8.012 Physics I: Classical Mechanics
Fall 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.012 | Fall 2005

FINAL EXAM
Wednesday, December 21, 2005

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MIT ID number: ____________

INSTRUCTIONS:

- Do all **SEVEN (7)** problems. You have 3 hours.
- **Show all work**, and circle your answer.
- All work must be done in this booklet.
- No books, notes, or calculators permitted.

USEFUL RELATIONS:

- Velocity in polar coordinates: $\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$
- Acceleration in polar coordinates: $\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\boldsymbol{\theta}}$
- Fictitious force for rotating coordinates: $\mathbf{F}_{\text{fict}} = -2m\,(\boldsymbol{\Omega}\times\mathbf{v}_{\text{rot}}) - m\,\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})$
- Effective potential for central force: $U_{\text{eff}}(r) = U(r) + \frac{L^2}{2\mu r^2}$

Problem	Maximum	Score	Grader
1	10		
2	15		
3	15		
4	15		
5	15		
6	15		
7	15		
TOTAL	100		

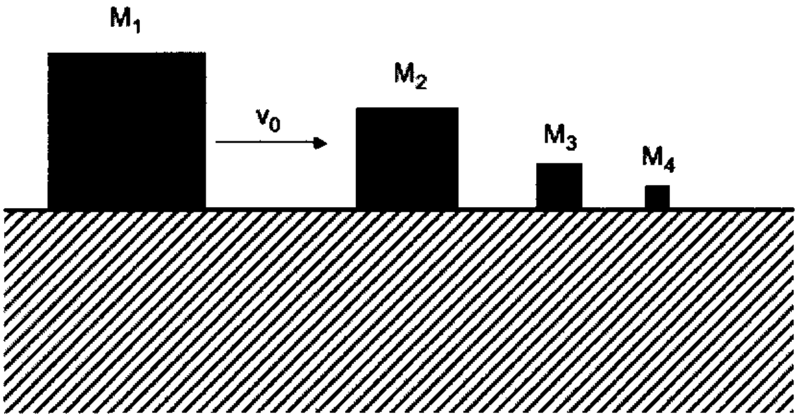
1. **Problem 1 of 7**
Colliding blocks. (10 points)

A mass M_1 is sliding with velocity v_0 along a frictionless table toward three masses M_2, M_3, and M_4 which are sitting at rest in the path of M_1, as shown. The masses get successively much smaller, so that

$$M_4 \ll M_3 \ll M_2 \ll M_1,$$

where the symbol "$\ll$" means "much smaller than". What is the velocity of mass M_4 after all the collisions have happened? Assume that all the collisions are perfectly elastic.

Hint: This problem is easiest if you consider each collision individually.



First collision: Two ways to approach problem. Let primed variables denote velocities after collision.

LAB FRAME

p: $M_1 V_0 = M_1 V_1' + M_2 V_2' \Rightarrow V_1' = \frac{M_1 V_0 - M_2 V_2'}{M_1}$

E: $\frac{1}{2} M_1 V_0^2 = \frac{1}{2} M_1 V_1'^2 + \frac{1}{2} M_2 V_2'^2$

$\Rightarrow M_1 V_0^2 = M_1 \left(\frac{M_1^2 V_0^2 - 2 M_1 M_2 V_0 V_2' + M_2^2 V_2'^2}{M_1^2} \right) + M_2 V_2'^2$

$M_2 << M_1$ $\quad = M_1 V_0^2 - 2 M_2 V_0 V_2' + \frac{M_2^2}{M_1} V_2'^2 + M_2 V_2'^2$

$\left(\frac{M_2}{M_1} + \cancelto{0}{\frac{M_2^2}{M_1^2}} \right) V_2'^2 = 2 \left(\frac{M_2}{M_1} \right) V_0 V_2'$

$\Rightarrow V_2' = 2 V_0$

CENTER-OF-MASS FRAME

$M_1 >> M_2 \Rightarrow M_1$ is center of mass

$V = V_0$ (CM velocity)

$V_{1c} = 0 \quad V_{2c} = -V_0$

$V_{1c}' = 0 \quad V_{2c}' = V_0$

$V_2' = V_{2c}' + V = 2 V_0$

$\Rightarrow V_2' = 2 V_0$

similarly, for each subsequent collision:

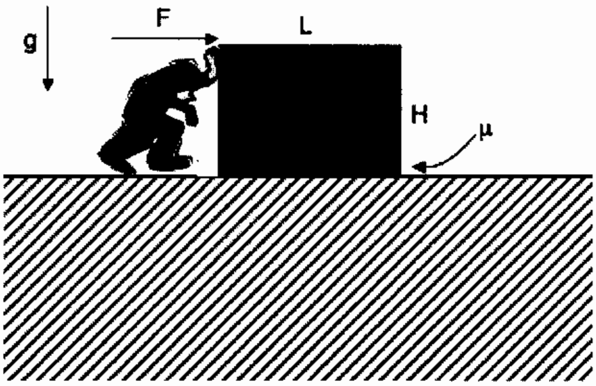
Second collision: $V_3' = 2(2V_0) = 4 V_0$

Third collision: $V_4' = 2(4V_0) = 8 V_0 \Rightarrow \boxed{V_4' = 8 V_0}$

2. **Problem 2 of 7**

Sliding a crate. (15 points)

A mover is trying to slide a uniformly filled crate of length L and height H across the floor. There is friction with coefficient μ between the crate and the floor. The mover exerts a horizontal force F at the upper back edge of the crate. If $\mu > \mu_0$, the crate will tip over before it slides. Calculate the critical friction coefficient μ_0. Gravity is directed downward.



Let M = mass of crate.

If crate is not sliding, then $F < \mu M g$ (*)

If crate tips over, it will be about lower right corner, so let's measure torques about this point. For tipping, clockwise torque must exceed counterclockwise torque:

$$FH > \frac{MgL}{2} \qquad (**)$$

Combining (*) and (**), we have (for tipping before sliding)

$$\frac{MgL}{2H} < F < \mu M g$$

$$\Rightarrow \mu > \frac{L}{2H}$$

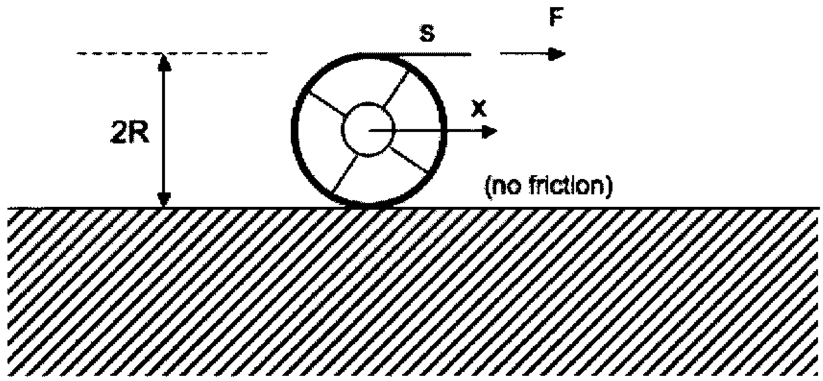
So, for critical condition ($\mu > \mu_0$),

$$\boxed{\mu_0 = \frac{L}{2H}}$$

3. **Problem 3 of 7**

Reel of tape on a frictionless surface. (15 points)

A long length of tape is wrapped around a reel which is initially at rest on a frictionless surface. The tape itself has negligible mass, but the reel has mass M, radius R, and moment of inertia I_0 about its center. The end of the tape is pulled horizontally with a constant force F, as shown in the figure. Calculate x, the horizontal displacement of the reel from its initial position when a length s of tape has unwound from the reel.



Translation of CM

$$F = Ma \Rightarrow a = \frac{F}{M}$$

$$x = \frac{1}{2}at^2 = \frac{1}{2}\left(\frac{F}{M}\right)t^2 \qquad (*)$$

Rotation about CM

$$\tau = FR = I_0\alpha \Rightarrow \alpha = \frac{FR}{I_0}$$

$$\theta = \frac{1}{2}\alpha t^2$$

$$s = \theta R = \frac{1}{2}\alpha R t^2 = \frac{1}{2}\left(\frac{FR^2}{I_0}\right)t^2$$

$$\Rightarrow t^2 = \frac{2I_0 s}{FR^2} \qquad (**)$$

Note that $a \neq \alpha R$, since we have both rolling AND slipping.

Combining $(*)$ and $(**)$, we have

$$x = \frac{1}{2}\left(\frac{F}{M}\right)\left(\frac{2I_0}{FR^2}\right)s$$

$$\Rightarrow \boxed{x = \left(\frac{I_0}{MR^2}\right)s}$$

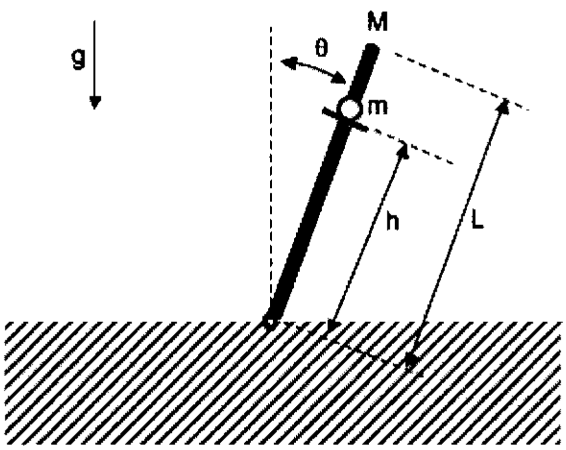
4. **Problem 4 of 7**
Falling rod. (15 points)

A uniform rod of length L and mass M is free to rotate about a pivot at its lower end. An attached bead of mass m (where $m \ll M$) is free to move without friction along the rod, but a massless collar fixed on the rod at a distance h from the pivot requires the bead's distance from the pivot to be $\geq h$. Initially, the rod is at rest and nearly vertical, and the bead is resting on the collar. The rod is then released and falls over. Gravity is directed downward.

(a) (7 points) Calculate the angular velocity ω of the rod when it has fallen to an angle θ from the vertical. Specify the full expression for any moment of inertia used in your calculation.

Hint: You may neglect m when computing the energy of the system, since $m \ll M$

(b) (8 points) Compute the angle θ_c at which the bead loses contact with the collar.



(a) From conservation of energy,

$$\frac{MgL}{2} = \frac{MgL}{2}\cos\theta + \frac{1}{2}I\omega^2$$

where $I = \frac{1}{3}ML^2$ for rod rotating about one end

$$= \frac{MgL}{2}\cos\theta + \frac{1}{6}ML^2\omega^2$$

$$3g = 3g\cos\theta + L\omega^2 \implies \boxed{\omega = \sqrt{\frac{3g(1-\cos\theta)}{L}}}$$

(b) Since bead can slide freely along rod, contact force on bead is purely in $\hat{\theta}$ direction. The only force available to provide centripetal acceleration for circular motion of bead is gravity. Bead loses contact when radial component of gravity too small:

$$mg\cos\theta < m\omega^2 h = \frac{3mg(1-\cos\theta)h}{L}$$

$$\cos\theta < \frac{3h}{L+3h} \implies \boxed{\cos\theta_c = \frac{3h}{L+3h}}$$

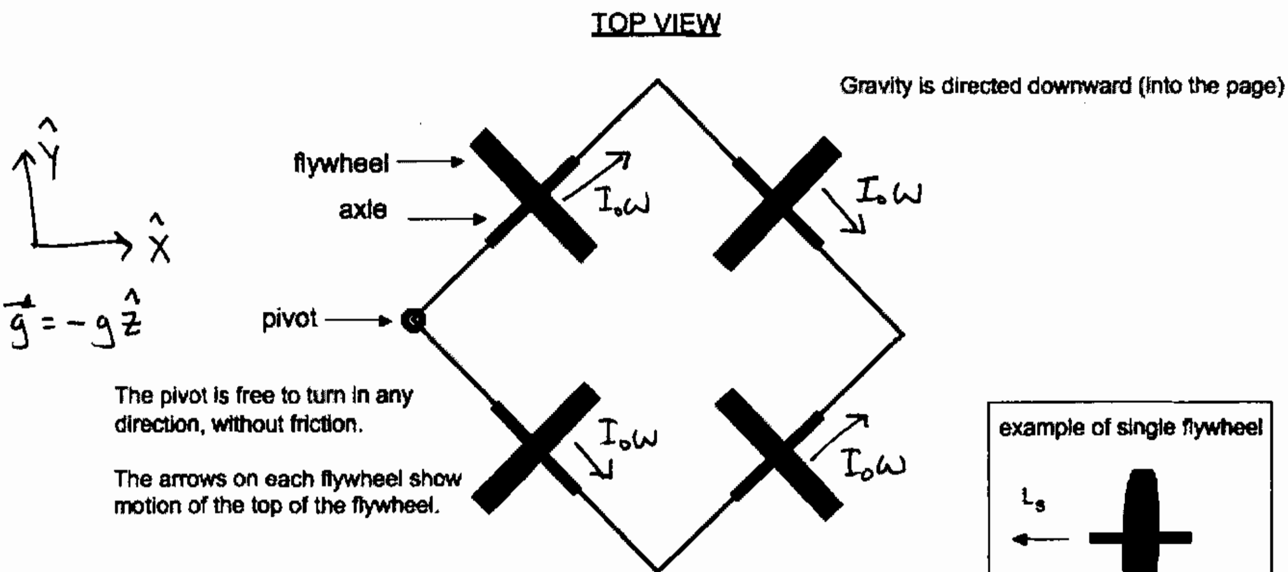
5. **Problem 5 of 7**

Compound gyroscope. (15 points)

A compound gyroscope consists of four identical flywheels mounted at the midpoints of the edges of a rigid square frame of length L. Each flywheel has moment of inertia I_0 and is spinning rapidly at the same angular velocity ω, with the sense of rotation shown in the figure. The entire assembly (frame plus flywheels) has total mass M, and the center of mass is at the geometric center of the square frame. One corner of the frame is attached to a frictionless pivot, so that the entire frame is free to rotate about the pivot in any direction. In the top-view figure, gravity is directed into the page.

(a) (10 points) Find the frequency of uniform precession Ω and indicate its direction.

(b) (5 points) Now suppose the entire compound gyroscope is placed inside an elevator. The elevator accelerates upward (opposite to gravity, so out of the page in the figure) with uniform acceleration A. Find the new frequency of uniform precession Ω_A.



(a) $\vec{L}_{s,tot} = \frac{I_0\omega}{\sqrt{2}}\left[(\hat{x}+\hat{y}) + (\hat{x}-\hat{y}) + (\hat{x}+\hat{y}) + (\hat{x}-\hat{y})\right]$

$= 2\sqrt{2}\, I_0\omega\,\hat{x}$

Torque with respect to pivot $\vec{\tau} = \vec{r}\times\vec{F} = \left(\frac{L}{\sqrt{2}}\hat{x}\right)\times\left(-Mg\hat{z}\right) = \frac{MgL}{\sqrt{2}}\hat{y}$

For uniform precession: $\tau = \Omega L_{s,tot} \Rightarrow \frac{MgL}{\sqrt{2}} = 2\sqrt{2}\, I_0\omega\Omega$

$\Rightarrow \boxed{\Omega = \frac{MgL}{4I_0\omega}, \text{ counter-clockwise}}$ $\Rightarrow \boxed{\vec{\Omega} = \frac{MgL}{4I_0\omega}\hat{z}}$

(b) Accelerating upward with $\vec{A} = A\hat{z}$, we have $\vec{F} = -(g+A)\hat{z}$, so

$\boxed{\vec{\Omega}_A = \frac{ML(g+A)}{4I_0\omega}\hat{z}}$ (counterclockwise)

6. **Problem 6 of 7**
Particle in a central force field. (15 points)

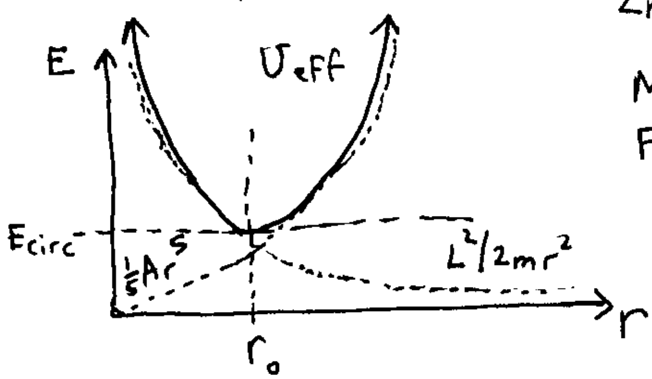
A mass m moves under the influence of an attractive central force Ar^4 with angular momentum L, where A and L are positive constants. Define the potential energy to be zero at the origin. For what total energy will the motion be circular, and what is the radius of this circular orbit?

Attractive force $\vec{F} = -Ar^4\hat{r}$

Potential energy $U(r) = -\int F\,dr = \int_0^r Ar'^4\,dr' = \frac{1}{5}Ar^5$

For a central force, we can define an effective potential

$$U_{eff}(r) = U(r) + \frac{L^2}{2mr^2} = \frac{1}{5}Ar^5 + \frac{L^2}{2mr^2}$$



Minimum of U_{eff} is at $r = r_0$.
For circular orbit, $E_{circ} = U_{eff}(r_0)$

$$0 = \left.\frac{dU_{eff}}{dr}\right]_{r_0} = Ar_0^4 - \frac{L^2}{mr_0^3} = 0 \implies \boxed{r_0 = \left(\frac{L^2}{mA}\right)^{1/7}}$$ radius of circular orbit

$$E_{circ} = U_{eff}(r_0) = \frac{1}{5}Ar_0^5 + \frac{L^2}{2mr_0^2} = \frac{1}{5}A\left(\frac{L^2}{mA}\right)^{5/7} + \frac{L^2}{2m}\left(\frac{mA}{L^2}\right)^{2/7}$$

$$= \boxed{\frac{7}{10}\left(\frac{L^{10}A^2}{m^5}\right)^{1/7}}$$ energy of circular orbit

7. **Problem 7 of 7**

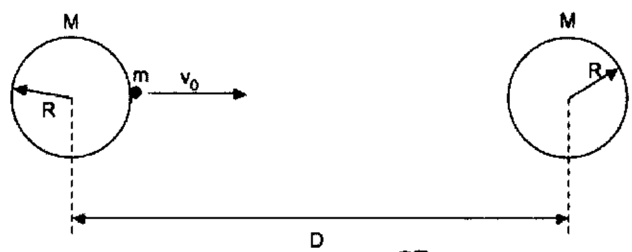
Projectile near two gravitating spheres. (15 points)

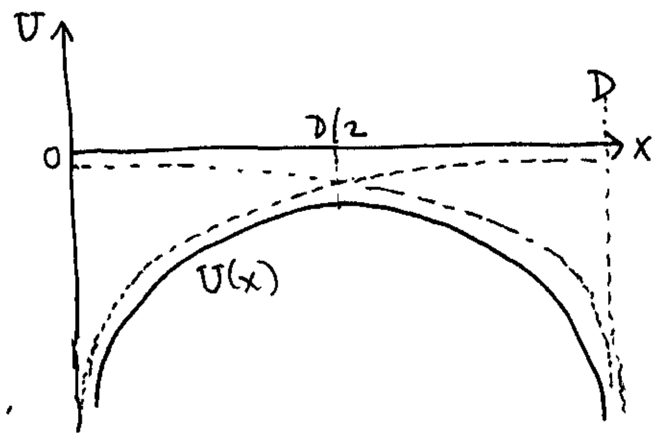
Two identical spheres of mass M and radius R are held fixed and are separated by a distance D. A small projectile of mass m is fired with initial velocity v_0 from the surface of one sphere directly toward the other. The projectile is attracted gravitationally by each sphere, and *no other forces are acting.*

(a) (8 points) Since the projectile moves only along the axis connecting the centers of the two spheres, we can treat this as a one-dimensional problem. Write down the potential energy $U(x)$ of the projectile (in 1-dimension) as a function of distance x from the center of the sphere on the left, measured along the axis between the two spheres. Sketch $U(x)$ as a function of x.

Hint: Does your answer exhibit the proper symmetry?

(b) (7 points) What is the minimum value of v_0 such that the projectile can reach the second sphere?





(a) $$U(x) = -\frac{GMm}{x} - \frac{GMm}{D-x}$$

Note deep potential wells at $x=0$ and $x=D$.

(b) For projectile to reach second sphere, need enough energy to exceed max P.E. at $x=\frac{1}{2}D$.

$$\frac{1}{2}mv_0^2 - \frac{GMm}{R} - \frac{GMm}{D-R} = -\frac{GMm}{D/2} - \frac{GMm}{D/2}$$

$$\Longrightarrow \boxed{v_0 = \left[2GM\left(\frac{1}{R} + \frac{1}{D-R} - \frac{4}{D}\right)\right]^{1/2}}$$