Lecture 18

1. Derive δW for general screw pinch
2. Derive Suydams criterion

Screw Pinch Equilibria

$$\mu_0 p' + \frac{B_z^2}{2}' + \frac{B_\theta}{r}(rB_\theta)' = 0$$

$$\mu_0 J_\theta = -B_z'$$

$$\mu_0 J_z = \frac{1}{r}(rB_\theta)'$$

Stability

$$\underline{\xi} = \underline{\xi}(r) e^{\iota m\theta + \iota kz}$$

$$\underline{\xi} = \xi_r \underline{e}_r + \xi_\theta \underline{e}_\theta + \xi_z \underline{e}_z = \underline{\xi_\perp} + \xi_{||} \underline{b}$$

Note: $\underline{b} = \frac{B_\theta}{B} \underline{e}_\theta + \frac{B_z}{B} \underline{e}_z$

$$\underline{e}_\eta = \underline{b} \times \underline{e}_r = \frac{B_z}{B} \underline{e}_\theta - \frac{B_\theta}{B} \underline{e}_z$$

$\underline{e}_r, \underline{e}_\eta, \underline{b} \rightarrow$ orthogonal unit vectors

1. Carry out calculation in terms of $\xi, \eta, \xi_{||} \rightarrow \xi_r, \xi_\theta, \xi_z$

$$\xi_{||} = \xi_\theta \frac{B_\theta}{B} + \xi_z \frac{B_z}{B}$$

$$\underline{\xi} = \underline{\xi_\perp} + \xi_{||} \underline{b}$$

$$\eta = \xi_\theta \frac{B_z}{B} - \xi_z \frac{B_\theta}{B}$$

$$\underline{\xi_\perp} = \xi \underline{e}_r + \eta \underline{e}_\eta$$

$$\xi = \xi_r$$

2. Check Incompressibility

a. $\nabla\cdot\underline{\xi} = \nabla\cdot\underline{\xi_\perp} + \nabla\cdot\left(\frac{\xi_\parallel}{B}\underline{B}\right) = \nabla\cdot\underline{\xi_\perp} + \underline{B}\cdot\nabla\frac{\xi_\parallel}{B}$

b. $\underline{B}\cdot\nabla\ \text{scalar} = \left(\frac{B_\theta}{r}\frac{\partial}{\partial\theta} + B_z\frac{\partial}{\partial z}\right)\ \text{scalar} = \left(\frac{\iota m B_\theta}{r} + \iota k B_z\right)\ \text{scalar}$

Define $F = \frac{mB_\theta}{r} + kB_z = \underline{k}\cdot\underline{B}$, $\quad \underline{k} = \frac{m}{r}\underline{e}_\theta + k\underline{e}_z$

$\therefore \underline{B}\cdot\nabla\ \text{scalar} = \iota F\ \text{scalar}$

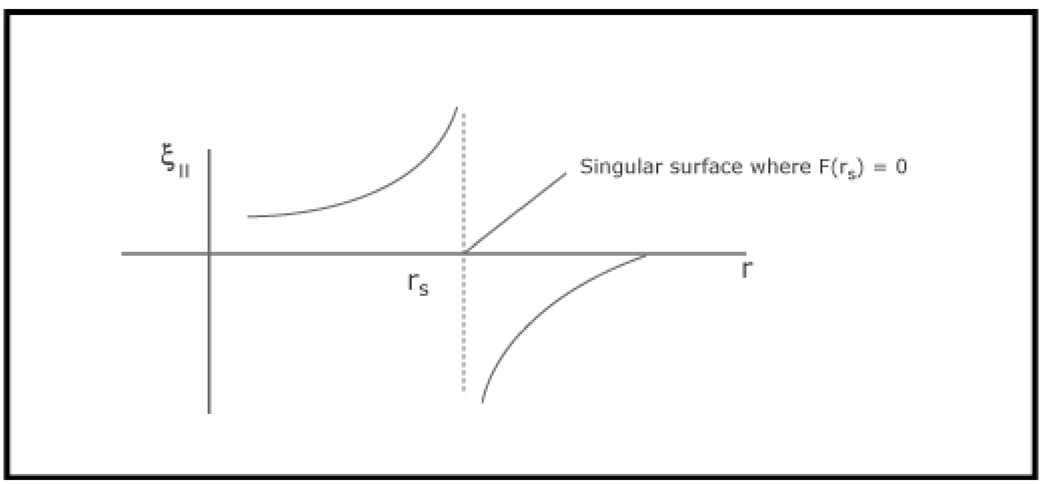
c. To make $\nabla\cdot\underline{\xi} = 0$ to minimize δW, we must choose $\xi_\parallel$ so that

$\nabla\cdot\underline{\xi_\perp} + \iota F\frac{\xi_\parallel}{B} = 0$ or

$$\boxed{\xi_\parallel = \frac{\iota B}{F}\nabla\cdot\underline{\xi_\perp}}$$

d. If k and m are such that $F \neq 0$ for $0 < r < a$, then $\xi_\parallel$ is bounded and we can choose $\nabla\cdot\underline{\xi} = 0$. This is the usual situation for external modes

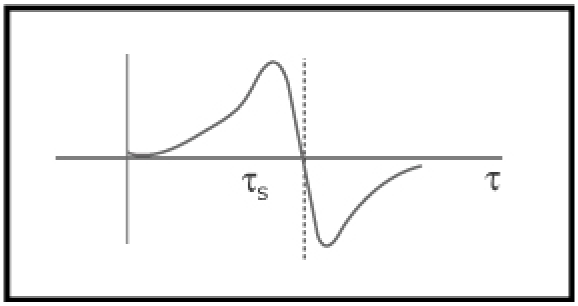
e. Suppose k and m are chosen so that F=0 at isolated internal points $0 < r < a$. Usual case for internal modes. $\xi_\parallel$ has the form



At $r_s, \xi_\parallel$ is not bounded (not allowable), but only at one point

f. Resolution: Choose $\xi_{\parallel} = \frac{\iota BF}{F^2 + \sigma^2} \nabla \cdot \xi_1$

$\xi_{\parallel}$ is now bounded, but $\nabla \cdot \underline{\xi}$ is no longer zero.



g. Calculate contribution to δW_F

$$\nabla \cdot \underline{\xi} = \nabla \cdot \underline{\xi_{\perp}} + \frac{\iota F \xi_{\parallel}}{B} = \nabla \cdot \underline{\xi_{\perp}} + \frac{\iota F}{B}\left(\frac{\iota BF}{F^2 + \sigma^2}\right)\nabla \cdot \underline{\xi_{\perp}} = \frac{\sigma^2}{F^2 + \sigma^2}\nabla \cdot \underline{\xi_{\perp}}$$

Assume now that σ is very small, but finite

Main contribution to $\gamma p \left|\nabla \cdot \underline{\xi}\right|^2$ comes from around $r = r_s$ where $F \approx 0$

h. Expand about $r = r_s$: $F = F(r_s) + F'(r_s)(r - r_s) \approx F'(r_s)\,x, \qquad x = r - r_s$, where $F(r_s) = 0$

$$\delta W_{\parallel} = \frac{1}{2}\int \gamma p \left|\nabla \cdot \underline{\xi}\right|^2 d\underline{r} = \frac{1}{2}\int \gamma p \left|\nabla \cdot \underline{\xi_{\perp}}\right|^2 \frac{\sigma^4}{\left(F^2 + \sigma^2\right)^2} r\, dr\, d\theta\, dz$$

$$= \pi L\left[\gamma p \left|\nabla \cdot \underline{\xi_{\perp}}\right|^2 r\right]_{r_s} \int dx \frac{\sigma^4}{\left(\sigma^2 + F'^2 x^2\right)^2}$$

$$= \pi^2 L\left[\frac{\gamma p \left|\nabla \cdot \underline{\xi_{\perp}}\right|^2 r}{\left|F'\right|}\right]_{r_s} \left|\sigma\right|$$

i. For small but finite $\sigma, \delta W_{\parallel} \to 0$

Conclusion: Even for isolated singular surfaces, the plasma compressibility term makes no contribution to δW

Minimize Remainder of δW

1. Separate terms

$$\underline{Q}_\perp = \left(\nabla \times \underline{\xi}_\perp \times \underline{B}\right)_\perp = Q_r \underline{e}_r + Q_\eta \underline{e}_\eta$$

$$Q_r = \iota F \xi$$

$$Q_\eta = \iota F \eta + \xi\left(\frac{B_z' B_\theta}{B} - r\frac{B_z}{B}\left(\frac{B_\theta}{r}\right)'\right)$$

2. $\underline{\kappa} = \underline{b} \cdot \nabla \underline{b} = -\dfrac{B_\theta^2}{rB^2}\underline{e}_r$

3. $\nabla \cdot \underline{\xi}_\perp + 2\underline{\xi}_\perp \cdot \underline{\kappa} = \dfrac{(r\xi)'}{r} - \dfrac{2B_\theta^2}{rB^2}\xi + \dfrac{\iota G \eta}{B}$ $\qquad G = \dfrac{mB_z}{r} - kB_\theta = \underline{e}_r \cdot \left(\underline{k} \times \underline{B}\right)$

4. $\left(\underline{\xi}_\perp \cdot \nabla p\right)\left(\underline{\xi}_\perp^* \cdot \underline{\kappa}\right) = -\dfrac{B_\theta^2}{rB^2}p'|\xi|^2$

5. $J_\parallel = \left(\underline{J} \cdot \underline{B}\right)/B = \dfrac{1}{B}\left[\dfrac{B_z}{r}(rB_\theta)' - B_\theta B_z'\right]$

6. $\underline{\xi}_\perp^* \times \underline{B} \cdot \underline{Q}_\perp = B\left(Q_r \eta^* - Q_\eta \xi^*\right)$

Substitute

$$\delta W_F = \frac{1}{2}\int d\underline{r}\left\{F^2|\xi|^2 + \left|\iota F_\eta + \xi\left[\frac{B_z' B_\theta}{B} - \frac{rB_z}{B}\left(\frac{B_\theta}{r}\right)'\right]\right|^2 \right. \qquad \text{line bending}$$

$$+B^2\left|\frac{(r\xi)'}{r} - \frac{2B_\theta^2}{rB^2}\xi + \frac{\iota G\eta}{B}\right|^2 \qquad \text{mag. comp.}$$

$$+\frac{2B_\theta^2}{rB^2}p'|\xi|^2 \qquad \text{pressure driven}$$

$$\left. -\frac{J_\parallel}{B}\left[B\left[\iota F\left(\xi\eta^* - \xi^*\eta\right)\right] - |\xi|^2\left[B_z' B_\theta - rB_z\left(\frac{B_\theta}{r}\right)'\right]\right]\right\} \qquad \text{kink}$$

Simplify

1. Note that η appears only algebraically. Complete the squares and minimize with respect to η

$$\boxed{\eta = \frac{i}{rk_0^2 B}\left[G(r\xi)' + 2kB_\theta \xi\right]}$$

$$k_0^2 = \frac{m^2}{r^2} + k^2$$

2. Remaining δW

$$\delta W_F = \underset{\theta}{\pi}\left(\underset{z}{2\pi R_0}\right)\int_0^a dr\left[a(r)\xi'^2 + b(r)\xi\xi' + c(r)\xi^2\right] \qquad (1)$$

   a. integrate (1) by parts

   b. lots of algebra, using equilibrium relation

3. Result:

$$\boxed{\begin{aligned} &\frac{\delta W_F}{2\pi^2 R_0/\mu_0} = \int_0^a dr\left[F\xi'^2 + g\xi^2\right] + \left[\frac{k^2r^2B_Z^2 - m^2B_\theta^2}{k_0^2r^2}\right]_a \xi^2(a) \\ &f = \frac{rF^2}{k_0^2} \\ &g = \frac{2k^2\mu_0 p'}{k_0^2} + \left(\frac{k_0^2r^2-1}{k_0^2r^2}\right)rF^2 + \frac{2k^2}{rk_0^4}\left(kB_z - \frac{mB_\theta}{r}\right)F \end{aligned}}$$

Complete Calculation by Computing $\delta W_s, \delta W_v$

1. Assume no surface currents: $\longrightarrow$ $\delta W_s = 0$

2. Vacuum Energy: $\delta W_v = \dfrac{1}{2\mu_0}\int \underline{\hat{B}}_1^2 d\underline{r}$ $\quad \nabla \times \underline{\hat{B}}_1 = \nabla \cdot \underline{\hat{B}}_1 = 0$

3. Write $\underline{\hat{B}}_1 = \nabla\phi_1$ with $\nabla^2\phi_1 = 0$

   B.C. a. Wall as $r = b \rightarrow \underline{n}\cdot\underline{\hat{B}}_1\Big|_b = 0 \quad \dfrac{\partial\phi_1}{\partial r}\Big|_b = 0$ (1)

b. $\underline{n}\cdot\underline{B}\big|_{a+\xi} = 0 \rightarrow \underline{n}\cdot\hat{\underline{B}}_1\big|_a = \underline{n}\cdot\nabla\times\left(\underline{\xi}_\perp\times B\right)\big|_a \qquad \left.\frac{\partial\phi}{\partial r}\right|_a = \iota F\xi(a) \qquad (2)$

Solution:

$$\phi_1 = \left[c_1 I_m(kr) + c_2 K_m(kr)\right]e^{\iota m\theta+\iota kz}$$

$$\frac{\partial\phi_1}{\partial r} = \left[kc_1\, I_m' + kc_2 K_m'\right]e^{\iota m\theta+\iota kz}$$

Choose c_1 and c_2 so that (1) and (2) are satisfied

Then $\delta W_v = \frac{1}{2\mu_0}\int\left|\hat{\underline{B}}_1^2\right| d\underline{r} = \frac{1}{2\mu_0}\int\nabla\phi^*\cdot\nabla\phi\, d\underline{r} = \frac{1}{2\mu_0}\int d\underline{r}\left[\nabla\cdot\left(\phi^*\nabla\phi\right) - \underbrace{\phi^*\nabla^2\phi}_{=0}\right]$

$$= \frac{1}{2\mu_0}\int dS\,\phi^*\,\hat{\underline{n}}\cdot\nabla\phi = -\frac{2\pi^2 R_0 a}{\mu_0}\left[\phi^*\,\frac{\partial\phi}{\partial r}\right]_a$$

Substitute

$$\frac{\delta W_v}{2\pi^2 R_0/\mu_0} = \left[\frac{r^2\Lambda F^2}{|m|}\right]_a \xi^2(a)$$

$$\Lambda = -\frac{|m|K_a}{kaK_a'}\left[\frac{1-\left(K_b' I_a\right)/\left(I_b' K_a\right)}{1-\left(K_b' I_a'\right)/\left(I_b' K_a'\right)}\right]$$

$$\approx \frac{1+(a/b)^{2|m|}}{1-(a/b)^{2|m|}} \quad kb \ll 1 \qquad \approx \frac{|m|}{ka} \quad \begin{matrix} ka\rightarrow\infty \\ kb\rightarrow\infty \end{matrix}$$

$$\approx 1 \quad \begin{matrix} kb\rightarrow\infty \\ ka\sim 1 \end{matrix}$$

Summary

δW for general screw pinch

$$\frac{\delta W}{2\pi^2 R_0/\mu_0} = \int_0^a\left[f\xi'^2 + g\xi^2\right]dr + \left[\left(\frac{krB_z - mB_\theta}{k_0^2 r^2}\right)rF + \frac{r_1^2\Lambda F^2}{|m|}\right]_a \xi^2(a)$$

internal modes: $\xi(a) = 0$

external modes: $\xi(a) \neq 0$

Suydam's Criterion

a. Necessary condition for stability

b. Tests against localized interchanges (external modes)

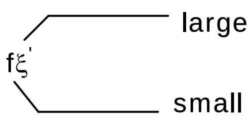
c. Only necessary, because a special "localized" trial function is used

Mathematical Motivation

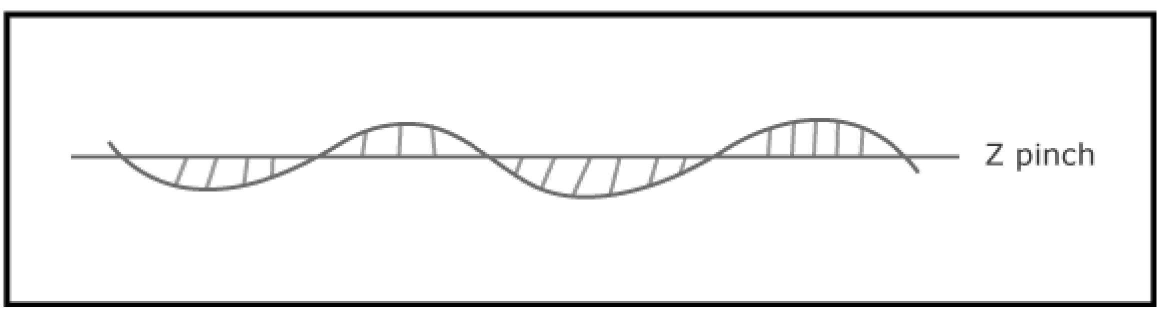
a. Choose k such that $F(r_s) = 0$ for same r_s in $0 < r_s < a$

b. Around this point $F \approx 0, g \approx \frac{2k^2}{k_0^2} p' < 0$ destabilizing

c. A localized mode can still give a finite contribution if ξ' is large.



Physical Motivation



a. interchange plasma and field: plasma wants to expand, field lines want to contract

b. interchange is more difficult with shear. As interchange takes place, <u>field lines are bent</u> from one surface to another.

Derivation

1. look as δW_F in the vicinity of $x = r - r_s$

2. assume internal mode so that $\xi(a) = 0$

3. assume localized internal mode $F(r) \approx F(r_s) + F'(r_s)x = F'(r_s)x$

$$\begin{matrix}\| \\ 0\end{matrix}$$

Then $f \approx \left[\dfrac{r^3 F'^2}{k^2 r^2 + m^2}\right]_{r_s} x^2$

$$g \approx \left[\frac{2k^2 r^2 p' \mu_0}{k^2 r^2 + m^2}\right]_{r_s}$$

and

$$\frac{\delta W_F}{2\pi^2 R_0 / \mu_0} = \left[\frac{r^3 F'^2}{k^2 r^2 + m^2}\right]_{r_s} \int dx \left[x^2 \left(\frac{d\xi}{dx}\right)^2 - D_s \xi^2\right]$$

$$D_s = -\left[\frac{2k^2 p' \mu_0}{r F'^2}\right]_{r_s}$$

4. Simplify D_s as $r = r_s$, $\left(kB_z + \dfrac{mB_\theta}{r}\right)_{r_s} = 0$ definition

5. Write $q(r) = \dfrac{rB_z}{R_0 B_\theta}$

Then $F(r) = kB_z\left(1 + \dfrac{mB_\theta}{krB_z}\right) = kB_z\left(1 + \dfrac{m}{kR_0}\dfrac{1}{q}\right)$

but, at $r = r_s$ $\quad \dfrac{kR_0}{m} = \left(\dfrac{R_0 B_\theta}{rB_z}\right)_{r_s} = \dfrac{1}{q(r_s)}$ resonant condition

so that $F(r) = kB_z(r)\left[1 - \dfrac{q(r_s)}{q(r)}\right]$

$$F'(r)\Big|_{r_s} = \underset{\substack{\| \\ 0}}{kB_z'\left[1 - \frac{q(r_s)}{q(r)}\right]_{r_s}} + kB_z(r_s)\, q(r_s)\left[\frac{q'}{q^2}\right]_{r_s}$$

$$= \left(kB_z \frac{q'}{q}\right)_{r_s}$$

$$\therefore \boxed{D_s = -\frac{2\mu_0 p' q^2}{r^2 B_z^2 q'^2}}$$ only a function of equilibrium quantities (no m's and k's)

6. $\delta W \propto \int dx \left(x^2 \xi'^2 - D_s \xi^2 \right)$

   a. if $p' > 0$, $D_s < 0 \rightarrow$ stability

   b. assume $p' < 0$ interesting case, $D_s > 0$ stability?

7. Vary $\xi \rightarrow \xi + \delta\xi$ to determine minimizing $\xi(r)$

$$\int dr \left(F\xi'^2 + g\xi^2 \right) \rightarrow \quad \left(F\xi' \right)' - g\xi = 0$$

$$\int dx \left[x^2 \xi'^2 - D_s \xi^2 \right] \rightarrow \boxed{\left(x^2 \xi' \right)' + D_s \xi = 0}$$

8. We can solve Euler–Lagrange equation: assume $\xi = x^P$

$$p(p+1) + D_s = 0$$

$$p_{1,2} = -\frac{1}{2} \pm \frac{1}{2}\left(1 - 4D_s\right)^{1/2}$$

9. Need to distinguish two cases: $D_s > 1/4$, $D_s < 1/4$

10. Note: $\int \left(x^2 \xi'^2 + D_s \xi^2 \right) dx = -x^2 \xi \xi' = -p x^{2p+1}$

$p > -\frac{1}{2}$ bounded $\rightarrow$ alternate function

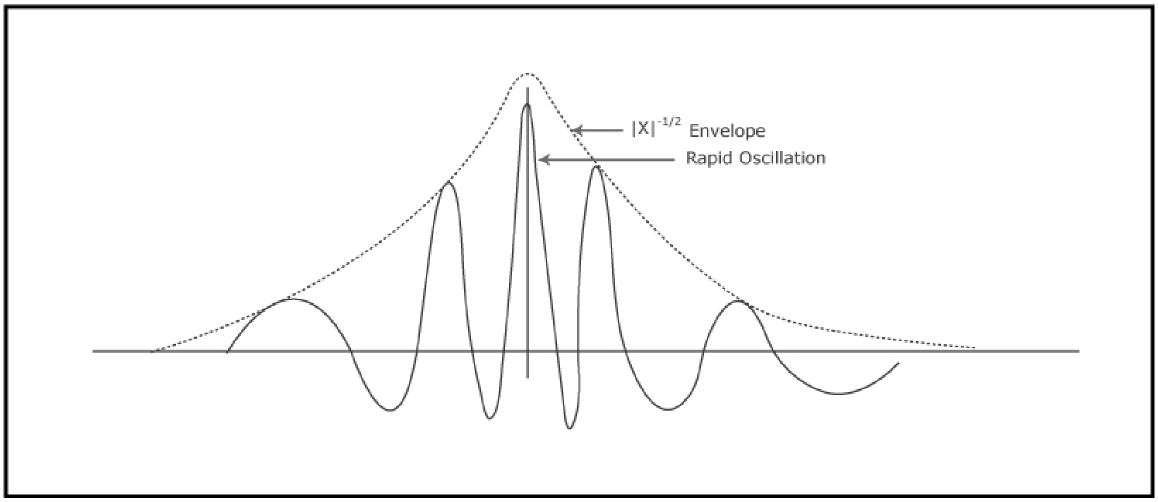
$p < -\frac{1}{2}$ unbounded $\rightarrow$ not allowable

$p = \frac{1}{2}$ logarithmic divergence $\rightarrow$ not bounded

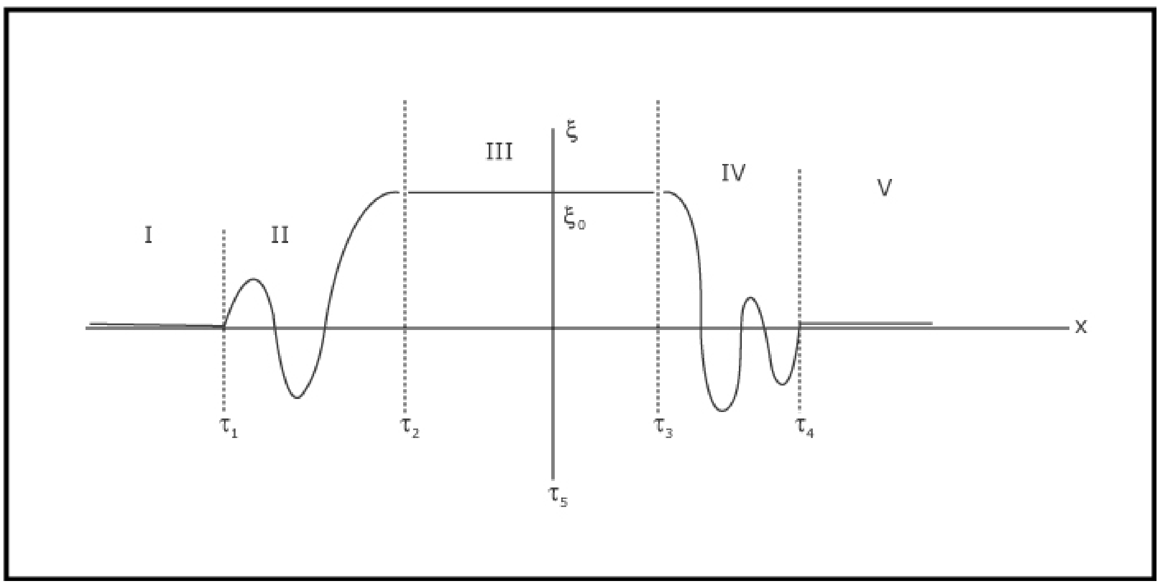
11. Consider $1 - 4D_s < 0$

$$\xi = \frac{1}{|x|^{1/2}} \left[c_1 \sin\left(k_r \ln|x|\right) + c_2 \cos\left(k_r \ln|x|\right) \right]$$

$$k_r = \frac{1}{2}\left(4D_s - 1\right)^{1/2}$$



12. Show oscillatory root always leads to instability by choosing a modified trial function



a. In I and V, $\xi = \xi' = 0 \rightarrow \delta W_I = \delta W_V = 0$

b. In II and IV ξ satisfies $\left(x^2\xi'\right)' + D_s\xi = 0$

$$0 = \int\left[\left(x^2\xi'\right)' + D_s\xi\right]\xi dx = \int dx\underbrace{\left[-x^2\xi'^2 + D_s\xi^2\right]}_{-\delta W} + x^2\xi\xi'$$

$$\therefore\ \delta W_{II} = x^2\xi\xi'\Big|_{r_1}^{r_2} = 0$$

$$\delta W_{IV} = x^2\xi\xi'\Big|_{r_3}^{r_4} = 0$$

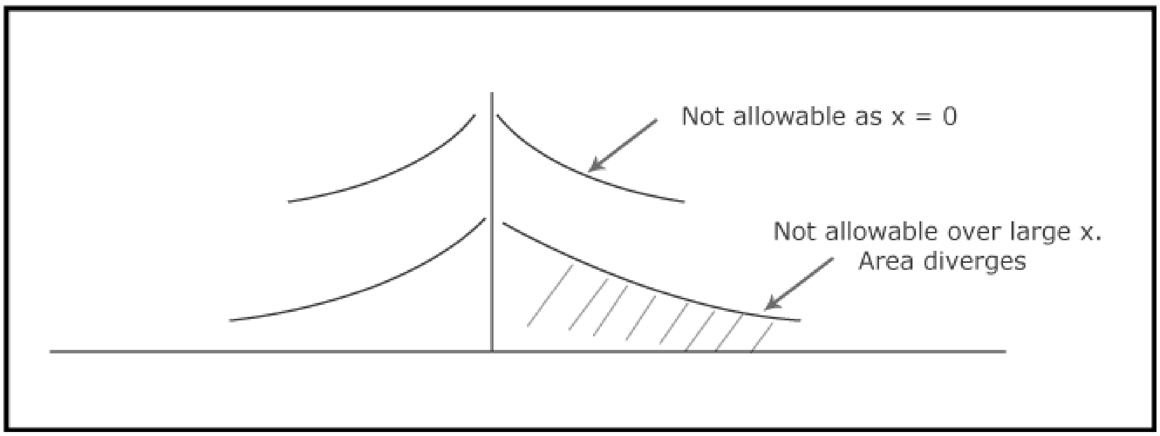
c. Region III $\xi = \xi_0 = \text{const},\ \xi' = 0$

$$\delta W_{III} = \int\left(x^2\xi'^2 - D_s\xi^2\right)dr = -D_s\xi_0^2\Delta r \qquad \Delta r = r_3 - r_2$$

d. by assumption $D_s > \frac{1}{4}$

$$\boxed{\therefore \delta W = -D_s\xi_0^2\Delta r < 0 \rightarrow \text{ instability}}$$

e. when $D_s < 1/4$ no oscillatory solutions exist. One root is not allowable, the other is allowable



Conclusion: when $D_s < 1/4$ not localized, oscillatory trial functions can be chosen. System is stable to localized interchanges

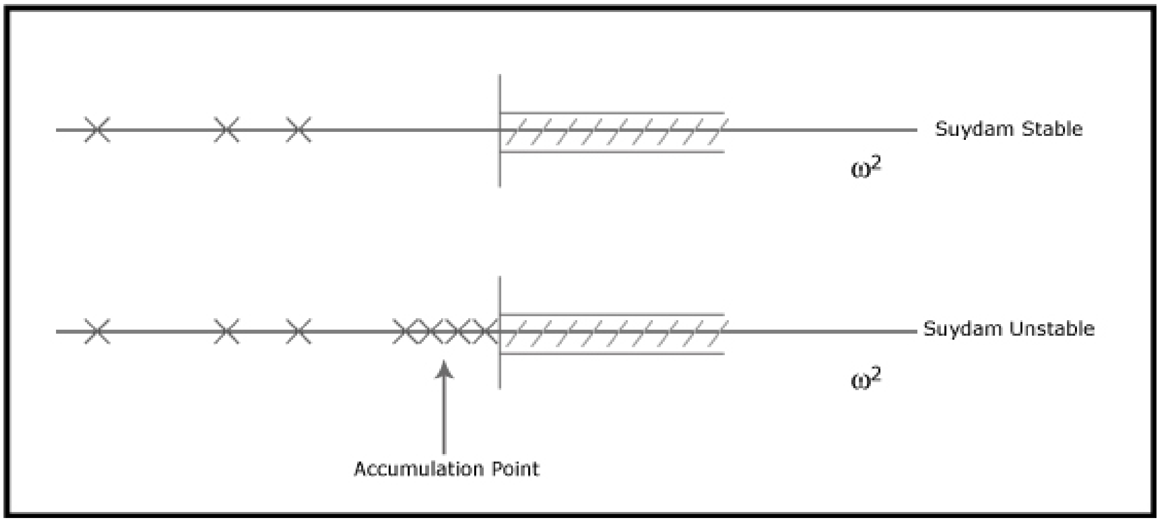
when $D_s > 1/4$ localized treat functions exist which make $\delta W < 0$

$D_s < \frac{1}{4}$ Suydams criterion

$$\boxed{rB_z^2\left(\frac{q'}{q}\right)^2 + 8\mu_0 p' > 0} \qquad \text{for stability}$$

Destabilizing term: $8\mu_0 p' \rightarrow$ pressure gradient, bad curvature

Stabilizing term: $rB_z^2 \frac{q'^2}{q^2} \rightarrow$ shear, line bending magnetic energy



Oscillation theorem

If suydams criterion is violated, there is always a zero mode, macroscopic mode with maximum growth rate.

This is significance of Suydams criterion.