22.611J, 6.651J, 8.613J

Practice problem Set Solution (#8)

1) The collision operator in Landau form is:
2)

$$C(f_\alpha, f_\beta) = \frac{2\pi e^4 \ln\Lambda}{m_\alpha^2 m} \vec{\nabla}_v \cdot \left[\int \frac{d^3v'}{g^3} (g^2 \bar{\bar{I}} - \vec{g}\vec{g}) \cdot (m_\beta f_\beta \vec{\nabla}_v f_\alpha - m_\alpha f_\alpha \vec{\nabla}_{v'} f_\beta) \right]$$

and so has the form

$$C(f_\alpha, f_\beta) = \Gamma \vec{\nabla}_v \cdot \left[\int \frac{d^3v'}{g^3} \cdots\cdots \right]$$

where $\vec{g} = \vec{v} - \vec{v}'$

See the last page for very helpful vector & tensor identities!

a) Particle conservation:

the easiest way to do this is to recognize that

since $f(\infty, \infty) = 0$

$$\int_{-\infty}^{\infty} d^3v \, \vec{\nabla}_v \cdot (\text{anything} \times f) = 0 \text{ always!}$$

Since the flux (i.e. $\vec{\nabla}_v$; divergence theorem) at $-\infty$ and ∞ has to be zero by definition!

hence, $\boxed{\int_{-\infty}^{\infty} d^3v \, C(f_\alpha, f_\beta) = 0}$ (for both like and un-like i.e. 1a) & 2a))

b) Momentum conservation:

We've

$$\vec{F}_{\alpha\beta} = \int d^3v \, m_\alpha \vec{v} \, C_{\alpha\beta}(f_\alpha, f_\beta)$$

where $f_\alpha = f_\alpha(v)$ & $f_\beta = f_\beta(v')$

1b) Con't

$$\vec{F} = \int d^3v\, m_\alpha \vec{v}\, \Gamma_{\alpha\beta} \vec{\nabla}_v \cdot \Big[\underbrace{\int \frac{d^3v'}{g^3} (g^2 \overline{\overline{I}} - \vec{g}\vec{g}) \cdot (m_\beta f_\beta \vec{\nabla}_v f_\alpha - m_\alpha f_\alpha \vec{\nabla}_{v'} f_\beta)}_{\text{represent this w/ } [\cdots]} \Big]$$

$$\vec{F} = \int d^3v\, m_\alpha \vec{v}\, \Gamma_{\alpha\beta} \vec{\nabla}_v \cdot [\cdots]$$

Integrating by parts, [redacted]

$$m_\alpha \Gamma_{\alpha\beta} \int d^3v\, \vec{v}\, \vec{\nabla}_v \cdot [\cdots]$$

$$+ m_\alpha \Gamma_{\alpha\beta} \int d^3v\, [\cdots] \cdot \vec{\nabla}_v \vec{v} = \underbrace{m_\alpha \Gamma_{\alpha\beta} \int d^3v\, \vec{\nabla}_v \cdot \vec{v}[\cdots]}_{=0}$$

so,

$$m_\alpha \Gamma_{\alpha\beta} \int d^3v\, \vec{v}\, \vec{\nabla}_v \cdot [\cdots] = -m_\alpha \Gamma_{\alpha\beta} \int d^3v\, [\cdots] \cdot \underbrace{\vec{\nabla}_v \vec{v}}_{\vec{\nabla}_v \vec{v} = \overline{\overline{I}}}$$

so,

$$\vec{F}_{\alpha\beta} = -m_\alpha \Gamma_{\alpha\beta} \int d^3v \Big[\int \frac{d^3v'}{g^3} (g^2 \overline{\overline{I}} - \vec{g}\vec{g}) \cdot (m_\beta f_\beta \vec{\nabla}_v f_\alpha - m_\alpha f_\alpha \vec{\nabla}_{v'} f_\beta) \Big]$$

$$\vec{F}_{\alpha\beta} = -m_\alpha \Gamma_{\alpha\beta} \Big\{ \int\int \frac{d^3v\, d^3v'}{g^3} (g^2 \overline{\overline{I}} - \vec{g}\vec{g}) \cdot (m_\beta f_\beta \vec{\nabla}_v f_\alpha) - \int \frac{d^3v\, d^3v'}{g^3} (g^2 \overline{\overline{I}} - \vec{g}\vec{g}) \cdot (m_\alpha f_\alpha \vec{\nabla}_{v'} f_\beta) \Big\}$$

b) Cont

i.e. $f_\alpha(v) = f_\beta(v)$

→ For like collisions (one species)
$m_\alpha = m_\beta$, interchange v and v' in the 1st term in $\vec{F}$ (since f_α & f_β are the same f except one depends on $\vec{v}$ & the other on $\vec{v}'$)(i.e. $\vec{v}$ & $\vec{v}'$ are dummy variables)
then, we've

$$\vec{F}_{\alpha\beta} \propto \left(\int \frac{d^3v\, d^3v'}{g^3} (g^2 \overline{\overline{I}} - \vec{g}\vec{g}) \cdot (m_\alpha f_\alpha \vec{\nabla}_{v'} f_\beta \quad) - \int \frac{d^3v\, d^3v'}{g^3} (g^2 \overline{\overline{I}} - \vec{g}\vec{g}) \cdot (m_\alpha f_\alpha \vec{\nabla}_{v'} f_\beta(v')) \right)$$

$$\boxed{\vec{F}_{\alpha\alpha} = 0 \qquad \text{conservation of momentum for one species.}}$$

→ For two species / un-like collisions
We need $\vec{F}_{\alpha\beta} + \vec{F}_{\beta\alpha} = 0$

$$\vec{F}_{\alpha\beta} = -m_\alpha \Gamma_{\alpha\beta} \left\{ \int \frac{d^3v\, d^3v'}{g^3} (g^2 \overline{\overline{I}} - \vec{g}\vec{g}) \cdot (m_\beta f_\beta \vec{\nabla}_v f_\alpha - m_\alpha f_\alpha \vec{\nabla}_{v'} f_\beta) \right\}$$

Here, $f_\alpha(v)$ & $f_\beta(v)$ are two different distributions; Hence, we can't just swith v & v' like we did above. (i.e. $f_\alpha(v) \neq f_\beta(v)$)

$$\vec{F}_{\beta\alpha} = -m_\beta \Gamma_{\beta\alpha} \left\{ \int \frac{d^3v\, d^3v'}{g^3} (g^2 \overline{\overline{I}} - \vec{g}\vec{g}) \cdot (m_\alpha f_\alpha \vec{\nabla}_{v'} f_\beta - m_\beta f_\beta \vec{\nabla}_v f_\alpha) \right\}$$

Hence, using

$$\boxed{\vec{F}_{\alpha\beta} + \vec{F}_{\beta\alpha} = 0}$$

Conservation of momentum for 2-species!

c) Energy Conservation

$$W_{\alpha\beta} = \int d^3v \, \frac{1}{2} m_\alpha v^2 C_{\alpha\beta}$$

$$= \int d^3v \, \frac{m_\alpha}{2} v^2 \Gamma_{\alpha\beta} \vec{\nabla}_v \cdot [\text{-----}]$$

Integrating by parts,

$$\int \frac{m_\alpha v^2}{2} \Gamma_{\alpha\beta} \vec{\nabla}_v \cdot [\text{----}] + \int \frac{m_\alpha}{2} [\text{----}] \Gamma_{\alpha\beta} \cdot \vec{\nabla}_v v^2$$

$$= \cancel{\int d^3v \, \frac{m_\alpha}{2} \Gamma_{\alpha\beta} \vec{\nabla}_v \cdot (v^2 [\text{-----}])} \to 0$$

since $\int d^3v \, \vec{\nabla}_v \cdot (\quad) = 0$

then $$W_{\alpha\beta} = - \int \frac{m_\alpha}{2} [\text{----}] \Gamma_{\alpha\beta} \cdot \underbrace{\vec{\nabla}_v v^2} \, d^3v$$

$$\vec{\nabla}_v v^2 = 2 \vec{v} \cdot \vec{\nabla}_v \vec{v}$$

$$v^2 = \vec{v} \cdot \vec{v} \qquad = 2 \vec{v} \cdot \overline{\overline{I}} = 2 \vec{v}$$

$$W_{\alpha\beta} = - \int m_\alpha \left(\vec{v} \cdot [\text{----}] \right) \Gamma_{\alpha\beta} \, d^3v$$

$$W_{\alpha\beta} = -\Gamma_{\alpha\beta} m_\alpha \int \frac{\vec{v} \cdot}{g^3} (g^2 \overline{\overline{I}} - \vec{g}\vec{g}) \cdot (m_\beta f_\beta \vec{\nabla}_v f_\alpha - m_\alpha f_\alpha \vec{\nabla}_{v'} f_\beta) \, d^3v \, d^3v'$$

→ like collisions (i.e. one species, $f_\alpha(v) = f_\beta(v)$)
$m_\beta = m_\alpha$
Interchange v & v' in both terms in $W_{\alpha\beta}$
take $\frac{1}{2}$ sum of original & interchanged $W_{\alpha\beta}$
and use $(\vec{v} - \vec{v}') \cdot (g^2 \overline{\overline{I}} - \vec{g}\vec{g}) = 0$

$$W_{\alpha\beta_I} = -\Gamma_{\alpha\beta} m_\alpha \int \frac{\vec{v}'}{g^3} \cdot (g^2 \overline{\overline{I}} - \vec{g}\vec{g}) \cdot (m_\alpha f_\alpha \vec{\nabla}_{v'} f_\beta - m_\beta f_\beta \vec{\nabla}_v f_\alpha) \, d^3v' \, d^3v$$

(Interchange v & v', okay since $f_\alpha = f_\beta$)

Again, $W_{\alpha\beta} = W_{\alpha\beta_I}$

$$W_{\alpha\beta} = \frac{1}{2}(W_{\alpha\beta} + W_{\alpha\beta_I})$$

$$\underbrace{\propto \int (\vec{v} - \vec{v}\,') \cdot \underbrace{(g^2 \overline{\overline{I}} - \vec{g}\vec{g})}_{=0} \cdot (\cdots\cdots)}_{=0}$$

Hence, $\boxed{W_{\alpha\beta} \Rightarrow W_{\alpha\alpha} = 0}$, energy is conserved.

→ 2 species; un-like collisions
We want $W_{\alpha\beta} + W_{\beta\alpha} = 0$

$$W_{\alpha\beta} = -\Gamma_{\alpha\beta} m_\alpha \int \frac{\vec{v}}{g^3} \cdot (g^2 \overline{\overline{I}} - \vec{g}\vec{g}) \cdot (m_\beta f_\beta \vec{\nabla}_v f_\alpha - m_\alpha f_\alpha \vec{\nabla}_{v'} f_\beta)$$

$$W_{\beta\alpha} = -\Gamma_{\beta\alpha} m_\beta \int \frac{\vec{v}\,'}{g^3} \cdot (g^2 \overline{\overline{I}} - \vec{g}\vec{g}) \cdot (m_\alpha f_\alpha \vec{\nabla}_{v'} f_\beta - m_\beta f_\beta \vec{\nabla}_v f_\alpha)$$

Hence, since $m_\alpha \Gamma_{\alpha\beta} = \Gamma_{\beta\alpha} m_\beta$

$$W_{\alpha\beta} + W_{\beta\alpha} \propto \int \underbrace{(\vec{v} - \vec{v}\,') \cdot (g^2 \overline{\overline{I}} - \vec{g}\vec{g})}_{=0} \cdot (\cdots\cdots)$$

so,

$\boxed{W_{\alpha\beta} + W_{\beta\alpha} = 0, \text{ and Energy is conserved for 2 species!}}$

3) From the kinetic equation,

$$\frac{\partial f_e}{\partial t} + \underbrace{\cancel{\vec{v}\cdot\frac{\partial}{\partial \vec{x}} f_e}}_{0 \text{ No spatial variation}} - \frac{e}{m}\left(\underbrace{\cancel{\vec{E} + \tfrac{1}{c}\vec{v}\times B}}_{0 \text{ No fields}}\right)\cdot\frac{\partial}{\partial \vec{v}} f_e = C_e(f_e)$$

$$\rightarrow \frac{\partial f_e}{\partial t} = C_e(f_e)$$

Now, assume $f_e = f_e^{max}$ & $f_i = f_i^{max}$ (but different T's)

$$\underbrace{\int d^3v \frac{1}{2} m_e v^2 \frac{\partial f_e^m}{\partial t}}= \int d^3v \frac{1}{2} m_e v^2 C_e(f_e^m)$$

$$= \frac{\partial}{\partial t}\frac{3}{2} n_e T_e = \int d^3v \frac{1}{2} m_e v^2 C_e(f_e^m)$$

Letting $f_e^m = A e^{-\frac{v^2}{v_e^2}}$, w/ $A = \frac{n}{(2\pi v_e)^{3/2}}$

$$C_e(f_e) \cong \underbrace{\cancel{C_{ei}^L(f_e)}}_{=0 \text{ since isotropic and no drifts}} + C_{ei}^E(f_e) = \nu_{ei}\frac{m_e}{2m_i} v_e^3\left[\frac{1}{v^2}\frac{\partial}{\partial v} + \frac{T_i}{m_e v^2}\frac{\partial}{\partial v}\left(\frac{1}{v}\left(\frac{\partial v}{\partial v}\right)\right)\right] f_e^m$$

We've

$$\frac{\partial f_e^m}{\partial v} = -\frac{2v}{v_e^2} A e^{-\frac{v^2}{v_e^2}} = -\frac{2v}{v_e^2} f_{em}$$

&

$$\frac{\partial}{\partial v}\left(\frac{1}{v}\left(\frac{\partial f_e^m}{\partial v}\right)\right) = \frac{\partial}{\partial v}\left(-\frac{2}{v_e^2} A e^{-\frac{v^2}{v_e^2}}\right) = \frac{4v}{v_e^4} A e^{-\frac{v^2}{v_e^2}}$$

3) Con't

$$C_{ei}^{E}(f_e) = \frac{\nu_{ei} m_e}{2m_i} v_e^3 \left[-\frac{2}{v v_e^2} f_e^m + \frac{T_i}{m_e v^2} \frac{4v^2}{v_e^4} f_e^m \right]$$

$$= \frac{\nu_{ei} m_e}{2m_i} \left[-\frac{2v_e}{v} f_e^m + \frac{T_i 4}{m_e v v_e} f_e^m \right]$$

then,

$$\int d^3v \frac{1}{2} m_e v^2 C_e(f_e^m) =$$

$$= \int d^3v \frac{1}{2} m_e v^2 \frac{\nu_{ei} m_e}{2m_i} \left[\frac{4T_i}{m_e v v_e} - \frac{2v_e}{v} \right] f_e^m$$

$$= \int d^3v \frac{1}{4} \frac{m_e^2 \nu_{ei}}{m_i} \left[\frac{4 T_i v}{m_e v_e} - 2 v_e v \right] f_e^m$$

$$= \int d^3v \frac{m_e^2 \nu_{ei}}{m_i} \left[\frac{T_i}{m_e v_e} - \frac{v_e}{2} \right] v f_e^m$$

$$= A \frac{m_e^2 \nu_{ei}}{m_i} \left[\frac{T_i}{m_e v_e} - \frac{v_e}{2} \right] 4\pi \int_0^\infty v^3 e^{-\frac{v^2}{v_e^2}} dv$$

$$= A \frac{m_e^2 \nu_{ei}}{m_i} \left[\frac{T_i}{m_e v_e} - \frac{v_e}{2} \right] 4\pi \frac{\Gamma(2)}{2\left(\frac{1}{v_e^2}\right)^2}$$

$$= A \frac{m_e^2 \nu_{ei}}{m_i} \left[\frac{T_i}{m_e v_e} - \frac{v_e}{2} \right] \frac{4\pi}{2} v_e^4$$

$$= \frac{n}{(2\pi v_e^2)^{3/2}} \frac{m_e^2 \nu_{ei}}{m_i} \left[\frac{T_i}{m_e} v_e^3 - \frac{v_e^5}{2} \right] \frac{4\pi}{2}$$

$$= \frac{n}{(2\pi)^{3/2}} \frac{m_e^2 \nu_{ei}}{m_i} \left[\frac{T_i}{m_e} - \frac{v_e^2}{2} \right] \frac{4\pi}{2}$$

3) Con't

Hence,

$$\frac{\partial}{\partial t}\frac{3}{2} n_e T_e = \frac{n}{(2\pi)^{3/2}} \frac{m_e}{m_i} \nu_{ei} \left[T_i - \frac{m_e v_E^2}{2} \right] \frac{2\cancel{4}\pi}{\cancel{x}}$$

$$\boxed{\frac{\partial}{\partial t}\frac{3}{2} n_e T_e = \frac{-n}{(2\pi)^{1/2}} \frac{m_e}{m_i} \nu_{ei} \left[-T_i + T_e \right]}$$

temperature equilibration!

Relative rate of angle scattering vs. thermalization:

$$\frac{C_{ei}^{L}(f_e^{m})}{C_{ei}^{E}(f_e^{m})} \approx \frac{\tau_p}{\tau_E} \approx \frac{m_i}{m_e}$$

$\vec{\nabla}_v(\vec{v}\cdot\vec{v}) = \vec{v}\cdot\vec{\nabla}_v\vec{v} + \vec{v}\cdot\vec{\nabla}_v v$

Course 22.616 | Vector Manipulations | P. J. Catto

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If you keep track of the $\cdot$'s, $\times$'s, and what the ∇'s operate on then you normally only have to remember the "bac-cab" identity:

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}\,\vec{a}\cdot\vec{c} - \vec{c}\,\vec{a}\cdot\vec{b}$$

Helpful sheet from transport!

Examples:

(1) $\nabla\cdot(S\vec{V}) = \vec{V}\cdot\nabla S + S\nabla\cdot\vec{V}$
(∇ operates on both S and $\vec{V}$, and ∇ and $\vec{V}$ dot)

(2) $\nabla(\vec{a}\cdot\vec{b}) = \nabla\vec{a}\cdot\vec{b} + \nabla\vec{b}\cdot\vec{a}$
(∇ operates on both $\vec{a}$ and $\vec{b}$, and $\vec{a}$ and $\vec{b}$ dot)

(3) $\vec{a}\times(\nabla\times\vec{c}) = (\nabla\vec{c})\cdot\vec{a} - \vec{a}\cdot\nabla\vec{c} = \nabla\vec{c}\cdot\vec{a} - \vec{a}\cdot\nabla\vec{c}$
(∇ only operates on $\vec{c}$, first term must be in the ∇ "direction" with dot between $\vec{a}$ and $\vec{c}$, and the second must be in the $\vec{c}$ direction with dot between $\vec{a}$ and ∇)

(4) $\nabla\times(\vec{b}\times\vec{c}) = \nabla\cdot(\vec{c}\,\vec{b}) - \nabla\cdot(\vec{b}\,\vec{c}) = \vec{b}\nabla\cdot\vec{c} + \vec{c}\cdot\nabla\vec{b} - \vec{c}\nabla\cdot\vec{b} - \vec{b}\cdot\nabla\vec{c}$
(∇ operates on both $\vec{b}$ and $\vec{c}$, first term must be in the $\vec{b}$ "direction" with dot between ∇ and $\vec{c}$, and the second must be in the $\vec{c}$ direction with dot between ∇ and $\vec{b}$)

More examples in velocity space: Consider $g \equiv |\vec{g}| \equiv (\vec{g}\cdot\vec{g})^{1/2}$ then

$$\nabla_g\vec{g} = \vec{I} \qquad \nabla_g\cdot\vec{g} = \nabla_g\vec{g}:\vec{I} = \vec{I}:\nabla_g\vec{g} = 3$$

$$\nabla_g g = \nabla_g(\vec{g}\cdot\vec{g})^{1/2} = \vec{g}/g \qquad \nabla_g(1/g) = -\vec{g}/g^3$$

$$\nabla_g\cdot(g^2\vec{I}) = \nabla_g g^2 = 2\vec{g}$$

$$\nabla_g\nabla_g g = \nabla_g(\vec{g}/g) = g^{-1}\nabla_g\vec{g} - g^{-2}(\nabla_g g)\vec{g} = g^{-3}(g^2\vec{I} - \vec{g}\vec{g})$$

$$\nabla_g\cdot\nabla_g g = \nabla_g^2 g = g^{-1}\nabla_g\cdot\vec{g} - g^{-2}\vec{g}\cdot\nabla_g g = 2/g$$

$$\nabla_g\cdot\nabla_g\nabla_g g = g^{-3}\nabla_g\cdot(g^2\vec{I} - \vec{g}\vec{g}) = g^{-3}(2\vec{g} - 3\vec{g} - \vec{g}) = -2\vec{g}/g^3$$

note $\vec{g}\cdot(g^2\vec{I} - \vec{g}\vec{g}) = (g^2\vec{I} - \vec{g}\vec{g})\cdot\vec{g} = 0$