8.012 Physics I: Classical Mechanics
Fall 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.012   Fall 2007

Exam 2
Thursday, November 15, 2007

NAME: _____ S O L U T I O N S _____

MIT ID number: _____

Instructions:
1. Do all **FIVE (5)** problems. You have **90 minutes**.
2. **SHOW ALL OF YOUR WORK**. Be sure to **circle your final answer**.
3. Read each question carefully.
4. All work must be done in this booklet. Extra blank pages are provided.
5. NO books, notes, calculators or computers are permitted.
6. **A sheet of useful equations is provided on the last page**.

Your Scores

Problem	Maximum	Score	Grader
1	10		
2	25		
3	25		
4	20		
5	20		
Total	100		

Problem 1: Quick Multiple Choice Questions [10 pts]

Answer all five problems. You do not need to show any work. **Do not spend a long time on this section**.

(a) What are the dimensions of the universal gravitational constant G? Recall that $[M]$ = mass, $[L]$ = length and $[T]$ = time

$[M][L][T]^{-2}$　　　　$[M][L]^2[T]^{-1}$　　　$[M]^{-1}[L]^2[T]^{-1}$　　　$\boxed{[M]^{-1}[L]^3[T]^{-2}}$

(b) A uniform disk of radius R has a circular hole of diameter R cut out of it. The hole lies between the outer edge of the disk and the center of the disk, as shown to the right. Mark with an "X" where the center of mass of this disk lies.

R　　　R

$R/6$
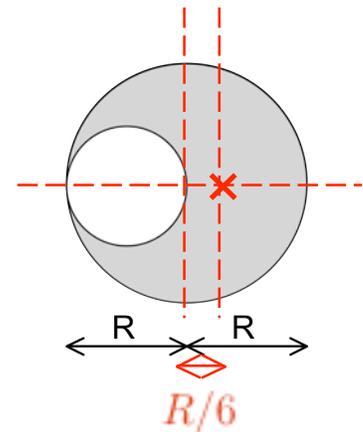
(c) A hollow cylinder rolls up an incline without slipping. In which direction is the friction force between the cylinder and incline surfaces acting?
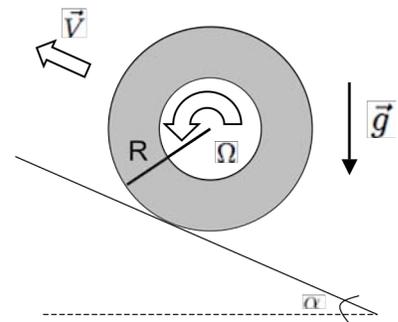
Down the incline　　　　$\boxed{\text{Up the incline}}$　　　　There is no friction force acting on the cylinder

(d) Consider the one-dimensional potential function shown below. Assume that an object in this potential starts at x = x_o at rest. Indicate on the plot the maximum value of x the object can attain and circle the stable equilibrium points in this range.
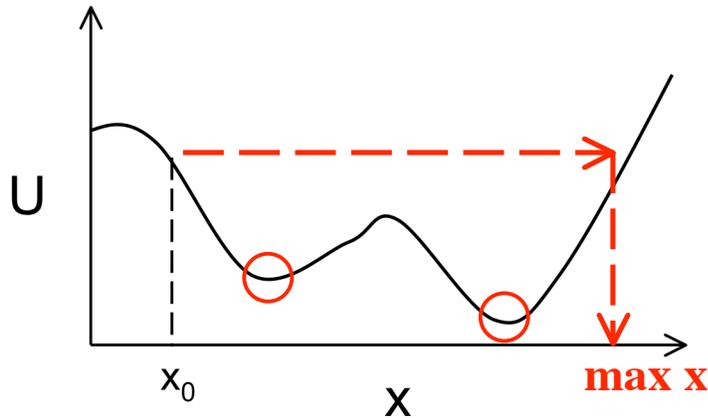
(e) A ball comes in from the left with speed 1 (in arbitrary units) and causes a series of collisions. The initial motion is shown below (the number in the circles indicates the objects' relative masses):
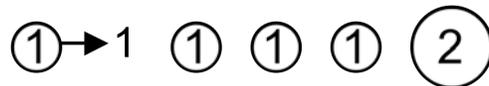
All collisions are elastic. Which of the following indicates the motions of the balls after all of the collisions are completed? (circle the correct diagram)
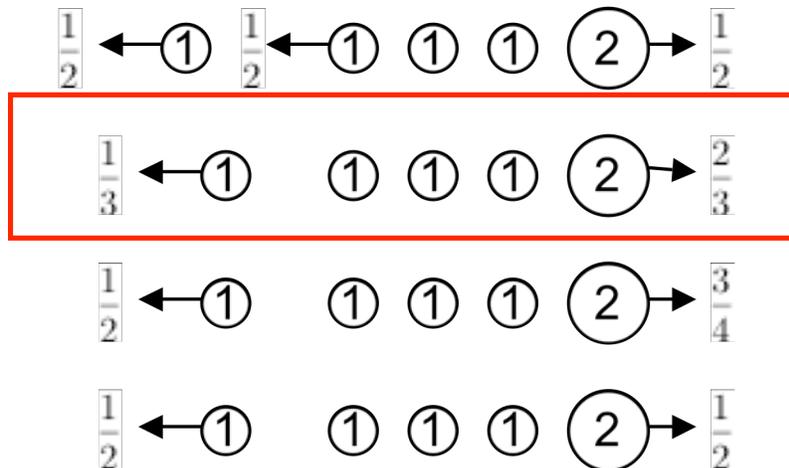
Problem 2: The Bowling Ball [25 pts]
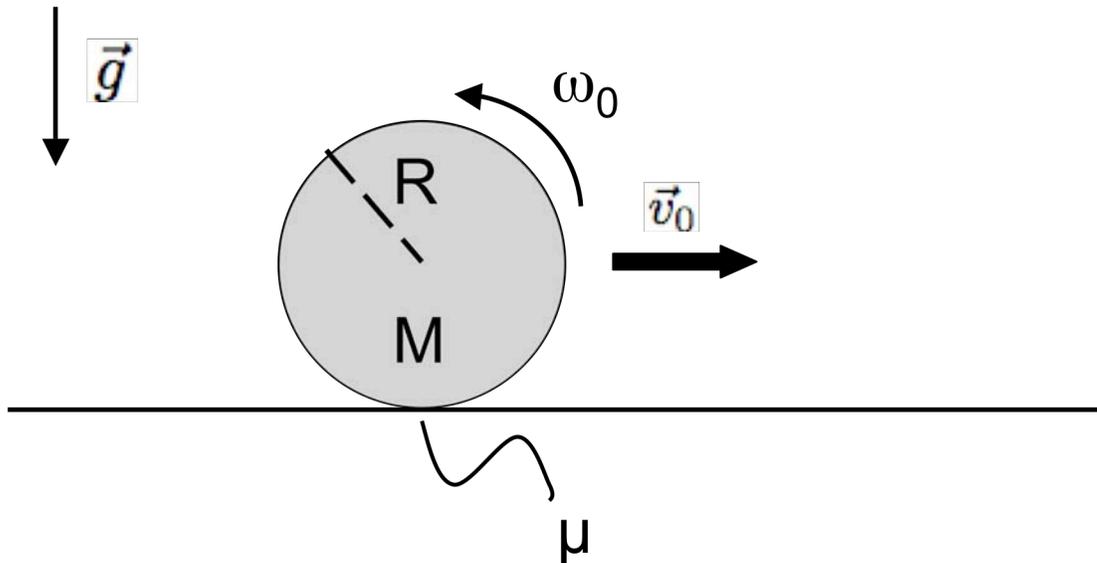
A bowling ball of radius R, mass M and uniform mass density is thrown down a lane with initial horizontal speed v_0. The ball is given some backspin – it is spun in the opposite direction of motion – with initial angular rate ω_0 as shown. The maximum coefficient of friction between the ball and lane surfaces is μ. Assume constant gravitational acceleration $\vec{g}$ points downward.

(a) [10 pts] Find the velocity $\vec{v}$ and angular rotation rate ω of the ball as a function of time after it makes contact with the lane, and up to the point when it starts to roll without slipping. Note: directions are important here!

(b) [10 pts] Find the "final" velocity and angular rotation rate of the ball when it starts to roll without slipping.

(c) [5 pts] How must the initial conditions, v_0 and ω_0, be related for the ball to come to rest?

Solutions

(a) This problem can solved by considering the translation of the center of mass and rotation about the center of mass of the ball separately. Referring to the force diagram to the right, the equations of motion are:
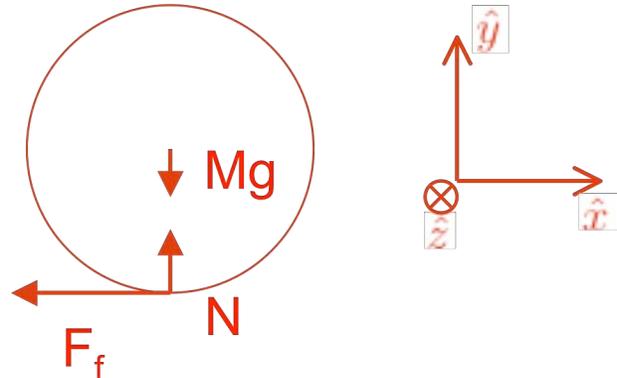
$$x : M\ddot{x} = -F_f \Rightarrow \ddot{x} = \frac{F_f}{M}$$

$$y : M\ddot{y} = -Mg + N = 0 \Rightarrow N = Mg$$

$$\theta : I\ddot{\theta} = -RF_f \Rightarrow \ddot{\theta} = -\frac{RF_f}{I}$$

Note that torque is working to slow down the rotation of the ball in this case, so there must be a minus sign in the angle equation if we set $\omega_0 > 0$. A ball is a sphere, so

$$I = \frac{2}{5}MR^2$$

While the ball is slipping, the friction force is at its maximum (constant) value of $F_f = \mu N = \mu Mg$, so the equations are easily solved. Using the initial conditions:

$$\dot{x}(0) = v_0$$
$$\dot{\theta}(0) = \omega(0) = \omega_0$$

we find (note the specific direction of velocity):

$$\vec{v}(t) = v_0\hat{x} - \mu g t \hat{x}$$
$$\omega(t) = \dot{\theta}(t) = \omega_0 - \frac{5}{2}\frac{\mu g}{R}t$$

(b) The condition of rolling without slipping occurs when

$$\omega(t) = -\frac{v(t)}{R}$$

Note the minus sign – this is because we defined positive rotation in the initial conditions such that the object has backspin, but when it is not slipping it must have forward spin. This condition can be used to find the time t_f when the ball starts to roll without slipping:

$$\omega_0 - \frac{5}{2}\frac{\mu g}{R}t_f = -\frac{v_0}{R} + \frac{\mu g}{R}t_f$$

$$\Rightarrow t_f = \frac{2}{7}\frac{R\omega_0 + v_0}{\mu g}$$

Plugging this back into the expressions from (a) we find:

$$\vec{v}_f = \frac{5}{7}v_0\hat{x} - \frac{2}{7}R\omega_0\hat{x}$$
$$\omega_f = \frac{2}{7}\omega_0 - \frac{5}{7}\frac{v_0}{R}$$

Note that after this point, the friction force ceases to act as the surfaces are no longer slipping relative to each other, and so the ball continues to roll/spin at these rates.

(c) The object comes to rest when its final velocity goes to 0. From the expression from (b) we find that this occurs if:

$$v_0 = \frac{2}{5} R\omega_0$$

Problem 3: A Generic Potential [25 pts]

Consider the following one-dimensional potential in which an object of mass m moves freely:

$$U(x) = U_0(2s^3 - 3s^2 - 4)$$

where U_0 is a constant with dimensions of energy and $s = x/x_0$ is a normalized dimension of length.

(a) [5 pts] Sketch this potential as a function of x in the range $-3x_0 < x < 3x_0$.

(b) [5 pts] Derive values for the equilibrium points and determine which of these are stable.

(c) [5 pts] Determine the period of small oscillations about the stable equilibrium point.

(d) [5 pts] If the mass starts from x = -∞ and moves toward positive x, what is the minimum total mechanical energy it needs to reach the stable equilibrium point? What is its velocity at the stable equilibrium point in this case?

(e) [5 pts] What is the magnitude and direction of the force acting on the mass when it is at x = $2x_o$?

Solution

(a) A sketch of the potential is shown below:
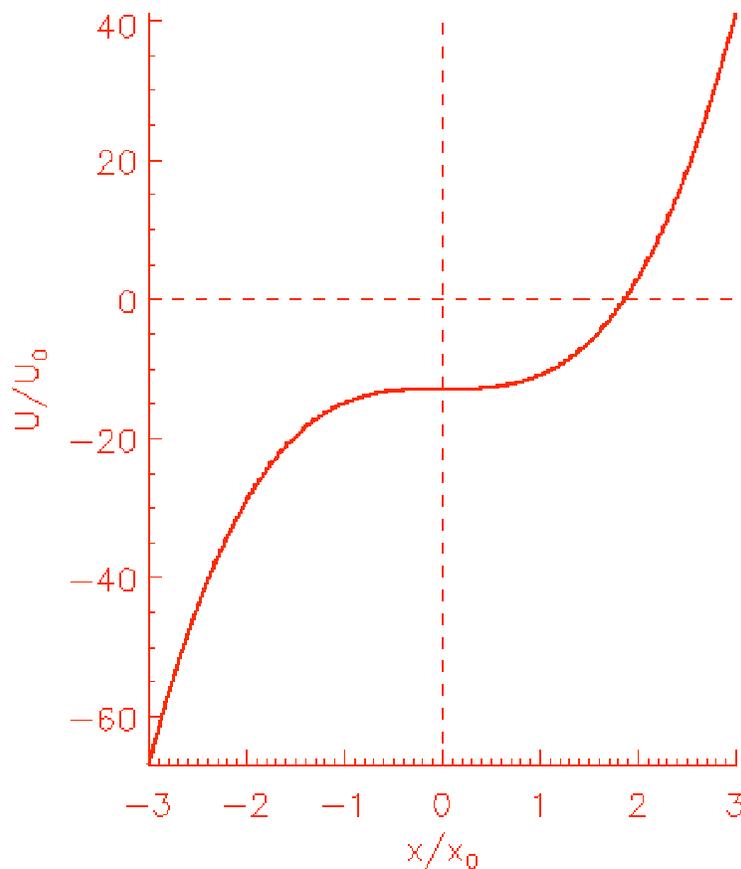
(b) At the equilibrium points the net force goes to zero, hence

$$F = -\frac{dU}{dx} = -\frac{1}{x_0}\frac{dU}{ds} = 0$$
$$\frac{dU}{ds} = U_0(6s^2 - 6s) = 0 \Rightarrow s_{eq} = 0, 1$$

$$\Rightarrow x_{eq} = 0, x_0$$

The stable equilibrium points satisfy

$$\frac{d^2U}{dx^2} > 0$$

By inspection of the sketch, it should be clear that $x = x_0$ (s=1) is the stable equilibrium point. Formally:

$$\frac{d^2U}{dx^2} = \frac{1}{x_0^2}\frac{d^2U}{ds^2} = \frac{U_0}{x_0^2}(12s - 6) = 6\frac{U_0}{x_0^2}$$

for s=1, which is > 0 for $U_0 > 0$.

(c) Period of small oscillation about the stable equilibrium point is:

$$P = \frac{2\pi}{\omega}$$

where

$$\omega = \sqrt{\frac{1}{m}\frac{d^2U}{dx^2}\bigg|_{x_{eq}}}$$

from (b) we find that:

$$P = 2\pi\sqrt{\frac{mx_0^2}{6U_0}}$$

(d) Moving from x = -∞, the mass must make it over the unstable equilibrium point at x=0, which has potential energy $U(0) = -4U_0$. Hence

$$E_{tot} > -4U_0$$

The total mechanical energy remains constant, so the loss in potential energy, $U(x_0) - U(0)$ where $U(x_0) = -5U_0$, goes into kinetic energy. If the mass has effectively zero kinetic energy at x = 0, then

$$\frac{1}{2}mv^2 = U_0 \Rightarrow v = \sqrt{\frac{2U_0}{m}}$$

(e) Using the expression for F from part (a) and setting $x = 2x_0$ we find:

$$F = -\frac{12U_0}{x_0}$$

The force must point toward negative x, since the potential barrier is increasing here.

Problem 4: A Ballistic Rotator [20 pts]
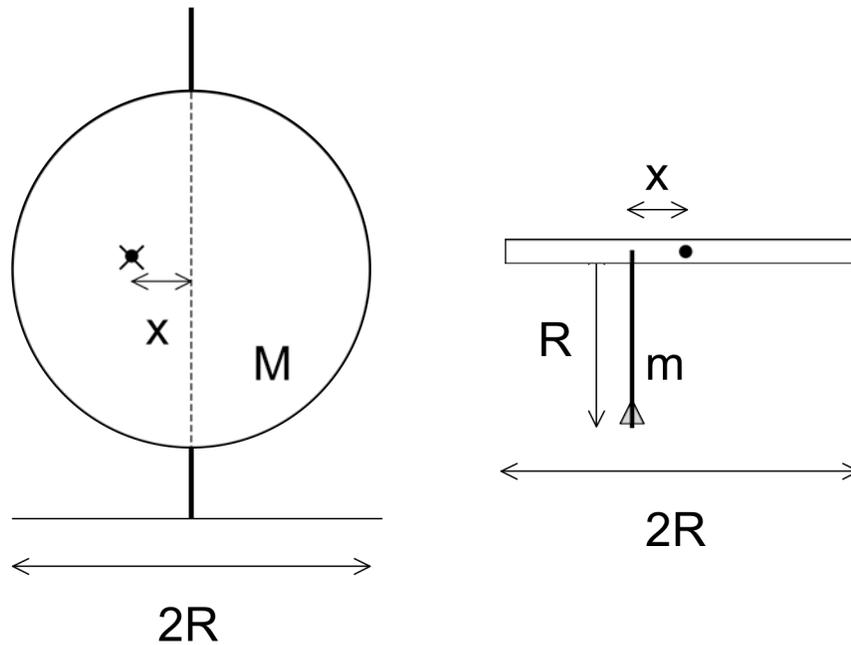
A thin arrow with length R, mass m and uniform linear mass density $\lambda = M/R$ is shot with velocity v into a circular target of radius R, mass M, uniform surface mass density $\sigma = M/\pi R^2$, and negligible thickness (i.e., it is essentially a thin disk). The arrow head sticks into the target just off center by a distance $x < R$ as shown above. The target is mounted onto a central frictionless shaft, about which it can spin freely. The target is at rest before it is struck by the arrow.

(a) [10 pts] Find the angular rotation rate of the target (with arrow embedded) after the arrow strikes it. You can express you answer in terms of I_T, the moment of inertia of the circular target *alone* about the rotation axis shown.

(b) [5 pts] Show that if m << M then to first order the rotation rate depends linearly on both x and v.

(c) [5 pts] Derive an expression for I_T in terms of M and R. **Warning!** This is a difficult problem, so save it for last!

Solution

(a) The arrow hitting the target is an example of an inelastic collision, since the arrow does not rebound off of the target. Hence energy s not conserved. Translational momentum is also not conserved, as the vertical shaft provides an impulse to prevent the target from moving. However, none of these forces present exert a torque on the arrow-target system during the collision (we can ignore gravity since it acts to slowly), so angular momentum is conserved. Let's measure angular momentum from the center of mass of the target. Then the angular momentum prior to the collision arises solely from the translational motion of the arrow relative to this point, i.e.,

$$\vec{L} = \vec{r} \times \vec{p} = mvx\hat{z} \Rightarrow L = mvx$$

where $\hat{z}$ points downward. After the collision all of the angular momentum is in rotation of the target and arrow about the shaft

$$L = (I_t + I_{arrow})\omega$$

(the direction of angular rotation should be such that L points downward) . The moment of inertia of the arrow can be found by successive applications of the parallax axis theorem, shifting from the center of mass of the arrow to its end, and then shifting from where the arrow sticks into the target to the rotation axis. Putting this together:

$$I_{arrow} = \frac{1}{12}mR^2 + m(R/2)^2 + mx^2 = \frac{1}{3}mR^2 + mx^2$$

So the angular rotation rate is:

$$\omega = \frac{mvx}{I_t + \frac{1}{3}mR^2 + mx^2}$$

(b) $I_t \approx MR^2$, $I_{arrow} \approx mR^2 + mx^2$ so if m << M, then $I_t >> I_{arrow}$, and the solution to (a) reduces to:

$$\omega \approx \frac{mvx}{I_t}$$

This depends linearly on both v and x, since I_t depends on neither of these quanitites.

(c) There are a number of ways of solving this, including using a fun trick the TA demonstrated in the review session. However, the most direct way is to use the standard integral expression for I:

$$I = \int_V \rho x^2 dV$$

where the factors in the integral, in polar coordinates, are:

$$x^2 = (r\cos\theta)^2$$

$$\rho = \frac{M}{\pi R^2}$$

$$dV = r\,dr\,d\theta$$

The volume integral is then:

$$I = \int_0^R \int_0^{2\pi} \frac{M}{\pi R^2} r^3 \cos^2 \theta \, dr \, d\theta$$

$$= \frac{1}{4\pi} M R^2 \int_o^{2\pi} \cos^2 \theta \, d\theta$$

The integral over θ can be computed by substituting the appropriate half-angle trig identity; an easier trick is to note that over 2π the integrals of $\cos^2$ and $\sin^2$ are the same (in both cases they are over a full period). Hence:

$$\int_o^{2\pi} \cos^2 \theta \, d\theta + \int_o^{2\pi} \sin^2 \theta \, d\theta = 2 \int_o^{2\pi} \cos^2 \theta \, d\theta = \int_o^{2\pi} d\theta = 2\pi$$

$$\Rightarrow \int_o^{2\pi} \cos^2 \theta \, d\theta = \pi$$

so

$$\boxed{I = \frac{1}{4} M R^2}$$

Problem 5: The Continuous Multi-stage Rocket [20 pts]
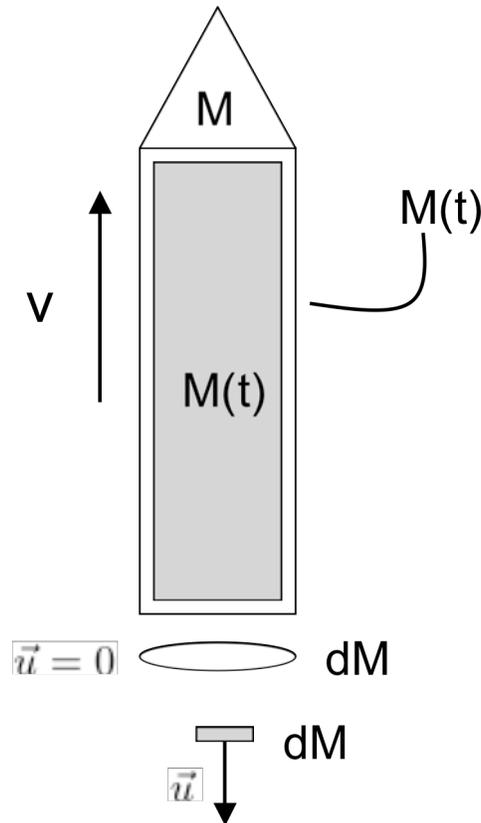
Consider a rocket of total mass 3M divided equally into payload (at top), rocket shell, and fuel. The rocket, initially at rest, ejects its fuel at a constant mass loss rate $\gamma = -dM/dt$ and with constant velocity $\vec{u}$ relative to the rocket. However, for each bit of fuel ejected, the rocket also sheds part of its shell, with the same mass loss rate as the fuel but with <u>zero</u> velocity relative to the rocket.

(a) [10 pts] Derive the equation of motion for this rocket, assuming it is moving in free space (i.e,. there are no external forces). Note: this is not necessary going to be the same as the standard rocket equation we learned in lecture.

(b) [10 pts] Determine the final speed of the payload after the fuel and shell are ejected in terms of M, u and γ.

Solution

(a) Let's simply the notation by letting $3M_0$ be the initial mass of the rocket. As there are no external forces, we can apply conservation of momentum when each small piece of fuel is ejected/part of rocket shell is lost:

$$p_i = p_f$$

$$\Rightarrow (2M(t) + M_0)v = (2M(t) - 2\Delta m + M_0)(v + \Delta v)$$
$$+ \Delta mv - \Delta m(u - v - \Delta v)$$

$$\Rightarrow 2M(t)\Delta v + M_0\Delta v = \Delta mu$$

$$\Rightarrow \Delta v = \frac{\Delta mu}{(M_0 + 2M(t))}$$

Because the rocket is losing mass, $\Delta m = -\Delta M$ so

$$\Rightarrow \Delta v = -\frac{\Delta Mu}{(M_0 + 2M(t))}$$

We can divide both sides by Δt and taking the limit for small differences, this equation reduces to derive the equation of motion:

$$\boxed{\frac{dv}{dt} = \frac{\gamma u}{M_0 + 2M(t)} = \frac{\gamma u}{3M_0 - 2\gamma t}}$$

Where we have used $M(t) = M_0 - \gamma t$ to eliminate the variable M(t).

(b) The solution from (a) can be integrated by multiplying both sides by Δt :

$$\Rightarrow \int_0^{v_f} dv = \int_0^{t_f} \frac{\gamma u dt}{3M_0 - 2\gamma t}$$

$$\Rightarrow v_f = u \int_0^{M_0/\gamma} \frac{\gamma dt}{3M_0 - 2\gamma t}$$

$$\Rightarrow v_f = \frac{u}{2} \ln\left(3M_0 - 2\gamma t\right)\Big|_0^{M_0/\gamma} = \frac{u}{2} \ln 3 \approx 0.55u$$

Note that this is just faster than final speed of a rocket of mass 3M that burns M of its fuel but keeps its shell:

$$v_f = u \ln \frac{3}{2} \approx 0.41u$$

USEFUL EQUATIONS

Trajectory for constant acceleration	$\vec{r}(t) = \frac{1}{2}\vec{a}t^2 + \vec{v}(0)t + \vec{r}(0)$
Velocity in polar coordinates	$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$
Acceleration in polar coordinates	$\ddot{\vec{r}} = \ddot{r}\hat{r} + r\ddot{\theta}\hat{\theta} + 2\dot{r}\dot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r}$
Center of mass of a rigid body	$\vec{R} = \frac{\sum m_i \vec{r}_i}{\sum m_i} = \frac{1}{M}\int \rho \vec{r} dV$
Volume element in cylindrical coordinates	$dV = r\,dr\,d\theta\,dz$
Kinetic energy	$K = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$
Work	$W = -\int \vec{F} \cdot d\vec{r} = -\int |\vec{\tau}|\,d\theta$
Potential Energy	$U = -\int \vec{F}_c \cdot d\vec{r}$ $\vec{F}_c = -\vec{\nabla}U$ where $\vec{\nabla} \times \vec{F}_c = 0$
Angular momentum	$\vec{L} = \vec{r} \times \vec{p} = I\dot{\theta}$
Torque	$\vec{\tau} = \vec{r} \times \vec{F} = I\ddot{\theta}$
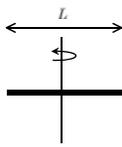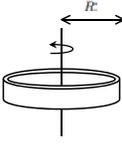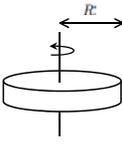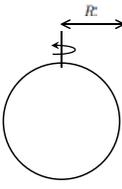
Moment of Inertia	$I_z = \int \rho(x^2 + y^2)dV$ $= \sum_i m_i(x_i^2 + y_i^2)$
Moment of inertia for a uniform bar (about COM)	$I = \frac{1}{12}ML^2$
Moment of inertia for a uniform hoop (about COM)	$I = MR^2$
Moment of inertia for a uniform disk (about COM)	$I = \frac{1}{2}MR^2$
Moment of inertia for a solid uniform sphere (about COM)	$I = \frac{2}{5}MR^2$
Parallel axis theorem	$I = I_{COM} + MR^2$
Taylor Expansion of f(x)	$f(x) = f(a) + \frac{1}{1!}\frac{df}{dx}\big|_a (x - a)$ $+ \frac{1}{2!}\frac{d^2f}{dx^2}\big|_a (x - a)^2 + \cdots$