Example 8.11: Capstan

A device called a capstan is used aboard ships in order to control a rope that is under great tension. The rope is wrapped around a fixed drum of radius R, usually for several turns (Figure 8.45 shows about three fourths turn as seen from overhead). The load on the rope pulls it with a force T_A, and the sailor holds the other end of the rope with a much smaller force T_B. The coefficient of static friction between the rope and the drum is μ_s. The sailor is holding the rope so that it is just about to slip. Show that $T_B = T_A e^{-\mu_s \theta_{BA}}$, where θ_{BA} is the angle subtended by the rope on the drum.

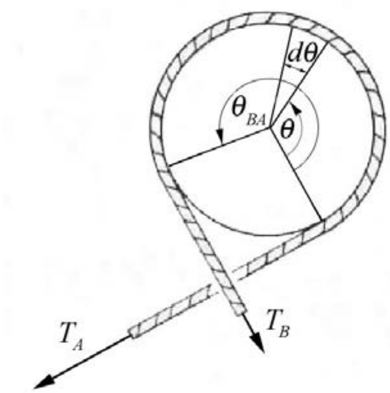

Figure 8.45 Capstan

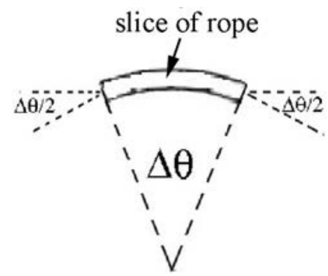

Figure 8.46 Small slice of rope

Solution: We begin by considering a small slice of rope of arc length $R\Delta\theta$, shown in the Figure 8.46. We choose unit vectors for the force diagram on this section of the rope and indicate them on Figure 8.47. The right edge of the slice is at angle θ and the left edge of the slice is at $\theta + \Delta\theta$. The angle edge end of the slice makes with the horizontal is $\Delta\theta / 2$. There are four forces acting on this section of the rope. The forces are the normal force between the capstan and the rope pointing outward, a static frictional force and the tensions at either end of the slice. The rope is held at the just slipping point, so if the load exerts a greater force the rope will slip to the right. Therefore the direction of the static frictional force between the capstan and the rope, acting on the rope, points to the left. The tension on the right side of the slice is denoted by $T(\theta) \equiv T$, while the tension on the left side of the slice is denoted by $T(\theta + \Delta\theta) \equiv T + \Delta T$. Does the tension in this slice from the right side to the left, increase, remain the same, or decrease? The tension decreases because the load on the left side is less than the load on the right side. Note that $\Delta T < 0$.

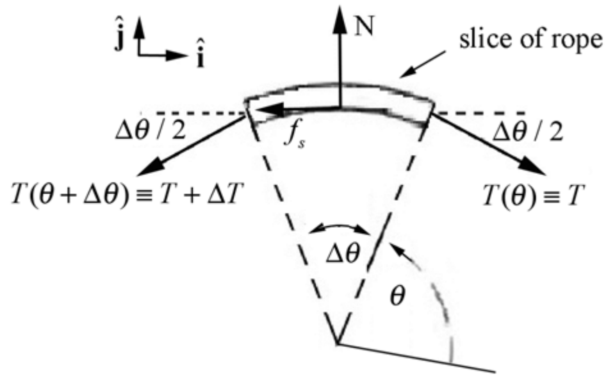

Figure 8.47 Free-body force diagram on small slice of rope

The vector decomposition of the forces is given by

$$\hat{\mathbf{i}}: T\cos(\Delta\theta/2) - f_s - (T+\Delta T)\cos(\Delta\theta/2) \tag{8.6.89}$$

$$\hat{\mathbf{j}}: -T\sin(\Delta\theta/2) + N - (T+\Delta T)\sin(\Delta\theta/2)\,. \tag{8.6.90}$$

For small angles $\Delta\theta$, $\cos(\Delta\theta/2)\cong 1$ and $\sin(\Delta\theta/2)\cong \Delta\theta/2$. Using the small angle approximations, the vector decomposition of the forces in the x-direction (the $+\hat{\mathbf{i}}$-direction) becomes

$$\begin{aligned} T\cos(\Delta\theta/2) - f_s - (T+\Delta T)\cos(\Delta\theta/2) &\simeq T - f_s - (T+\Delta T) \\ &= -f_s - \Delta T \end{aligned}\,. \tag{8.6.91}$$

By the static equilibrium condition the sum of the x-components of the forces is zero,

$$-f_s - \Delta T = 0\,. \tag{8.6.92}$$

The vector decomposition of the forces in the y-direction (the $+\hat{\mathbf{j}}$-direction) is approximately

$$\begin{aligned} -T\sin(\Delta\theta/2) + N - (T+\Delta T)\sin(\Delta\theta/2) &\simeq -T\Delta\theta/2 + N - (T+\Delta T)\Delta\theta/2 \\ &= -T\Delta\theta + N - \Delta T\Delta\theta/2 \end{aligned}\,. \tag{8.6.93}$$

In the last equation above we can ignore the terms proportional to $\Delta T\,\Delta\theta$ because these are the product of two small quantities and hence are much smaller than the terms proportional to either ΔT or $\Delta\theta$. The vector decomposition in the y-direction becomes

$$-T\Delta\theta + N\,. \tag{8.6.94}$$

Static equilibrium implies that this sum of the y-components of the forces is zero,

$$-T\Delta\theta + N = 0\,. \tag{8.6.95}$$

We can solve this equation for the magnitude of the normal force

$$N = T\Delta\theta\,. \tag{8.6.96}$$

The just slipping condition is that the magnitude of the static friction attains its maximum value

$$f_s = (f_s)_{\max} = \mu_s N\,. \tag{8.6.97}$$

We can now combine the Equations (8.6.92) and (8.6.97) to yield

$$\Delta T = -\mu_s N\,. \tag{8.6.98}$$

Now substitute the magnitude of the normal force, Equation (8.6.96), into Equation (8.6.98), yielding

$$-\mu_s T\Delta\theta - \Delta T = 0\,. \tag{8.6.99}$$

Finally, solve this equation for the ratio of the change in tension to the change in angle,

$$\frac{\Delta T}{\Delta\theta} = -\mu_s T\,. \tag{8.6.100}$$

The derivative of tension with respect to the angle θ is defined to be the limit

$$\frac{dT}{d\theta} \equiv \lim_{\Delta\theta \to 0} \frac{\Delta T}{\Delta\theta}\,, \tag{8.6.101}$$

and Equation (8.6.100) becomes

$$\frac{dT}{d\theta} = -\mu_s T\,. \tag{8.6.102}$$

This is an example of a first order linear differential equation that shows that the rate of change of tension with respect to the angle θ is proportional to the negative of the tension at that angle θ. This equation can be solved by integration using the technique of separation of variables. We first rewrite Equation (8.6.102) as

$$\frac{dT}{T} = -\mu_s\, d\theta\,. \tag{8.6.103}$$

Integrate both sides, noting that when $\theta = 0$, the tension is equal to force of the load T_A, and when angle $\theta = \theta_{A,B}$ the tension is equal to the force T_B the sailor applies to the rope,

$$\int_{T=T_A}^{T=T_B} \frac{dT}{T} = -\int_{\theta=0}^{\theta=\theta_{BA}} \mu_s \, d\theta \, . \tag{8.6.104}$$

The result of the integration is

$$\ln\left(\frac{T_B}{T_A}\right) = -\mu_s \theta_{BA} \, . \tag{8.6.105}$$

Note that the exponential of the natural logarithm

$$\exp\left(\ln\left(\frac{T_B}{T_A}\right)\right) = \frac{T_B}{T_A} \, , \tag{8.6.106}$$

so exponentiating both sides of Equation (8.6.105) yields

$$\frac{T_B}{T_A} = e^{-\mu_s \theta_{BA}} \, ; \tag{8.6.107}$$

the tension decreases exponentially,

$$T_B = T_A \, e^{-\mu_s \theta_{BA}} \, , \tag{8.6.108}$$

Because the tension decreases exponentially, the sailor need only apply a small force to prevent the rope from slipping.

Example 8.12 Free Fall with Air Drag

Consider an object of mass *m* that is in free fall but experiencing air resistance. The magnitude of the drag force is given by Eq. (8.6.1), where ρ is the density of air, A is