6.2 Circular Motion: Velocity and Angular Velocity

We begin our description of circular motion by choosing polar coordinates. In Figure 6.1 we sketch the position vector $\vec{\mathbf{r}}(t)$ of the object moving in a circular orbit of radius r.

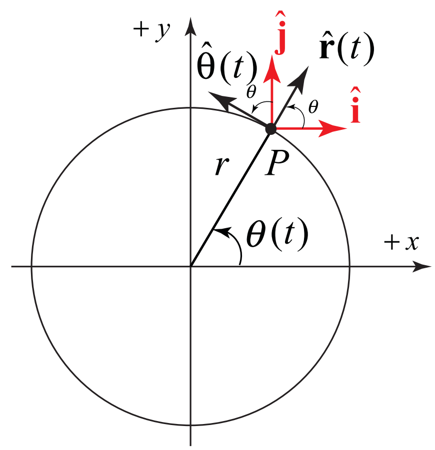

Figure 6.1 A circular orbit with unit vectors.

At time t, the particle is located at the point P with coordinates $(r, \theta(t))$ and position vector given by

$$\vec{\mathbf{r}}(t) = r\,\hat{\mathbf{r}}(t). \tag{6.2.1}$$

At the point P, consider two sets of unit vectors ($\hat{\mathbf{r}}(t)$, $\hat{\boldsymbol{\theta}}(t)$) and ($\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$), as shown in Figure 6.1. The vector decomposition expression for $\hat{\mathbf{r}}(t)$ and $\hat{\boldsymbol{\theta}}(t)$ in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ is given by

$$\hat{\mathbf{r}}(t) = \cos\theta(t)\,\hat{\mathbf{i}} + \sin\theta(t)\,\hat{\mathbf{j}}, \tag{6.2.2}$$

$$\hat{\boldsymbol{\theta}}(t) = -\sin\theta(t)\,\hat{\mathbf{i}} + \cos\theta(t)\,\hat{\mathbf{j}}. \tag{6.2.3}$$

Before we calculate the velocity, we shall calculate the time derivatives of Eqs. (6.2.2) and (6.2.3). Let's first begin with $d\hat{\mathbf{r}}(t)/dt$:

$$\frac{d\hat{\mathbf{r}}(t)}{dt} = \frac{d}{dt}(\cos\theta(t)\,\hat{\mathbf{i}} + \sin\theta(t)\,\hat{\mathbf{j}}) = (-\sin\theta(t)\frac{d\theta(t)}{dt}\,\hat{\mathbf{i}} + \cos\theta(t)\frac{d\theta(t)}{dt}\,\hat{\mathbf{j}})$$
$$= \frac{d\theta(t)}{dt}(-\sin\theta(t)\,\hat{\mathbf{i}} + \cos\theta(t)\,\hat{\mathbf{j}}) = \frac{d\theta(t)}{dt}\hat{\boldsymbol{\theta}}(t) \quad ; \tag{6.2.4}$$

where we used the chain rule to calculate that

$$\frac{d}{dt}\cos\theta(t) = -\sin\theta(t)\frac{d\theta(t)}{dt} \quad , \tag{6.2.5}$$

$$\frac{d}{dt}\sin\theta(t) = \cos\theta(t)\frac{d\theta(t)}{dt} \quad . \tag{6.2.6}$$

The calculation for $d\hat{\boldsymbol{\theta}}(t)/dt$ is similar:

$$\frac{d\hat{\boldsymbol{\theta}}(t)}{dt} = \frac{d}{dt}(-\sin\theta(t)\,\hat{\mathbf{i}} + \cos\theta(t\,\hat{\mathbf{j}}) = (-\cos\theta(t)\frac{d\theta(t)}{dt}\,\hat{\mathbf{i}} - \sin(t)\frac{d\theta(t)}{dt}\,\hat{\mathbf{j}})$$
$$= \frac{d\theta(t)}{dt}(-\cos\theta(t)\,\hat{\mathbf{i}} - \sin\theta(t)\,\hat{\mathbf{j}}) = -\frac{d\theta(t)}{dt}\hat{\mathbf{r}}(t) \quad . \tag{6.2.7}$$

The velocity vector is then

$$\vec{\mathbf{v}}(t) = \frac{d\vec{\mathbf{r}}(t)}{dt} = r\frac{d\hat{\mathbf{r}}}{dt} = r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}(t) = v_\theta\,\hat{\boldsymbol{\theta}}(t), \tag{6.2.8}$$

where the $\hat{\boldsymbol{\theta}}$-component of the velocity is given by

$$v_\theta = r\frac{d\theta}{dt}, \tag{6.2.9}$$

a quantity we shall refer to as the *tangential component of the velocity*. Denote the magnitude of the velocity by $v \equiv \left|\vec{\mathbf{v}}\right|$, The angular speed is the magnitude of the rate of change of angle with respect to time, which we denote by the Greek letter ω,

$$\omega \equiv \left|\frac{d\theta}{dt}\right|. \tag{6.2.10}$$

6.2.1 Geometric Derivation of the Velocity for Circular Motion

Consider a particle undergoing circular motion. At time t, the position of the particle is $\vec{\mathbf{r}}(t)$. During the time interval Δt, the particle moves to the position $\vec{\mathbf{r}}(t+\Delta t)$ with a displacement $\Delta\vec{\mathbf{r}}$.

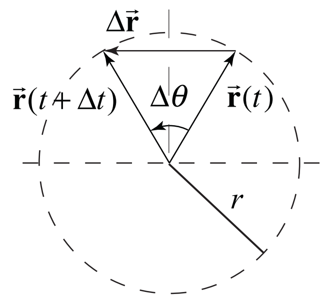

Figure 6.2 Displacement vector for circular motion

The magnitude of the displacement, $\left|\Delta\vec{\mathbf{r}}\right|$, is represented by the length of the horizontal vector $\Delta\vec{\mathbf{r}}$ joining the heads of the displacement vectors in Figure 6.2 and is given by

$$\left|\Delta\vec{\mathbf{r}}\right| = 2r\sin(\Delta\theta/2). \tag{6.2.11}$$

When the angle $\Delta\theta$ is small, we can approximate

$$\sin(\Delta\theta/2) \cong \Delta\theta/2. \tag{6.2.12}$$

This is called the *small angle approximation*, where the angle $\Delta\theta$ (and hence $\Delta\theta/2$) is measured in radians. This fact follows from an infinite power series expansion for the sine function given by

$$\sin\left(\frac{\Delta\theta}{2}\right) = \frac{\Delta\theta}{2} - \frac{1}{3!}\left(\frac{\Delta\theta}{2}\right)^3 + \frac{1}{5!}\left(\frac{\Delta\theta}{2}\right)^5 - \cdots. \tag{6.2.13}$$

When the angle $\Delta\theta/2$ is small, only the first term in the infinite series contributes, as successive terms in the expansion become much smaller. For example, when $\Delta\theta/2 = \pi/30 \cong 0.1$, corresponding to 6°, $(\Delta\theta/2)^3/3! \cong 1.9\times10^{-4}$; this term in the power series is three orders of magnitude smaller than the first and can be safely ignored for small angles.

Using the small angle approximation, the magnitude of the displacement is

$$\left|\Delta\vec{\mathbf{r}}\right| \cong r\,\Delta\theta\,. \tag{6.2.14}$$

This result should not be too surprising since in the limit as $\Delta\theta$ approaches zero, the length of the chord approaches the arc length $r\,\Delta\theta$.

The magnitude of the velocity, v, is proportional to the rate of change of the magnitude of the angle with respect to time,

$$v \equiv \left|\vec{\mathbf{v}}(t)\right| = \lim_{\Delta t\to 0}\frac{\left|\Delta\vec{\mathbf{r}}\right|}{\Delta t} = \lim_{\Delta t\to 0}\frac{r\left|\Delta\theta\right|}{\Delta t} = r\lim_{\Delta t\to 0}\frac{\left|\Delta\theta\right|}{\Delta t} = r\left|\frac{d\theta}{dt}\right| = r\omega\,. \tag{6.2.15}$$

The direction of the velocity can be determined by considering that in the limit as $\Delta t \to 0$ (note that $\Delta\theta \to 0$), the direction of the displacement $\Delta\vec{\mathbf{r}}$ approaches the direction of the tangent to the circle at the position of the particle at time t (Figure 6.3).

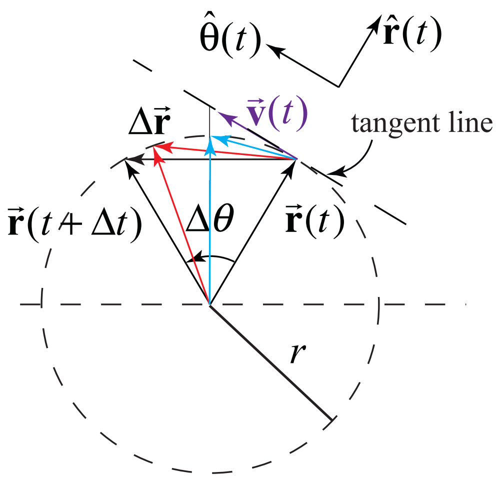

Figure 6.3 Direction of the displacement approaches the direction of the tangent line

Thus, in the limit $\Delta t \to 0$, $\Delta\vec{\mathbf{r}} \perp \vec{\mathbf{r}}$, and so the direction of the velocity $\vec{\mathbf{v}}(t)$ at time t is perpendicular to the position vector $\vec{\mathbf{r}}(t)$ and tangent to the circular orbit in the $+\hat{\boldsymbol{\theta}}$-direction for the case shown in Figure 6.3.