 = -\frac{\ell^4 B_0^2 v_0}{c^2 R}$$

Problem #8 (Power in a coaxial cable) (25 pts)



A coaxial cable transmits DC power from a battery to a load. The cable consists of two concentric long hollow cylinders of zero resistance; the inner has radius a, the outer has radius b, and the length of both is L. The battery provides an *EMF* $\mathcal{E}$ between the two conductors at one end of the cable, and the load is a resistance R connected between the two conductors at the other end of the cable.

(a) How much power P is dissipated in the resistor?

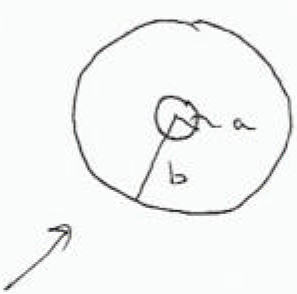
$$\boxed{P = \mathcal{E}^2 / R}$$

(b) What are $\boldsymbol{E}$ and $\boldsymbol{B}$ in the cable (in terms of $\varepsilon, R, c,$ etc.)? You'll need to determine the cable capacitance in order to figure out $\boldsymbol{E}$.

$$B(r) = \frac{2\mathcal{E}}{crR} \qquad a \le r \le b$$

$$= 0 \qquad \text{elsewhere}$$

Clockwise



Batteryend: current is inward on center pipe
outward on outside pipe.

$$E = \frac{2Q}{rL}$$

$$\rightarrow V = \frac{2Q}{L} \ln(b/a) = \varepsilon$$

$$Q = \frac{\varepsilon L}{2 \ln(b/a)}$$

$$\boxed{\begin{aligned} E &= \frac{\varepsilon}{r \ln(b/a)} \quad a \le r \le b \\ &= 0 \quad \text{elsewhere} \end{aligned}}$$

radially outward from center

(c) What is the Poynting vector **S** in the cable?

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}$$

$$\boxed{|\vec{S}| = \frac{\varepsilon^2}{2\pi r^2 R \ln(b/a)}}$$

Points down pipe towards resistor

(d) Show that $\int \vec{S} \cdot d\vec{a} = P$.

$$\int \vec{S} \cdot d\vec{A} = \frac{\varepsilon^2}{2\pi R \ln(b/a)} \int_a^b \frac{2\pi r\, dr}{r^2}$$

$$= \frac{\varepsilon^2}{R \ln(b/a)} \int_a^b \frac{dr}{r}$$

$$\boxed{\int \vec{S} \cdot d\vec{A} = \frac{\varepsilon^2}{R}}$$