14.9 Worked Examples

Example 14.2 Escape Velocity of Toro

The asteroid Toro, discovered in 1964, has a radius of about $R = 5.0\,\text{km}$ and a mass of about $m_t = 2.0 \times 10^{15}\,\text{kg}$. Let's assume that Toro is a perfectly uniform sphere. What is the escape velocity for an object of mass m on the surface of Toro? Could a person reach this speed (on earth) by running?

Solution: The only potential energy in this problem is the gravitational potential energy. We choose the zero point for the potential energy to be when the object and Toro are an infinite distance apart, $U^G(\infty) \equiv 0$. With this choice, the potential energy when the object and Toro are a finite distance r apart is given by

$$U^G(r) = -\frac{Gm_t\,m}{r} \tag{14.8.1}$$

with $U^G(\infty) \equiv 0$. The expression ***escape velocity*** refers to the minimum speed necessary for an object to escape the gravitational interaction of the asteroid and move off to an infinite distance away. If the object has a speed less than the escape velocity, it will be unable to escape the gravitational force and must return to Toro. If the object has a speed greater than the escape velocity, it will have a non-zero kinetic energy at infinity. The condition for the escape velocity is that the object will have exactly zero kinetic energy at infinity.

We choose our initial state, at time t_i, when the object is at the surface of the asteroid with speed equal to the escape velocity. We choose our final state, at time t_f, to occur when the separation distance between the asteroid and the object is infinite.

The initial kinetic energy is $K_i = (1/2)mv_{esc}^2$. The initial potential energy is $U_i = -Gm_t m / R$, and so the initial mechanical energy is

$$E_i = K_i + U_i = \frac{1}{2}mv_{esc}^2 - \frac{Gm_t m}{R} \ . \tag{14.8.2}$$

The final kinetic energy is $K_f = 0$, because this is the condition that defines the escape velocity. The final potential energy is zero, $U_f = 0$ because we chose the zero point for potential energy at infinity. The final mechanical energy is then

$$E_f = K_f + U_f = 0 \ . \tag{14.8.3}$$

There is no non-conservative work, so the change in mechanical energy is zero

$$0 = W_{nc} = \Delta E_m = E_f - E_i \ . \tag{14.8.4}$$

Therefore

$$0 = -\left(\frac{1}{2}mv_{esc}^2 - \frac{Gm_t m}{R} \right) . \tag{14.8.5}$$

This can be solved for the escape velocity,

$$\begin{aligned} v_{esc} &= \sqrt{\frac{2Gm_t}{R}} \\ &= \sqrt{\frac{2(6.67\times10^{-11}\,\text{N}\cdot\text{m}^2\cdot\text{kg}^{-2})(2.0\times10^{15}\,\text{kg})}{(5.0\times10^3\,\text{m})}} = 7.3 \ \text{m}\cdot\text{s}^{-1}. \end{aligned} \tag{14.8.6}$$

Considering that Olympic sprinters typically reach velocities of 12 $\text{m} \cdot \text{s}^{-1}$, this is an easy speed to attain by running on earth. It may be harder on Toro to generate the acceleration necessary to reach this speed by pushing off the ground, since any slight upward force will raise the runner's center of mass and it will take substantially more time than on earth to come back down for another push off the ground.

Example 14.3 Spring-Block-Loop-the-Loop

A small block of mass m is pushed against a spring with spring constant k and held in place with a catch. The spring is compressed an unknown distance x (Figure 14.12). When the catch is removed, the block leaves the spring and slides along a frictionless circular loop of radius r. When the block reaches the top of the loop, the force of the loop on the block (the normal force) is equal to twice the gravitational force on the mass. (a) Using conservation of energy, find the kinetic energy of the block at the top of the loop. (b) Using Newton's Second Law, derive the equation of motion for the block when it is at the top of the loop. Specifically, find the speed v_{top} in terms of the gravitation constant g and the loop radius r. (c) What distance was the spring compressed?
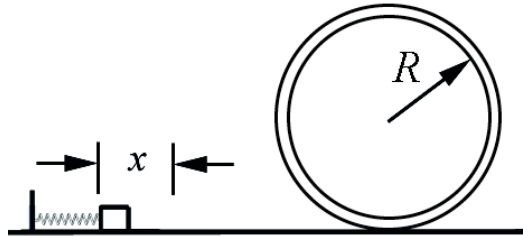

Figure 14.12 Initial state for spring-block-loop-the-loop system

Solution: a) Choose for the initial state the instant before the catch is released. The initial kinetic energy is $K_i = 0$. The initial potential energy is non-zero, $U_i = (1/2)k\,x^2$. The initial mechanical energy is then

$$E_i = K_i + U_i = \frac{1}{2}k\,x^2 \,. \tag{14.8.7}$$

Choose for the final state the instant the block is at the top of the loop. The final kinetic energy is $K_f = (1/2)mv_{\text{top}}^2$; the block is in motion with speed v_{top}. The final potential energy is non-zero, $U_f = (mg)(2R)$. The final mechanical energy is then

$$E_f = K_f + U_f = 2mgR + \frac{1}{2}mv_{\text{top}}^2 \,. \tag{14.8.8}$$

Because we are assuming the track is frictionless and neglecting air resistance, there is no non-conservative work. The change in mechanical energy is therefore zero,

$$0 = W_{nc} = \Delta E_m = E_f - E_i .$$
(14.8.9)

Mechanical energy is conserved, $E_f = E_i$, therefore

$$2mgR + \frac{1}{2}mv_{top}^2 = \frac{1}{2}kx^2 .$$
(14.8.10)

From Equation (14.8.10), the kinetic energy at the top of the loop is

$$\frac{1}{2}mv_{top}^2 = \frac{1}{2}kx^2 - 2mgR .$$
(14.8.11)

b) At the top of the loop, the forces on the block are the gravitational force of magnitude mg and the normal force of magnitude N, both directed down. Newton's Second Law in the radial direction, which is the downward direction, is

$$-mg - N = -\frac{mv_{top}^2}{R} .$$
(14.8.12)

In this problem, we are given that when the block reaches the top of the loop, the force of the loop on the block (the normal force, *downward* in this case) is equal to twice the weight of the block, $N = 2mg$. The Second Law, Eq. (14.8.12), then becomes

$$3mg = \frac{mv_{top}^2}{R} .$$
(14.8.13)

We can rewrite Equation (14.8.13) in terms of the kinetic energy as

$$\frac{3}{2}mg\,R = \frac{1}{2}mv_{top}^2 .$$
(14.8.14)

The speed at the top is therefore
$$v_{top} = \sqrt{3mg\,R} .$$
(14.8.15)

c) Combing Equations (14.8.11) and (14.8.14) yields

$$\frac{7}{2}mg\,R = \frac{1}{2}kx^2 .$$
(14.8.16)

Thus the initial displacement of the spring from equilibrium is

$$x = \sqrt{\frac{7mg\,R}{k}} \,.$$
(14.8.17)

Example 14.4 Mass-Spring on a Rough Surface

A block of mass m slides along a horizontal table with speed v_0. At $x = 0$ it hits a spring with spring constant k and begins to experience a friction force. The coefficient of friction is variable and is given by $\mu = bx$, where b is a positive constant. Find the loss in mechanical energy when the block first momentarily comes to rest.
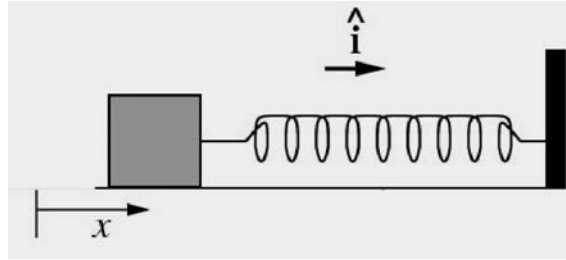

Figure 14.13 Spring-block system

Solution: From the model given for the frictional force, we could find the non-conservative work done, which is the same as the loss of mechanical energy, if we knew the position x_f where the block first comes to rest. The most direct (and easiest) way to find x_f is to use the work-energy theorem. The initial mechanical energy is $E_i = mv_i^2 / 2$ and the final mechanical energy is $E_f = k x_f^2 / 2$ (note that there is no potential energy term in E_i and no kinetic energy term in E_f). The difference between these two mechanical energies is the non-conservative work done by the frictional force,

$$W_{nc} = \int_{x=0}^{x=x_f} F_{nc}\, dx = \int_{x=0}^{x=x_f} -F_{friction}\, dx = \int_{x=0}^{x=x_f} -\mu\, N\, dx$$

$$= -\int_0^{x_f} b\, x\, mg\, dx = -\frac{1}{2} bmg\, x_f^2.$$
(14.8.18)

We then have that

$$W_{nc} = \Delta E_m$$
$$W_{nc} = E_f - E_i$$
(14.8.19)
$$-\frac{1}{2} bmg\, x_f^2 = \frac{1}{2} k x_f^2 - \frac{1}{2} mv_i^2.$$

Solving the last of these equations for x_f^2 yields

$$x_f^2 = \frac{mv_0^2}{k + bmg}. \qquad (14.8.20)$$

Substitute Eq. (14.8.20) into Eq. (14.8.18) gives the result that

$$W_{nc} = -\frac{bmg}{2}\frac{mv_0^2}{k+bmg} = -\frac{mv_0^2}{2}\left(1+\frac{k}{bmg}\right)^{-1}. \qquad (14.8.21)$$

It is worth checking that the above result is dimensionally correct. From the model, the parameter b must have dimensions of inverse length (the coefficient of friction μ must be dimensionless), and so the product bmg has dimensions of force per length, as does the spring constant k; the result is dimensionally consistent.

Example 14.5 Cart-Spring on an Inclined Plane

An object of mass m slides down a plane that is inclined at an angle θ from the horizontal (Figure 14.14). The object starts out at rest. The center of mass of the cart is a distance d from an unstretched spring that lies at the bottom of the plane. Assume the spring is massless, and has a spring constant k. Assume the inclined plane to be frictionless. (a) How far will the spring compress when the mass first comes to rest? (b) Now assume that the inclined plane has a coefficient of kinetic friction μ_k. How far will the spring compress when the mass first comes to rest? The friction is primarily between the wheels and the bearings, not between the cart and the plane, but the friction force may be modeled by a coefficient of friction μ_k. (c) In case (b), how much energy has been lost to friction?
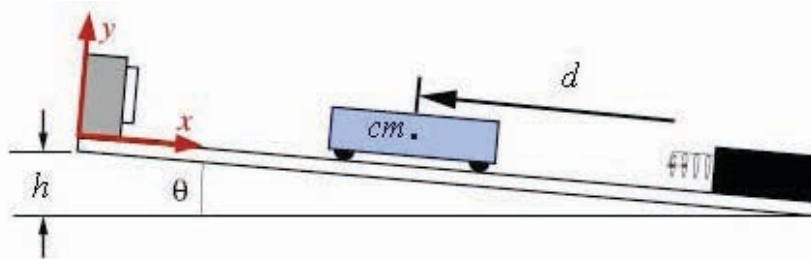

Figure 14.14 Cart on inclined plane

Solution: Let x denote the displacement of the spring from the equilibrium position. Choose the zero point for the gravitational potential energy $U^g(0) = 0$ not at the very bottom of the inclined plane, but at the location of the end of the unstretched spring. Choose the zero point for the spring potential energy where the spring is at its equilibrium position, $U^s(0) = 0$.

a) Choose for the initial state the instant the object is released (Figure 14.15). The initial kinetic energy is $K_i = 0$. The initial potential energy is non-zero, $U_i = mg\,d\sin\theta$. The initial mechanical energy is then

$$E_i = K_i + U_i = mg\,d\sin\theta \qquad (14.8.22)$$

Choose for the final state the instant when the object first comes to rest and the spring is compressed a distance x at the bottom of the inclined plane (Figure 14.16). The final kinetic energy is $K_f = 0$ since the mass is not in motion. The final potential energy is non-zero, $U_f = k x^2 / 2 - x\,mg\sin\theta$. Notice that the gravitational potential energy is negative because the object has dropped below the height of the zero point of gravitational potential energy.
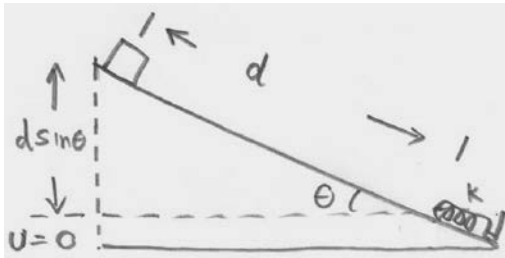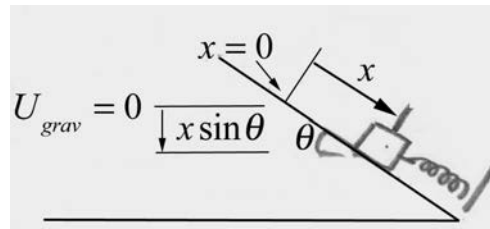

Figure 14.15 Initial state   **Figure 14.16** Final state

The final mechanical energy is then

$$E_f = K_f + U_f = \frac{1}{2}k x^2 - x\,mg\sin\theta. \qquad (14.8.23)$$

Because we are assuming the track is frictionless and neglecting air resistance, there is no non- conservative work. The change in mechanical energy is therefore zero,

$$0 = W_{nc} = \Delta E_m = E_f - E_i. \qquad (14.8.24)$$

Therefore

$$d\,mg\sin\theta = \frac{1}{2}k x^2 - x\,mg\sin\theta. \qquad (14.8.25)$$

This is a quadratic equation in x,

$$x^2 - \frac{2mg\sin\theta}{k}x - \frac{2d\,mg\sin\theta}{k} = 0. \qquad (14.8.26)$$

In the quadratic formula, we want the positive choice of square root for the solution to ensure a positive displacement of the spring from equilibrium,

$$x = \frac{mg\sin\theta}{k} + \left(\frac{m^2 g^2 \sin^2\theta}{k^2} + \frac{2d\,mg\sin\theta}{k} \right)^{1/2}$$

(14.8.27)

$$= \frac{mg}{k}(\sin\theta + \sqrt{1 + 2(k\,d\,/\,mg)\sin\theta}).$$

(What would the solution with the negative root represent?)

b) The effect of kinetic friction is that there is now a non-zero non-conservative work done on the object, which has moved a distance, $d + x$, given by

$$W_{nc} = -f_k(d + x) = -\mu_k N(d + x) = -\mu_k mg\cos\theta(d + x).$$

(14.8.28)

Note the normal force is found by using Newton's Second Law in the perpendicular direction to the inclined plane,

$$N - mg\cos\theta = 0.$$

(14.8.29)

The change in mechanical energy is therefore

$$W_{nc} = \Delta E_m = E_f - E_i,$$

(14.8.30)

which becomes

$$-\mu_k mg\cos\theta(d + x) = \left(\frac{1}{2}k\,x^2 - x\,mg\sin\theta \right) - d\,mg\sin\theta.$$

(14.8.31)

Equation (14.8.31) simplifies to

$$0 = \left(\frac{1}{2}k\,x^2 - x\,mg(\sin\theta - \mu_k\cos\theta) \right) - d\,mg(\sin\theta - \mu_k\cos\theta).$$

(14.8.32)

This is the same as Equation (14.8.25) above, but with $\sin\theta \to \sin\theta - \mu_k\cos\theta$. The maximum displacement of the spring is when there is friction is then

$$x = \frac{mg}{k}((\sin\theta - \mu_k\cos\theta) + \sqrt{1 + 2(k\,d\,/\,mg)(\sin\theta - \mu_k\cos\theta)}).$$

(14.8.33)

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c) The energy lost to friction is given by $W_{nc} = -\mu_k mg\cos\theta(d + x)$, where x is given in part b).

Example 14.6 Object Sliding on a Sphere

A small point like object of mass m rests on top of a sphere of radius R. The object is released from the top of the sphere with a negligible speed and it slowly starts to slide (Figure 14.17). Let g denote the gravitation constant. (a) Determine the angle θ_1 with

respect to the vertical at which the object will lose contact with the surface of the sphere. (b) What is the speed v_1 of the object at the instant it loses contact with the surface of the sphere.
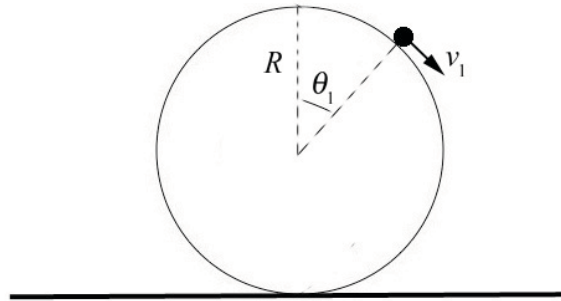

Figure 14.17 Object sliding on surface of sphere

Solution: We begin by identifying the forces acting on the object. There are two forces acting on the object, the gravitation and radial normal force that the sphere exerts on the particle that we denote by N. We draw a free-body force diagram for the object while it is sliding on the sphere. We choose polar coordinates as shown in Figure 14.18.
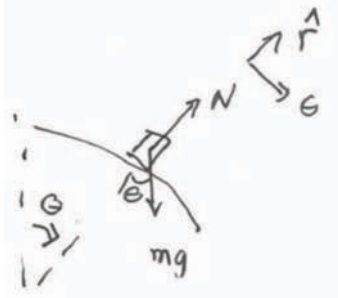

Figure 14.18 Free-body force diagram on object

The key constraint is that when the particle just leaves the surface the normal force is zero,

$$N(\theta_1) = 0,$$ (14.8.34)

where θ_1 denotes the angle with respect to the vertical at which the object will just lose contact with the surface of the sphere. Because the normal force is perpendicular to the displacement of the object, it does no work on the object and hence conservation of energy does not take into account the constraint on the motion imposed by the normal force. In order to analyze the effect of the normal force we must use the radial component of Newton's Second Law,

$$N - mg\cos\theta = -m\frac{v^2}{R}.$$ (14.8.35)

Then when the object just loses contact with the surface, Eqs. (14.8.34) and (14.8.35) require that

$$mg\cos\theta_1 = m\frac{v_1^{\,2}}{R}.$$ (14.8.36)

where v_1 denotes the speed of the object at the instant it loses contact with the surface of the sphere. Note that the constrain condition Eq. (14.8.36) can be rewritten as

$$mgR\cos\theta_1 = mv_1^{\,2}.$$ (14.8.37)

We can now apply conservation of energy. Choose the zero reference point $U = 0$ for potential energy to be the midpoint of the sphere.

Identify the initial state as the instant the object is released (Figure 14.19). We can neglect the very small initial kinetic energy needed to move the object away from the top of the sphere and so $K_i = 0$. The initial potential energy is non-zero, $U_i = mgR$. The initial mechanical energy is then
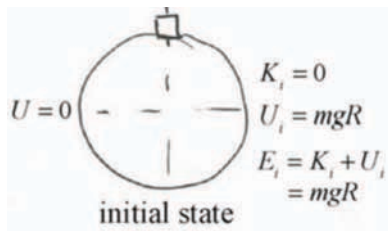
$$E_i = K_i + U_i = mgR.$$ (14.8.38)
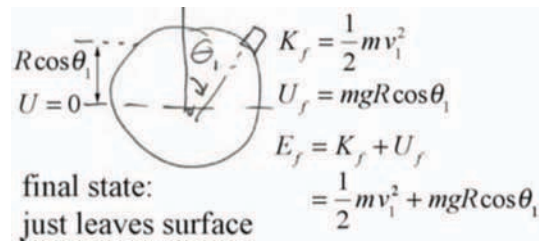

Figure 14.19 Initial state

Figure 14.20 Final state

Choose for the final state the instant the object leaves the sphere (Figure 14.20). The final kinetic energy is $K_f = mv_1^2 / 2$; the object is in motion with speed v_1. The final potential energy is non-zero, $U_f = mgR\cos\theta_1$. The final mechanical energy is then

$$E_f = K_f + U_f = \frac{1}{2}mv_1^2 + mgR\cos\theta_1.$$ (14.8.39)

Because we are assuming the contact surface is frictionless and neglecting air resistance, there is no non-conservative work. The change in mechanical energy is therefore zero,

$$0 = W_{nc} = \Delta E_m = E_f - E_i.$$ (14.8.40)

Therefore

$$\frac{1}{2}mv_1^2 + mgR\cos\theta_1 = mgR.$$ (14.8.41)

We now solve the constraint condition Eq. (14.8.37) into Eq. (14.8.41) yielding

$$\frac{1}{2}mgR\cos\theta_1 + mgR\cos\theta_1 = mgR .$$

(14.8.42)

We can now solve for the angle at which the object just leaves the surface

$$\theta_1 = \cos^{-1}(2/3) .$$

(14.8.43)

We now substitute this result into Eq. (14.8.37) and solve for the speed

$$v_1 = \sqrt{2gR/3} .$$

(14.8.44)

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