12.1 Introduction

So far we have restricted ourselves to considering systems consisting of discrete objects or point-like objects that have fixed amounts of mass. We shall now consider systems in which material flows between the objects in the system, for example we shall consider coal falling from a hopper into a moving railroad car, sand leaking from railroad car fuel, grain moving forward into a railroad car, and fuel ejected from the back of a rocket, In each of these examples material is continuously flows into or out of an object. We have already shown that the total external force causes the momentum of a system to change,

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = \frac{d\vec{\mathbf{p}}_{\text{system}}}{dt}. \qquad (12.2.1)$$

We shall analyze how the momentum of the constituent elements our system change over a time interval $[t, t+\Delta t]$, and then consider the limit as $\Delta t \to 0$. We can then explicit calculate the derivative on the right hand side of Eq. (12.2.1) and Eq. (12.2.1) becomes

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = \frac{d\vec{\mathbf{p}}_{\text{system}}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\mathbf{p}}_{\text{system}}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{p}}_{\text{system}}(t+\Delta t) - \vec{\mathbf{p}}_{\text{system}}(t)}{\Delta t}. \qquad (12.2.2)$$

We need to be very careful how we apply this generalized version of Newton's Second Law to systems in which mass flows between constituent objects. In particular, when we isolate elements as part of our system we must be careful to identify the mass Δm of the material that continuous flows in or out of an object that is part of our system during the time interval Δt under consideration.

We shall consider four categories of mass flow problems that are characterized by the momentum transfer of the material of mass Δm.

12.1.1 Transfer of Material into an Object, but no Transfer of Momentum

Consider for example rain falling vertically downward with speed u into car of mass m moving forward with speed v. A small amount of falling rain Δm_r has no component of momentum in the direction of motion of the car. There is a transfer of rain into the car but no transfer of momentum in the direction of motion of the car (Figure 12.1).

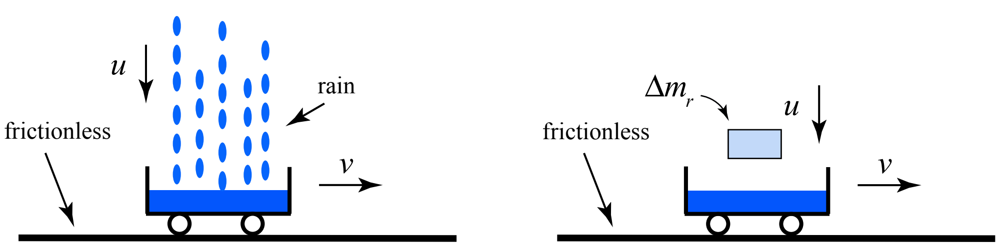


Figure 12.1 Transfer of rain mass into the car but no transfer of momentum in direction of motion

12.1.2 Transfer of Material Out of an Object, but no Transfer of Momentum

The material continually leaves the object but it does not transport any momentum away from the object in the direction of motion of the object (Figure 12.2). Consider an ice skater gliding on ice at speed v holding a bag of sand that is leaking straight down with respect to the moving skater. The sand continually leaves the bag but it does not transport any momentum away from the bag in the direction of motion of the object. In Figure 12.2, sand of mass Δm_s leaves the bag.

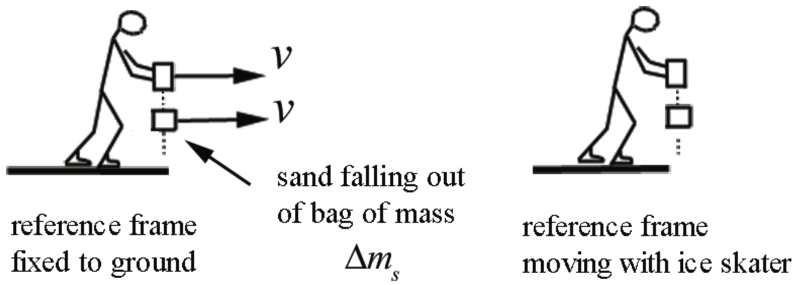


Figure 12.2 Transfer of mass out of object but no transfer of momentum in direction of motion

12.1.3 Transfer of Material Impulses Object Via Transfer of Momentum

Suppose a fire hose is used to put out a fire on a boat of mass m_b. Assume the column of water moves horizontally with speed u. The incoming water continually hits the boat propelling it forward. During the time interval Δt, a column of water of mass Δm_s will hit the boat that is moving forward with speed v increasing it's speed (Figure 12.3).

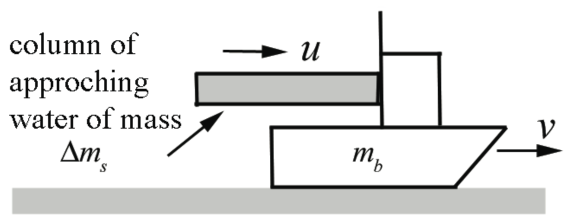
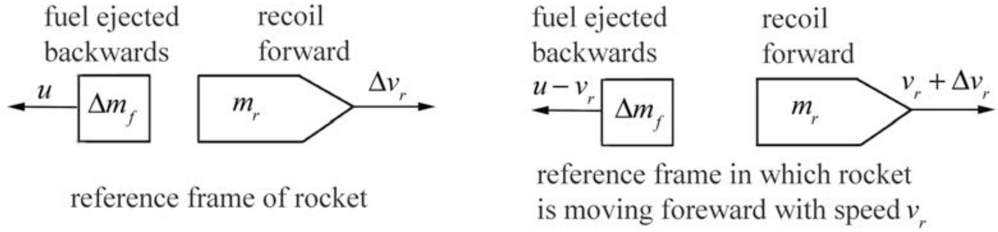


Figure 12.3 Transfer of mass of water increases speed of boat

12.1.4 Material Continually Ejected From Object results in Recoil of Object

When fuel of mass Δm_f is ejected from the back of a rocket with speed u relative to the rocket, the rocket of mass m_r recoils forward. Figure 12.4a shows the recoil of the rocket in the reference frame of the rocket. The rocket recoils forward with speed Δv_r. In a reference frame in which the rocket is moving forward with speed v_r, then the speed after recoil is $v_r + \Delta v_r$. The speed of the backwardly ejected fuel is $u - v_r$ (Figure 12.4b).



(a) (b)

Figure 12.4 Transfer of mass out of rocket provides impulse on rocket in (a) reference frame of rocket, (b) reference frame in which rocket moves with speed v_r

We must carefully identify the momentum of the object and the material transferred at time t in order to determine $\vec{\mathbf{p}}_{\text{system}}(t)$. We must also identify the momentum of the object and the material transferred at time $t+\Delta t$ in order to determine $\vec{\mathbf{p}}_{\text{system}}(t+\Delta t)$ as well. Recall that when we defined the momentum of a system, we assumed that the mass of the system remain constant. Therefore we cannot ignore the momentum of the transferred material at time $t+\Delta t$ even though it may have left the object; it is still part of our system (or at time t even though it has not flowed into the object yet).

12.2 Worked Examples

Example 12.1 Filling a Coal Car

An empty coal car of mass m_0 starts from rest under an applied force of magnitude F. At the same time coal begins to run into the car at a steady rate b from a coal hopper at rest along the track (Figure 12.5). Find the speed when a mass m_c of coal has been transferred.

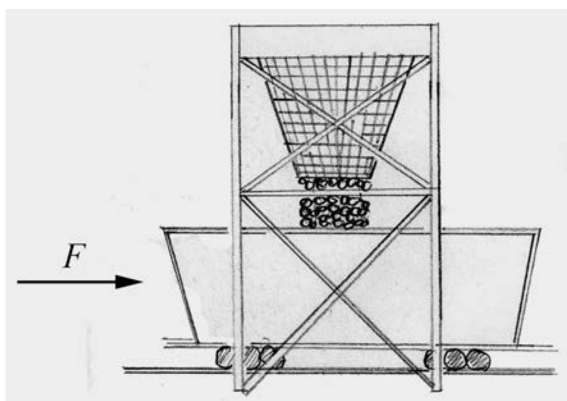


Figure 12.5 Filling a coal car

Solution: We shall analyze the momentum changes in the horizontal direction, which we call the x-direction. Because the falling coal does not have any horizontal velocity, the falling coal is not transferring any momentum in the x-direction to the coal car. So we shall take as our system the empty coal car and a mass m_c of coal that has been transferred. Our initial state at $t=0$ is when the coal car is empty and at rest before any coal has been transferred. The x-component of the momentum of this initial state is zero,

$$p_x(0) = 0. \qquad (12.3.1)$$

Our final state at $t = t_f$ is when all the coal of mass $m_c = bt_f$ has been transferred into the car that is now moving at speed v_f. The x-component of the momentum of this final state is

$$p_x(t_f) = (m_0 + m_c)v_f = (m_0 + bt_f)v_f . \quad (12.3.2)$$

There is an external constant force $F_x = F$ applied through the transfer. The momentum principle applied to the x-direction is

$$\int_0^{t_f} F_x dt = \Delta p_x = p_x(t_f) - p_x(0) . \quad (12.3.3)$$

Because the force is constant, the integral is simple and the momentum principle becomes

$$Ft_f = (m_0 + bt_f)v_f . \quad (12.3.4)$$

So the final speed is

$$v_f = \frac{Ft_f}{(m_0 + bt_f)} . \quad (12.3.5)$$

Example 12.2 Emptying a Freight Car

A freight car of mass m_c contains sand of mass m_s. At $t = 0$ a constant horizontal force of magnitude F is applied in the direction of rolling and at the same time a port in the bottom is opened to let the sand flow out at the constant rate $b = dm_s / dt$. Find the speed of the freight car when all the sand is gone (Figure 12.6). Assume that the freight car is at rest at $t = 0$.

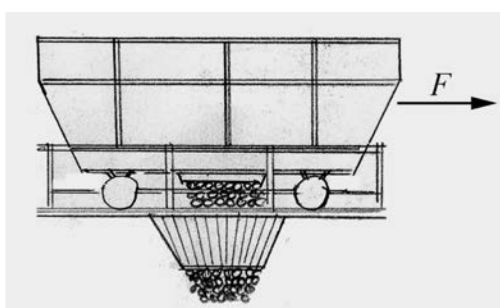


Figure 12.6 Emptying a freight car

Solution: Choose the positive x-direction to point in the direction that the car is moving. Choose for the system the amount of sand in the fright car at time t, $m_c(t)$. At time t, the car is moving with velocity $\vec{\mathbf{v}}_c(t) = v_c(t)\hat{\mathbf{i}}$. The momentum diagram for the system at time t is shown in the diagram on the left in Figure 12.7.

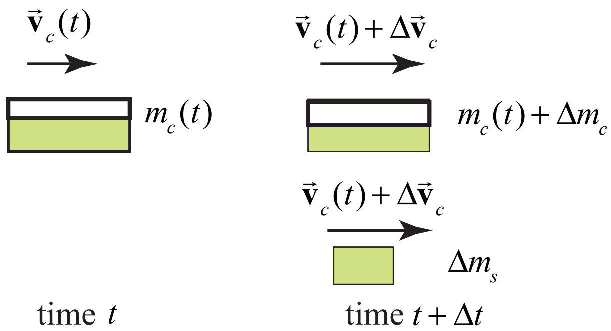


Figure 12.7 Momentum diagram at time t and at time $t + \Delta t$

The momentum of the system at time t is given by

$$\vec{\mathbf{p}}_{sys}(t) = m_c(t)\vec{\mathbf{v}}_c(t). \tag{12.3.6}$$

During the time interval $[t, t + \Delta t]$, an amount of sand of mass Δm_s leaves the freight car and the mass of the freight car changes by $m_c(t + \Delta t) = m_c(t) + \Delta m_c$, where $\Delta m_c = -\Delta m_s$. At the end of the interval the car is moving with velocity $\vec{\mathbf{v}}_c(t + \Delta t) = \vec{\mathbf{v}}_c(t) + \Delta\vec{\mathbf{v}}_c = (v_c(t) + \Delta v_c)\hat{\mathbf{i}}$. The momentum diagram for the system at time $t + \Delta t$ is shown in the diagram on the right in Figure 12.7. The momentum of the system at time $t + \Delta t$ is given by

$$\vec{\mathbf{p}}_{sys}(t + \Delta t) = (\Delta m_s + m_c(t) + \Delta m_c)(\vec{\mathbf{v}}_c(t) + \Delta\vec{\mathbf{v}}_c) = m_c(t)(\vec{\mathbf{v}}_c(t) + \Delta\vec{\mathbf{v}}_c). \tag{12.3.7}$$

Note that the sand that leaves the car is shown with velocity $\vec{\mathbf{v}}_c(t) + \Delta\vec{\mathbf{v}}_c$. This implies that all the sand leaves the car with the velocity of the car at the end of the interval. This is an approximation. Because the sand leaves continuous, the velocity will vary from $\vec{\mathbf{v}}_c(t)$ to $\vec{\mathbf{v}}_c(t) + \Delta\vec{\mathbf{v}}_c$ but so does the change in mass of the car and these two contributions to the system's moment exactly cancel. The change in momentum of the system is then

$$\Delta\vec{\mathbf{p}}_{sys} = \vec{\mathbf{p}}_{sys}(t + \Delta t) - \vec{\mathbf{p}}_{sys}(t) = m_c(t)(\vec{\mathbf{v}}_c(t) + \Delta\vec{\mathbf{v}}_c) - m_c(t)\vec{\mathbf{v}}_c(t) = m_c(t)\Delta\vec{\mathbf{v}}_c. \tag{12.3.8}$$

Throughout the interval a constant force $\vec{\mathbf{F}} = F\hat{\mathbf{i}}$ is applied to the system so the momentum principle becomes

$$\vec{\mathbf{F}} = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{p}}_{sys}(t+\Delta t) - \vec{\mathbf{p}}_{sys}(t)}{\Delta t} = \lim_{\Delta t \to 0} m_c(t) \frac{\Delta \vec{\mathbf{v}}_c}{\Delta t} = m_c(t) \frac{d\vec{\mathbf{v}}_c}{dt} . \qquad (12.3.9)$$

Because the motion is one-dimensional, Eq. (12.3.9) written in terms of x-components becomes

$$F = m_c(t) \frac{dv_c}{dt} . \qquad (12.3.10)$$

Denote by initial mass of the car by $m_{c,0} = m_c + m_s$ where m_c is the mass of the car and m_s is the mass of the sand in the car at $t = 0$. The mass of the sand that has left the car at time t is given by

$$m_s(t) = \int_0^t \frac{dm_s}{dt} dt = \int_0^t b dt = bt . \qquad (12.3.11)$$

Thus

$$m_c(t) = m_{c,0} - bt = m_c + m_s - bt . \qquad (12.3.12)$$

Therefore Eq. (12.3.10) becomes

$$F = (m_c + m_s - bt) \frac{dv_c}{dt} . \qquad (12.3.13)$$

This equation can be solved for the x-component of the velocity at time t, $v_c(t)$, (which in this case is the speed) by the method of separation of variables. Rewrite Eq. (12.3.13) as

$$dv_c = \frac{Fdt}{(m_c + m_s - bt)} . \qquad (12.3.14)$$

Then integrate both sides of Eq. (12.3.14) with the limits as shown

$$\int_{v'_c=0}^{v'_c=v_c(t)} dv'_c = \int_{t'=0}^{t'=t} \frac{Fdt'}{m_c + m_s - bt'} . \qquad (12.3.15)$$

Integration yields the speed of the car as a function of time

$$v_c(t) = -\frac{F}{b} \ln(m_c + m_s - bt')\Big|_{t'=0}^{t'=t} = -\frac{F}{b} \ln\left(\frac{m_c + m_s - bt}{m_c + m_s}\right) = \frac{F}{b} \ln\left(\frac{m_c + m_s}{m_c + m_s - bt}\right) . \qquad (12.3.16)$$

In writing Eq. (12.3.16), we used the property that $\ln(a) - \ln(b) = \ln(a/b)$ and therefore $\ln(a/b) = -\ln(b/a)$. Note that $m_c + m_s \geq m_c + m_s - bt$, so the term $\ln\left(\dfrac{m_c + m_s}{m_c + m_s - bt}\right) \geq 0$, and the speed of the car increases as we expect.

Example 12.3 Filling a Freight Car

Grain is blown into car A from car B at a rate of b kilograms per second. The grain leaves the chute vertically downward, so that it has the same horizontal velocity, u as car B, (Figure 12.8). Car A is initially at rest before any grain is transferred in and has mass $m_{A,0}$. At the moment of interest, car A has mass m_A and speed v. Determine an expression for the speed car A as a function of time t.

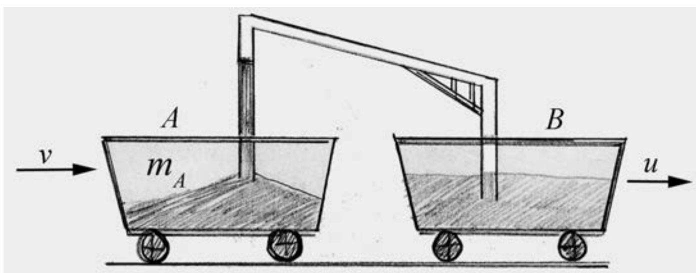


Figure 12.8 Filling a freight car

Solution: Choose positive x-direction to the right in the direction the cars are moving. Define the system at time t to be the car and grain that is already in it, which together has mass $m_A(t)$, and the small amount of material of mass Δm_g that is blown into car A during the time interval $[t, t + \Delta t]$. At time that is moving with x-component of the velocity v_A. At time t, car A is moving with velocity $\vec{\mathbf{v}}_A(t) = v_A(t)\hat{\mathbf{i}}$, and the material blown into car is moving with velocity $\vec{\mathbf{u}} = u\hat{\mathbf{i}}$ At time $t + \Delta t$, car A is moving with velocity $\vec{\mathbf{v}}_A(t) + \Delta\vec{\mathbf{v}}_A = (v_A(t) + \Delta v_A)\hat{\mathbf{i}}$, and the mass of car A is $m_A(t + \Delta t) = m_A(t) + \Delta m_A$, where $\Delta m_A = \Delta m_g$. The momentum diagram for times t and for $t + \Delta t$ is shown in Figure 12.9.

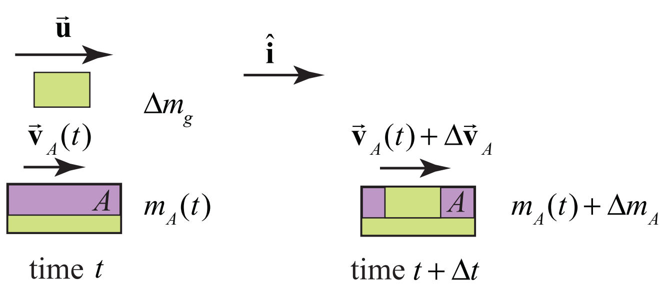


Figure 12.9 Momentum diagram at times t and $t + \Delta t$

The momentum at time t is

$$\vec{\mathbf{P}}_{sys}(t) = m_A(t)\vec{\mathbf{v}}_A(t) + \Delta m_g \vec{\mathbf{u}} \; . \tag{12.3.17}$$

The momentum at time $t + \Delta t$ is

$$\vec{\mathbf{P}}_{sys}(t+\Delta t) = (m_A(t) + \Delta m_A)(\vec{\mathbf{v}}_A(t) + \Delta \vec{\mathbf{v}}_A) \; . \tag{12.3.18}$$

There are no external forces acting on the system in the x-direction and the external forces acting on the system perpendicular to the motion sum to zero, so the momentum principle becomes

$$\vec{\mathbf{0}} = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{P}}_{sys}(t+\Delta t) - \vec{\mathbf{P}}_{sys}(t)}{\Delta t} \; . \tag{12.3.19}$$

Using the results above (Eqs. (12.3.17) and (12.3.18), the momentum principle becomes

$$\vec{\mathbf{0}} = \lim_{\Delta t \to 0} \frac{(m_A(t) + \Delta m_A)(\vec{\mathbf{v}}_A(t) + \Delta \vec{\mathbf{v}}_A) - (m_A(t)\vec{\mathbf{v}}_A(t) + \Delta m_g \vec{\mathbf{u}})}{\Delta t} \; . \tag{12.3.20}$$

which after using the condition that $\Delta m_A = \Delta m_g$ and some rearrangement becomes

$$\vec{\mathbf{0}} = \lim_{\Delta t \to 0} \frac{m_A(t)\Delta \vec{\mathbf{v}}_A}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m_A(\vec{\mathbf{v}}_A(t) - \vec{\mathbf{u}})}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m_A \Delta \vec{\mathbf{v}}_A}{\Delta t} \; . \tag{12.3.21}$$

In the limit as , the product $\Delta m_A \Delta \vec{\mathbf{v}}_A$ is a second order differential (the product of two first order differentials) and the term $\Delta m_A \Delta \vec{\mathbf{v}}_A / \Delta t$ approaches zero, therefore the momentum principle yields the differential equation

$$\vec{\mathbf{0}} = m_A(t)\frac{d\vec{\mathbf{v}}_A}{dt} + \frac{dm_A}{dt}(\vec{\mathbf{v}}_A(t) - \vec{\mathbf{u}}) \; . \tag{12.3.22}$$

The x-component of Eq. (12.3.22) is then

$$0 = m_A(t)\frac{dv_A}{dt} + \frac{dm_A}{dt}(v_A(t) - u) \; . \tag{12.3.23}$$

Rearranging terms and using the fact that the material is blown into car A at a constant rate $b \equiv dm_A / dt$, we have that the rate of change of the x-component of the velocity of car A is given by

$$\frac{dv_A(t)}{dt} = \frac{b(u - v_A(t))}{m_A(t)} \; . \tag{12.3.24}$$

We cannot directly integrate Eq. (12.3.24) with respect to dt because the mass of the car A is a function of time. In order to find the x-component of the velocity of car A we need to know the relationship between the mass of car A and the x-component of the velocity of the car A. There are two approaches. In the first approach we separate variables in Eq. (12.3.24) where we have suppressed the dependence on t in the expressions for m_A and v_A yielding

$$\frac{dv_A}{u - v_A} = \frac{dm_A}{m_A}, \tag{12.3.25}$$

which becomes the integral equation

$$\int_{v'_A=0}^{v'_A=v_A(t)} \frac{dv'_A}{u - v'_A} = \int_{m'_A=m_{A,0}}^{m'_A=m_A(t)} \frac{dm'_A}{m'_A}, \tag{12.3.26}$$

where $m_{A,0}$ is the mass of the car before any material has been blown in. After integration we have that

$$\ln\frac{u}{u - v_A(t)} = \ln\frac{m_A(t)}{m_{A,0}}. \tag{12.3.27}$$

Exponentiate both side yields

$$\frac{u}{u - v_A(t)} = \frac{m_A(t)}{m_{A,0}}. \tag{12.3.28}$$

We can solve this equation for the x-component of the velocity of the car

$$v_A(t) = \frac{m_A(t) - m_{A,0}}{m_A(t)}u. \tag{12.3.29}$$

Because the material is blown into the car at a constant rate $b \equiv dm_A / dt$, the mass of the car as a function of time is given by

$$m_A(t) = m_{A,0} + bt. \tag{12.3.30}$$

Therefore substituting Eq. (12.3.30) into Eq. (12.3.29) yields the x-component of the velocity of the car as a function of time

$$v_A(t) = \frac{bt}{m_{A,0} + bt}u. \tag{12.3.31}$$

In a second approach, we substitute Eq. (12.3.30) into Eq. (12.3.24) yielding

$$\frac{dv_A}{dt} = \frac{b(u - v_A)}{m_{A,0} + bt} . \tag{12.3.32}$$

Separate variables in Eq. (12.3.32):

$$\frac{dv_A}{u - v_A} = \frac{bdt}{m_{A,0} + bt} , \tag{12.3.33}$$

which then becomes the integral equation

$$\int_{v_A'=0}^{v_A'=v_A(t)} \frac{dv_A'}{u - v_A'} = \int_{t'=0}^{t'=t} \frac{dt'}{m_{A,0} + bt'} . \tag{12.3.34}$$

Integration yields

$$\ln \frac{u}{u - v_A(t)} = \ln \frac{m_{A,0} + bt}{m_{A,0}} . \tag{12.3.35}$$

Again exponentiate both sides resulting in

$$\frac{u}{u - v_A(t)} = \frac{m_{A,0} + bt}{m_{A,0}} . \tag{12.3.36}$$

After some algebraic manipulation we can find the speed of the car as a function of time

$$v_A(t) = \frac{bt}{m_{A,0} + bt} u . \tag{12.3.37}$$

in agreement with Eq. (12.3.31).

Check result:

We can rewrite Eq. (12.3.37) as

$$(m_{A,0} + bt) v_A(t) = btu , \tag{12.3.38}$$

which illustrates the point that the momentum of the system at time t is equal to the momentum of the grain that has been transferred to the system during the interval $[0,t]$.

Example 12.4 Boat and Fire Hose

A burning boat of mass m_0 is initially at rest. A fire fighter stands on a bridge and sprays water onto the boat. The water leaves the fire hose with a speed u at a rate α (measured in $\text{kg} \cdot \text{s}^{-1}$). Assume that the motion of the boat and the water jet are horizontal, that gravity does not play any role, and that the river can be treated as a frictionless surface. Also assume that the change in the mass of the boat is only due to the water jet and that all the water from the jet is added to the boat, (Figure 12.10).

Figure 12.10 Example 12.4

a) In a time interval $[t, t+\Delta t]$, an amount of water Δm hits the boat. Choose a system. Is the total momentum constant in your system? Write down a differential equation that results from the analysis of the momentum changes inside your system.

b) Integrate the differential equation you found in part a), to find the velocity $v(m)$ as a function of the increasing mass m of the boat, m_0, and u.

Solution: Let's take as our system the boat, the amount of water of mass Δm_w that enters the boat during the time interval $[t, t+\Delta t]$ and whatever water is in the boat at time t. The water from the fire hose has a speed u. Denote the mass of the boat (including some water) at time t by $m_b \equiv m_b(t)$, and the speed of the boat by $v \equiv v_b(t)$. At time $t+\Delta t$ the speed of the boat is $v+\Delta v$. Choose the positive x- direction in the direction that the boat is moving. Then the x-components of the momentum of the system at time t and $t+\Delta t$ are shown in Figure 12.11.

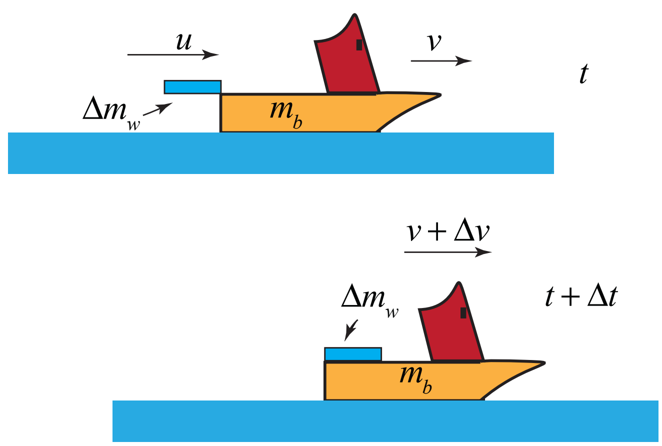


Figure 12.11 Momentum diagrams for burning boat

Because we are assuming that the burning boat slides with negligible resistance and that gravity has a negligible effect on the arc of the water jet, there are no external forces acting on the system in the x-direction. Therefore the x-component of the momentum of the system is constant during the interval $[t, t+\Delta t]$ and so

$$0 = \lim_{\Delta t \to 0} \frac{p_x(t+\Delta t) - p_x(t)}{\Delta t}. \quad (12.3.39)$$

Using the information from the figure above, Eq. (12.3.39) becomes

$$0 = \lim_{\Delta t \to 0} \frac{(m_b + \Delta m_w)(v + \Delta v) - (\Delta m_w u + m_b v)}{\Delta t}. \quad (12.3.40)$$

Eq. (12.3.40) simplifies to

$$0 = \lim_{\Delta t \to 0} m_b \frac{\Delta v}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m_w}{\Delta t} v + \lim_{\Delta t \to 0} \frac{\Delta m_w \Delta v}{\Delta t} - \lim_{\Delta t \to 0} \frac{\Delta m_w}{\Delta t} u. \quad (12.3.41)$$

The third term vanishes when we take the limit $\Delta t \to 0$ because it is of second order in the infinitesimal quantities (in this case $\Delta m_w \Delta v$) and so when dividing by Δt the quantity is of first order and hence vanishes since both $\Delta m_w \to 0$ and $\Delta v \to 0$. Eq. (12.3.41) becomes

$$0 = \lim_{\Delta t \to 0} m_b \frac{\Delta v}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m_w}{\Delta t} v - \lim_{\Delta t \to 0} \frac{\Delta m_w}{\Delta t} u. \quad (12.3.42)$$

We now use the definition of the derivatives:

$$\lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}; \ \lim_{\Delta t \to 0} \frac{\Delta m_w}{\Delta t} = \frac{dm_w}{dt}. \quad (12.3.43)$$

in Eq. (12.3.42) to fund the differential equation describing the relation between the acceleration of the boat and the time rate of change of the mass of water entering the boat

$$0 = m_b \frac{dv}{dt} + \frac{dm_w}{dt}(v - u). \quad (12.3.44)$$

The mass of the boat is increasing due to the addition of the water. Let $m_w(t)$ denote the mass of the water that is in the boat at time t. Then the mass of the boat can be written as

$$m_b(t) = m_0 + m_w(t), \quad (12.3.45)$$

where m_0 is the mass of the boat before any water entered. Note we are neglecting the effect of the fire on the mass of the boat. Differentiating Eq. (12.3.45) with respect to time yields

$$\frac{dm_b}{dt} = \frac{dm_w}{dt}, \quad (12.3.46)$$

Then Eq. (12.3.44) becomes

$$0 = m_b \frac{dv}{dt} + \frac{dm_b}{dt}(v - u). \quad (12.3.47)$$

(b) We can integrate this equation through the separation of variable technique. Rewrite Eq. (12.3.47) as (cancel the common factor dt)

$$\frac{dv}{v-u} = -\frac{dm_b}{m_b}. \tag{12.3.48}$$

We can then integrate both sides of Eq. (12.3.48) with the limits as shown

$$\int_{v=0}^{v(t)} \frac{dv}{v-u} = -\int_{m_0}^{m_b(t)} \frac{dm_b}{m_b} \tag{12.3.49}$$

Integration yields

$$\ln\left(\frac{v(t)-u}{-u}\right) = -\ln\left(\frac{m_b(t)}{m_0}\right) \tag{12.3.50}$$

Recall that $\ln(a/b) = -\ln(b/a)$ so Eq. (12.3.50) becomes

$$\ln\left(\frac{v(t)-u}{-u}\right) = \ln\left(\frac{m_0}{m_b(t)}\right) \tag{12.3.51}$$

Also recall that $\exp(\ln(a/b)) = a/b$ and so exponentiating both sides of Eq. (12.3.51) yields

$$\frac{v(t)-u}{-u} = \frac{m_0}{m_b(t)} \tag{12.3.52}$$

So the speed of the boat at time t can be expressed as

$$v(t) = u\left(1 - \frac{m_0}{m_b(t)}\right) \tag{12.3.53}$$

Check result:

We can rewrite Eq. (12.3.52) as

$$m_b(t)(v(t)-u) = -m_0 u \Rightarrow m_b(t)v(t) = (m_b(t) - m_0)u \ . \tag{12.3.54}$$

Recall that the mass of the water that enters the car during the interval $[0,t]$ is $m_w(t) = m_b(t) - m_0$. Therefore Eq. (12.3.54) becomes

$$m_b(t)v(t) = m_w(t)u \ . \tag{12.3.55}$$

During the interaction between the jet of water and the boat, the water transfers an amount of momentum $m_w(t)u$ to the boat and car producing a momentum $m_b(t)v(t)$. Because all the water that collides with the boat ends up in the boat, all the interaction forces between the jet of water and the boat are internal forces. The boat recoils forward and the water recoils backward and through collisions with the boat stays in the boat. Therefore if we choose as our system, all of the water that eventually ends up in the boat and the boat then the momentum principle states

$$p_{sys}(t) = p_{sys}(0) \ , \tag{12.3.56}$$

where $p_{sys}(0) = m_w(t)u$ is the momentum of all of the water that eventually ends up in the boat.

Note that the problem didn't ask to find the speed of the boat as a function t. We shall now show how to find that. We begin by observing that

$$\frac{dm_b}{dt} = \frac{dm_w}{dt} \neq \alpha \tag{12.3.57}$$

where the constant α is measured in $\text{kg}\cdot\text{s}^{-1}$ and is specified as a given constant according to the information in the problem statement. The reason is that α is the rate that the water is ejected from the hose but not the rate that the water enters the boat.

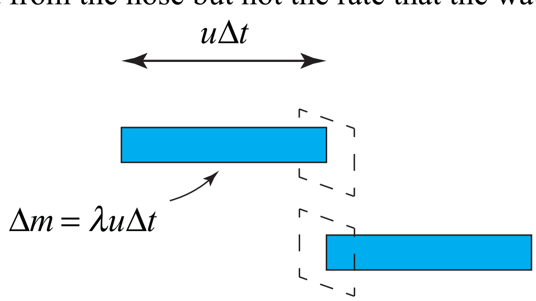


Figure 12.12 Mass per unit length of water jet

Consider a small amount of water that is moving with speed u that, in a time interval Δt, flows through a cross sectional area oriented perpendicular to the flow (see Figure 12.12). The area is larger than the cross sectional area of the jet of water. The amount of water that floes through the area element $\Delta m = \lambda u \Delta t$, where λ is the mass per unit length of the jet and $u\Delta t$ is the length of the jet that flows through the area in the interval Δt. The mass rate of water that flows through the cross sectional area element is then

$$\alpha = \frac{\Delta m}{\Delta t} = \lambda u \ . \tag{12.3.58}$$

In the Figure 12.13 we consider a small length $u\Delta t$ of the water jet that is just behind the boat at time t. During the time interval $[t, t+\Delta t]$, the boat moves a distance $v\Delta t$.

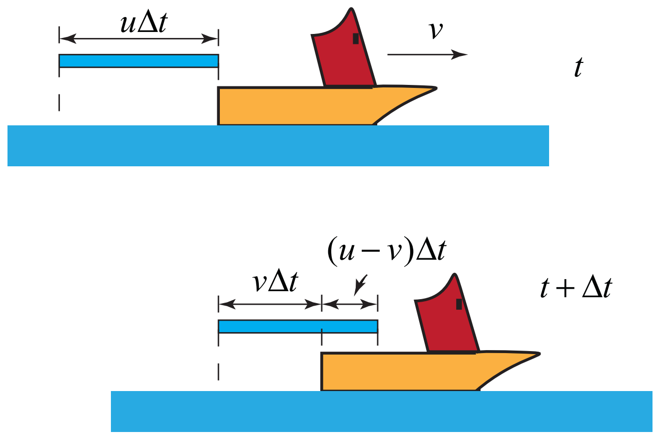


Figure 12.13 Amount of water that enter boat in time interval $[t, t+\Delta t]$

Only a fraction of the length $u\Delta t$ of water enters the boat and is given by

$$\Delta m_w = \lambda(u-v)\Delta t = \frac{\alpha}{u}(u-v)\Delta t \qquad (12.3.59)$$

Dividing Eq. (12.3.59) through by Δt and taking limits we have that

$$\frac{dm_w}{dt} = \lim_{\Delta t \to 0} \frac{\Delta m_w}{\Delta t} = \frac{\alpha}{u}(u-v) = \alpha(1-\frac{v}{u}). \qquad (12.3.60)$$

Substituting Eq. (12.3.53) and Eq. (12.3.46) into Eq. (12.3.60) yields

$$\frac{dm_b}{dt} = \alpha(1-\frac{v}{u}) = \alpha\frac{m_0}{m_b(t)}. \qquad (12.3.61)$$

We can integrate this equation by separating variables to find an integral expression for the mass of the boat as a function of time

$$\int_{m_0}^{m_b(t)} m_b\, dm_b = \alpha m_0 \int_{t=0}^{t} dt\,. \qquad (12.3.62)$$

We can easily integrate both sides of Eq. (12.3.62) yielding

$$\frac{1}{2}\left(m_b(t)^2 - m_0{}^2\right) = \alpha m_{b,0} t \,. \tag{12.3.63}$$

The mass of the boat as a function of time is then

$$m_b(t) = m_0\sqrt{1 + 2\frac{\alpha t}{m_0}} \,. \tag{12.3.64}$$

We now substitute Eq. (12.3.64) into Eq. (12.3.65)yielding the speed of the burning boat as a function of time

$$v(t) = u\left(1 - \frac{1}{\sqrt{1 + 2\dfrac{\alpha t}{m_{b,0}}}}\right) \tag{12.3.66}$$

12.3 Rocket Propulsion

A rocket at time $t = t_i$ is moving with velocity $\vec{\mathbf{v}}_{r,i}$ with respect to a fixed reference frame. During the time interval $[t_i, t_f]$ the rocket continuously burns fuel that is continuously ejected backwards with velocity $\vec{\mathbf{u}}$ relative to the rocket. This exhaust velocity is independent of the velocity of the rocket. The rocket must exert a force to accelerate the ejected fuel backwards and therefore by Newton's Third law, the fuel exerts a force that is equal in magnitude but opposite in direction accelerating the rocket forward. The rocket velocity is a function of time, $\vec{\mathbf{v}}_r(t)$. Because fuel is leaving the rocket, the mass of the rocket is also a function of time, $m_r(t)$, and is decreasing at a rate dm_r / dt. Let $\vec{\mathbf{F}}_{ext}$ denote the total external force acting on the rocket. We shall use the momentum principle, to determine a differential equation that relates $d\vec{\mathbf{v}}_r / dt$, dm_r / dt, $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}_r(t)$, and $\vec{\mathbf{F}}_{ext}$, an equation known as the rocket equation.

We shall apply the momentum principle during the time interval $[t, t + \Delta t]$ with Δt taken to be a small interval (we shall eventually consider the limit that $\Delta t \to 0$), and $t_i < t < t_f$. During this interval, choose as our system the mass of the rocket at time t,

$$m_{sys} = m_r(t) = m_{r,d} + m_f(t) \,, \tag{12.3.67}$$

where $m_{r,d}$ is the dry mass of the rocket and $m_f(t)$ is the mass of the fuel in the rocket at time t. During the time interval $[t, t + \Delta t]$, a small amount of fuel of mass Δm_f (in the limit that $\Delta t \to 0$, $\Delta m_f \to 0$) is ejected backwards with velocity $\vec{\mathbf{u}}$ to the rocket. Before the fuel is ejected, it is traveling at the velocity of the rocket and so during the time interval $[t, t+\Delta t]$, the elected fuel undergoes a change in momentum and the rocket recoils forward. At time $t+\Delta t$ the rocket has velocity $\vec{\mathbf{v}}_r(t+\Delta t)$. Although the ejected fuel continually changes its velocity, we shall assume that the fuel is all ejected at the instant $t+\Delta t$ and then consider the limit as $\Delta t \to 0$. Therefore the velocity of the ejected fuel with respect to the fixed reference frame is the vector sum of the relative velocity of the fuel with respect to the rocket and the velocity of the rocket, $\vec{\mathbf{u}} + \vec{\mathbf{v}}_r(t+\Delta t)$. Figure 12.14 represents momentum diagrams for our system at time t and $t+\Delta t$ relative to a fixed inertial reference frame in which velocity of the rocket at time t is $\vec{\mathbf{v}}_r(t)$.

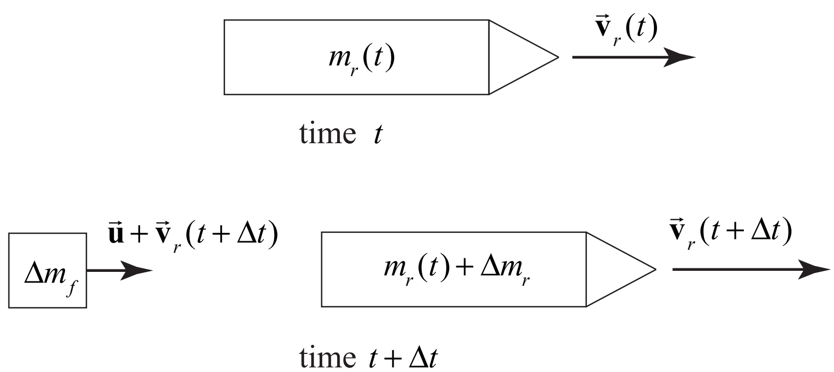


Figure 12.14 Momentum diagrams for system at time t and $t+\Delta t$

The momentum of the system at time t is

$$\vec{\mathbf{p}}_{sys}(t) = m_r(t)\vec{\mathbf{v}}_r(t). \tag{12.3.68}$$

Note that the mass of the system at time t is

$$m_{sys} = m_r(t). \tag{12.3.69}$$

The momentum of the system at time $t+\Delta t$ is

$$\vec{\mathbf{p}}_{sys}(t+\Delta t) = m_r(t+\Delta t)\vec{\mathbf{v}}_r(t+\Delta t) + \Delta m_f(\vec{\mathbf{u}} + \vec{\mathbf{v}}_r(t+\Delta t)), \tag{12.3.70}$$

where $m_r(t+\Delta t) = m_r(t) + \Delta m_r$. With this notation the mass of the system at time $t+\Delta t$ is given by

$$m_{sys} = m_r(t+\Delta t) + \Delta m_f = m_r(t) + \Delta m_r + \Delta m_f. \tag{12.3.71}$$

Because the mass of the system is constant, setting Eq. (12.3.69) equal to Eq. (12.3.71) requires that

$$\Delta m_r = -\Delta m_f\,. \tag{12.3.72}$$

The momentum of the system at time $t+\Delta t$ (Eq. (12.3.70)) can be rewritten as

$$\begin{aligned}\vec{\mathbf{p}}_{sys}(t+\Delta t) &= (m_r(t)+\Delta m_r)\vec{\mathbf{v}}_r(t+\Delta t) - \Delta m_r(\vec{\mathbf{u}}+\vec{\mathbf{v}}_r(t+\Delta t))\\ \vec{\mathbf{p}}_{sys}(t+\Delta t) &= m_r(t)\vec{\mathbf{v}}_r(t+\Delta t) - \Delta m_r\vec{\mathbf{u}}\end{aligned}\,, \tag{12.3.73}$$

We can now apply Newton's Second Law in the form of the momentum principle,

$$\begin{aligned}\vec{\mathbf{F}}_{ext} &= \lim_{\Delta t\to 0}\frac{(m_r(t)\vec{\mathbf{v}}_r(t+\Delta t)-\Delta m_r\vec{\mathbf{u}}) - m_r(t)\vec{\mathbf{v}}_r(t)}{\Delta t}\\ &= m_r(t)\lim_{\Delta t\to 0}\frac{\vec{\mathbf{v}}_r(t+\Delta t)-\vec{\mathbf{v}}_r(t)}{\Delta t} - \lim_{\Delta t\to 0}\frac{\Delta m_r}{\Delta t}\vec{\mathbf{u}}\end{aligned}\,.. \tag{12.3.74}$$

We now take the limit as

$$\vec{\mathbf{F}}_{ext} = m_r(t)\frac{d\vec{\mathbf{v}}_r}{dt} - \frac{dm_r}{dt}\vec{\mathbf{u}}\,. \tag{12.3.75}$$

Eq. (12.3.75) is known as the ***rocket equation***.

Suppose the rocket is moving in the positive x-direction with an external force given by $\vec{\mathbf{F}}_{ext} = F_{ext,x}\hat{\mathbf{i}}$ Then $\vec{\mathbf{u}} = -u\hat{\mathbf{i}}$, where $u>0$ is the relative speed of the fuel and it is moving in the negative x-direction, $\vec{\mathbf{v}}_r = v_{r,x}\hat{\mathbf{i}}$. Then the rocket equation (Eq. (12.3.75)) becomes

$$F_{ext,x} = m_r(t)\frac{dv_{r,x}}{dt} + \frac{dm_r}{dt}u\,. \tag{12.3.76}$$

Note that the rate of decrease of the mass of the rocket, dm_r/dt, is equal to the negative of the rate of increase of the exhaust fuel

$$\frac{dm_r}{dt} = -\frac{dm_f}{dt}\,. \tag{12.3.77}$$

We can rewrite Eq. (12.3.76) as

$$F_{ext,x} - \frac{dm_r}{dt}u = m_r(t)\frac{dv_{r,x}}{dt}\,. \tag{12.3.78}$$

The second term on the left-hand-side of Eq. (12.3.78) is called the ***thrust***

$$F_{thrust,x} = -\frac{dm_r}{dt}u = \frac{dm_f}{dt}u\,. \tag{12.3.79}$$

Note that this is not an extra force but the result of the forward recoil due to the ejection of the fuel. Because we are burning fuel at a positive rate $dm_f / dt > 0$ and the speed $u > 0$, the direction of the thrust is in the positive x-direction.

12.3.1 Rocket Equation in Gravity-free Space

We shall first consider the case in which there are no external forces acting on the system, then Eq. (12.3.78) becomes

$$-\frac{dm_r}{dt}u = m_r(t)\frac{dv_{r,x}}{dt}\,. \tag{12.3.80}$$

In order to solve this equation, we separate the variable quantities $v_{r,x}(t)$ and $m_r(t)$ and multiply both sides by dt yielding

$$dv_{r,x} = -u\frac{dm_r}{m_r(t)}\,. \tag{12.3.81}$$

We now integrate both sides of Eq. (12.3.81) with limits corresponding to the values of the x-component of the velocity and mass of the rocket at times t_i when the ejection of the burned fuel began and the time t_f when the process stopped,

$$\int_{v'_{r,x}=v_{r,x,i}}^{v'_{r,x}=v_{r,x,f}} dv'_{r,x} = -\int_{m'_r=m_{r,i}}^{m'_r=m_{r,f}} \frac{u}{m'_r}dm'_r\,. \tag{12.3.82}$$

Performing the integration and substituting in the values at the endpoints yields

$$v_{r,x,f} - v_{r,x,i} = -u\ln\left(\frac{m_{r,f}}{m_{r,i}}\right). \tag{12.3.83}$$

Because the rocket is losing fuel, $m_{r,f} < m_{r,i}$, we can rewrite Eq. (12.3.83) as

$$v_{r,x,f} - v_{r,x,i} = u\ln\left(\frac{m_{r,i}}{m_{r,f}}\right). \tag{12.3.84}$$

We note $\ln(m_{r,i} / m_{r,f}) > 1$. Therefore $v_{r,x,f} > v_{r,x,i}$, as we expect. After a slight rearrangement of Eq. (12.3.84), we have an expression for the x-component of the velocity of the rocket as a function of the mass m_r of the rocket

$$v_{r,x,f} = v_{r,x,i} + u\ln\left(\frac{m_{r,i}}{m_{r,f}}\right). \tag{12.3.85}$$

Let's examine our result. First, let's suppose that all the fuel was burned and ejected. Then $m_{r,f} \equiv m_{r,d}$ is the final dry mass of the rocket (empty of fuel). The ratio

$$R = \frac{m_{r,i}}{m_{r,d}} \tag{12.3.86}$$

is the ratio of the initial mass of the rocket (including the mass of the fuel) to the final dry mass of the rocket (empty of fuel). The final velocity of the rocket is then

$$v_{r,x,f} = v_{r,x,i} + u\ln R\,. \tag{12.3.87}$$

This is why multistage rockets are used. You need a big container to store the fuel. Once all the fuel is burned in the first stage, the stage is disconnected from the rocket. During the next stage the dry mass of the rocket is much less and so R is larger than the single stage, so the next burn stage will produce a larger final speed then if the same amount of fuel were burned with just one stage (more dry mass of the rocket). In general rockets do not burn fuel at a constant rate but if we assume that the burning rate is constant where

$$b = \frac{dm_f}{dt} = -\frac{dm_r}{dt} \tag{12.3.88}$$

then we can integrate Eq. (12.3.88)

$$\int_{m'_r = m_{r,i}}^{m'_r = m_r(t)} dm'_r = -b\int_{t'=t_i}^{t'=t} dt' \tag{12.3.89}$$

and find an equation that describes how the mass of the rocket changes in time

$$m_r(t) = m_{r,i} - b(t - t_i)\,. \tag{12.3.90}$$

For this special case, if we set $t_f = t$ in Eq. (12.3.85), then the velocity of the rocket as a function of time is given by

$$v_{r,x,f} = v_{r,x,i} + u\ln\left(\frac{m_{r,i}}{m_{r,i} - bt}\right). \tag{12.3.91}$$

Example 12.4 Single-Stage Rocket

Before a rocket begins to burn fuel, the rocket has a mass of $m_{r,i} = 2.81\times10^{7}\,\text{kg}$, of which the mass of the fuel is $m_{f,i} = 2.46\times10^{7}\,\text{kg}$. The fuel is burned at a constant rate with total burn time is $510\,\text{s}$ and ejected at a speed $u = 3000\,\text{m/s}$ relative to the rocket. If the rocket starts from rest in empty space, what is the final speed of the rocket after all the fuel has been burned?

Solution: The dry mass of the rocket is $m_{r,d} \equiv m_{r,i} - m_{f,i} = 0.35\times10^{7}\,\text{kg}$, hence $R = m_{r,i} / m_{r,d} = 8.03$. The final speed of the rocket after all the fuel has burned is

$$v_{r,f} = \Delta v_r = u\ln R = 6250\,\text{m/s} \ . \tag{12.3.92}$$

Example 12.5 Two-Stage Rocket

Now suppose that the same rocket in Example 12.4 burns the fuel in two stages ejecting the fuel in each stage at the same relative speed. In stage one, the available fuel to burn is $m_{f,1,i} = 2.03\times10^{7}\,\text{kg}$ with burn time $150\,\text{s}$. Then the empty fuel tank and accessories from stage one are disconnected from the rest of the rocket. These disconnected parts have a mass $m = 1.4\times10^{6}\,\text{kg}$. All the remaining fuel with mass is burned during the second stage with burn time of $360\,\text{s}$. What is the final speed of the rocket after all the fuel has been burned?

Solution: The mass of the rocket after all the fuel in the first stage is burned is $m_{r,1,d} = m_{r,1,i} - m_{f,1,i} = 0.78\times10^{7}\,\text{kg}$ and $R_1 = m_{r,1,i} / m_{r,1,d} = 3.60$. The change in speed after the first stage is complete is

$$\Delta v_{r,1} = u\ln R_1 = 3840\,\text{m/s}. \tag{12.3.93}$$

After the empty fuel tank and accessories from stage one are disconnected from the rest of the rocket, the remaining mass of the rocket is $m_{r,2,d} = 2.1\times10^{6}\,\text{kg}$. The remaining fuel has mass $m_{f,2,i} = 4.3\times10^{6}\,\text{kg}$. The mass of the rocket plus the unburned fuel at the beginning of the second stage is $m_{r,2,i} = 6.4\times10^{6}\,kg$. Then $R_2 = m_{r,2,i} / m_{r,2,d} = 3.05$. Therefore the rocket increases its speed during the second stage by an amount

$$\Delta v_{r,2} = u\ln R_2 = 3340\,\text{m/s}. \tag{12.3.94}$$

The final speed of the rocket is the sum of the change in speeds due to each stage,

$$v_f = \Delta v_r = u \ln R_1 + u \ln R_2 = u \ln(R_1 R_2) = 7190 \text{ m/s}, \qquad (12.3.95)$$

which is greater than if the fuel were burned in one stage. Plots of the speed of the rocket as a function time for both one-stage and two-stage burns are shown Figure 12.15.

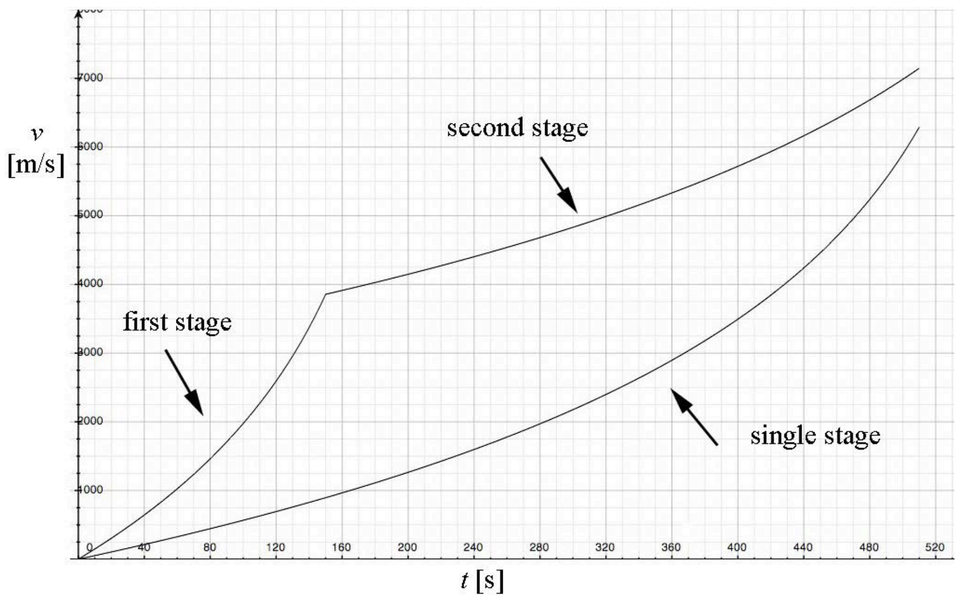


Figure 12.15 Plots of speed of rocket for both one-stage burn and two-stage burn

12.3.2 Rocket in a Constant Gravitational Field:

Now suppose that the rocket takes off from rest at time $t = 0$ in a constant gravitational field then the external force is

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = m_r \vec{\mathbf{g}} . \qquad (12.3.96)$$

Choose the positive x-axis in the upward direction then $F_{ext,x}(t) = -m_r(t)g$. Then the rocket equation (Eq. (12.3.75) becomes

$$-m_r(t)g - \frac{dm_r}{dt}u = m_r(t)\frac{dv_{r,x}}{dt} . \qquad (12.3.97)$$

Multiply both sides of Eq. (12.3.97) by dt, and divide both sides by $m_r(t)$. Then Eq. (12.3.97) can be written as

$$dv_{r,x} = -gdt - \frac{dm_r}{m_r(t)}u . \qquad (12.3.98)$$

We now integrate both sides

$$\int_{v_{r,x,i}=0}^{v_{r,x}(t)} dv'_{r,x} = -u \int_{m_{r,i}}^{m_r(t)} \frac{dm'_r}{m'_r} - g\int_0^t dt' , \tag{12.3.99}$$

where $m_{r,i}$ is the initial mass of the rocket and the fuel. Integration yields

$$v_{r,x}(t) = -u \ln\left(\frac{m_r(t)}{m_{r,i}}\right) - gt = u \ln\left(\frac{m_{r,i}}{m_r(t)}\right) - gt . \tag{12.3.100}$$

After all the fuel is burned at $t = t_f$, the mass of the rocket is equal to the dry mass $m_{r,f} = m_{r,d}$ and so

$$v_{r,x}(t_f) = u \ln R - gt_f . \tag{12.3.101}$$

The first term on the right hand side is independent of the burn time. However the second term depends on the burn time. The shorter the burn time, the smaller the negative contribution from the third turn, and hence the rocket ends up with a larger final speed. So the rocket engine should burn the fuel as fast as possible in order to obtain the maximum possible speed.