PS #7 22.611J, 6.651J, 8.613J Fall 2002

Answer Key

(The Landau prob. & Coll. Dissipation-II, Dec 3, 2002)

1) From the notes, the linearized e^- plasma waves consists of:

(In cgs units)

$$\frac{\partial \tilde{f}_e}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{x}} \tilde{f}_e + \frac{e}{m_e} \frac{\partial \tilde{\phi}}{\partial \vec{x}} \cdot \frac{\partial}{\partial \vec{v}} \bar{f}_e = 0 \quad (1)$$

$$-\nabla^2 \tilde{\phi} = 4\pi e \int d^3v \, \tilde{f}_e \quad (2)$$

In our case, we've to add a point source test charge at $x=0$ and $t=0$. Hence, (2) changes to:

$$-\nabla^2 \tilde{\phi} = 4\pi e \int d^3v \, \tilde{f}_e + 4\pi q_T \, \delta(\vec{x}) H(t) \quad (3)$$

where $H(t)$ is the Heaviside function representing a step function at $t=0$.

Now, Fourier-Laplace transforming, we've

$$-f_k(t=0, \vec{v}) - i(\omega - \vec{k}\cdot\vec{v}) f_{k\omega}(\vec{v}) + i\frac{e}{m}\phi_{k\omega} \, \vec{k} \cdot \frac{\partial}{\partial \vec{v}} \bar{f}_e = 0 \quad (4)$$

$$k^2 \phi_{k\omega} + 4\pi e \int d^3v \, f_{k\omega}(\vec{v}) + \frac{4\pi q_T}{i\omega} = 0 \quad (5)$$

Since $\mathcal{L}[H(t)] = \dfrac{-e^{i\omega t_0}}{i\omega}$ where $t_0 = 0$ for our case

and $\mathcal{F}[\delta(\vec{x})] = 1$

so, rearranging (4) & (5), (as in pg. 1 → 2 of the notes (12/3/02))

$$k^2 \epsilon \, \phi_{k\omega} = -4\pi e \int d^3v \frac{i}{\omega - \vec{k}\cdot\vec{v}} f_k(0, \vec{v}) - \frac{4\pi q_T}{i\omega}$$

defining $$S_0(\vec{k}, \omega) = -4\pi e \int \frac{d^3v \, i \, f_k(0, \vec{v})}{\omega - \vec{k}\cdot\vec{v}}$$

and $$S_T(\vec{k}, \omega) = \frac{4\pi i q_T}{\omega}$$

We get

$$k^2 \epsilon \phi_{k\omega} = S_0 + S_T$$

$$\Rightarrow \boxed{\phi_{k\omega} = \frac{S_0 + S_T}{k^2 \epsilon}}$$ (as requested in # 1)

where $\epsilon(\vec{k},\omega) = 1 + \frac{1}{k^2 \lambda_D^2}\left[1 + \frac{\omega}{k v_e} Z\left(\frac{\omega}{k v_e}\right)\right]$

→ Now, we've to find Φ at $t \to \infty$.
We can do this by invoking the final value theorem; for Φ at $t \to \infty$, $\phi_{k\omega}$ goes to $\omega \to 0$.

→ hence, we see that $S_0(\vec{k},\omega)$ does not contribute compared w/ $S_T(\vec{k},\omega)$, since for $\omega \to 0$,

$$S_0 \to -4\pi e \int d^3v \frac{i f_k}{-\vec{k}\cdot\vec{v}}$$

but

$$S_T \to \frac{4\pi i q_T}{0} \to \infty$$

(also, $Z\left(\frac{\omega}{k v_e}\right)\frac{\omega}{k v_e}$ as $\omega \to 0$ gives $Z \sim 0$)

So, for $\omega \to 0$, the inversion formula for ϕ is simply

$$\underset{w/\, t \gg 0}{\phi(\vec{x},t)} = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d^3k\, e^{i\vec{k}\cdot\vec{x}}\left(\frac{1}{2\pi}\int_C d\omega\, e^{-i\omega t}\frac{S_T(k,\omega)}{k^2\left(1 + \frac{1}{k^2\lambda_D^2}\right)}\right)$$

Let's do the Laplace inversion first....

$$\phi_k(t) = \frac{1}{2\pi}\int_C d\omega\, e^{-i\omega t}\frac{4\pi i q_T}{\omega k^2\left(1 + \frac{1}{k^2 \lambda_D^2}\right)}$$

w/ $t \gg 0$

$$\phi_k(t) = \frac{i2q_T}{(k^2+\frac{1}{\lambda_D^2})}\int_C d\omega\, e^{-i\omega t}\frac{1}{\omega}$$

so we've a simple pole at $\omega = 0$

using the residue formula, we get

$$\int_C d\omega \frac{e^{-i\omega t}}{\omega} = 2\pi i\left(\lim_{\omega\to 0}(\cancel{\omega}-0)\frac{e^{-i\omega t}}{\cancel{\omega}}\right) = 2\pi i$$

$$\phi_k(t) = \frac{-4\pi q_T}{k^2+\frac{1}{\lambda_D^2}}$$ (...this Fourier transform is looking a lot like PS # 6's #2 !)

w/ $t \gg 0$

For the Fourier inversion, we've

$$\phi(\vec{x},t) = \frac{-1}{(2\pi)^3}\int_{-\infty}^{\infty} d^3k\, e^{i\vec{k}\cdot\vec{x}}\left(\frac{4\pi q_T}{k^2+\frac{1}{\lambda_D^2}}\right)$$

which is exactly the same integral we did in PS #6, problem #2 ! (letting $k_0^2 = \frac{1}{\lambda_D^2}$)

$$\boxed{\phi(\vec{x},t) = \frac{e^{\left(-\frac{r}{\lambda_D}\right)}}{r} \quad \text{for } t \gg 0}$$

which is, of course, just Debye shielding.

→ Note on $t \gg 0$:

this is really just the time for Debye shielding to occur, which is of order $\tau \sim \frac{\lambda_D}{v_e}$ .... which is clearly a very short time in terms of real-world applications!

We can also get Debye shielding the old way:
Using Eq. (2),

$$-\nabla^2\tilde{\phi}(\vec{x},t) = 4\pi e\int d^3v\,\tilde{f}_e$$

$$\text{w/ } \tilde{f}_e = f_{max}\frac{e\tilde{\phi}}{T_e}$$

(the adiabatic response)

$$-\nabla^2\tilde{\phi} - 4\pi\frac{e^2\tilde{\phi}}{T_e}\int d^3v\, f_{max} = 0$$

$$\nabla^2\tilde{\phi} + \frac{4\pi e^2 n_e}{T_e}\tilde{\phi} = 0$$

$$\nabla^2\tilde{\phi} + \frac{\tilde{\phi}}{\lambda_D^2} = 0$$

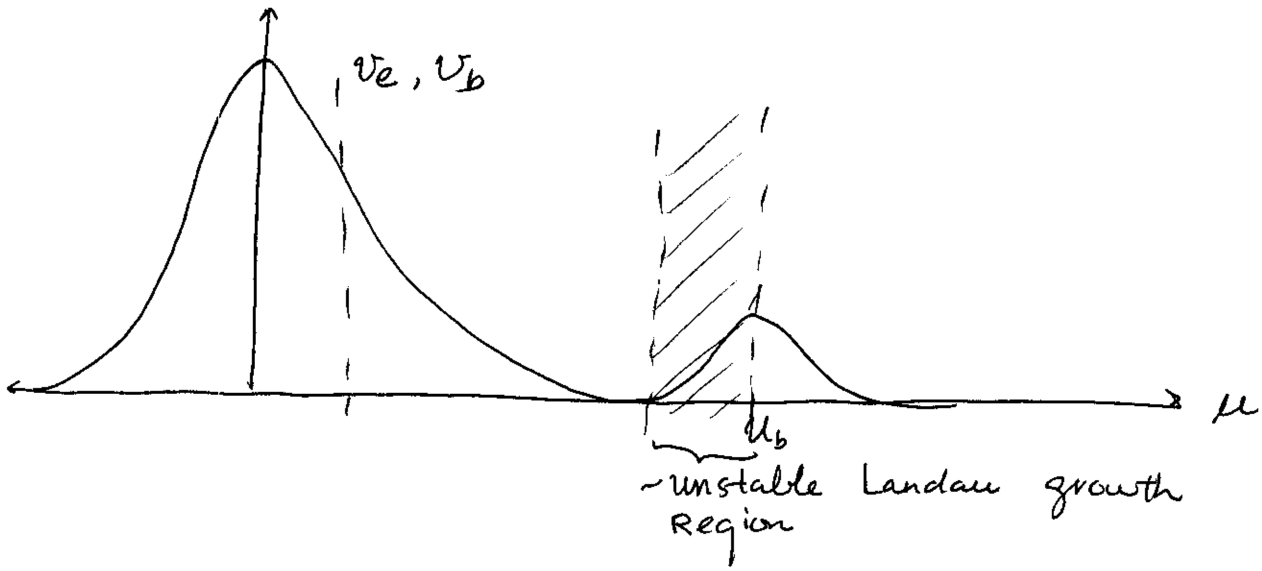
(which is again our formula for debye shielding)

$$\boxed{\tilde{\phi} = \frac{e^{-\frac{r}{\lambda_D}}}{r}}$$

→ What this solution represents in terms of physics:

- Debye shielding is a fundamental process in plasmas — it does not ~~com~~ require collisions ....
- We probably shouldn't be surprised to get this out of kinetic theory, since the fluid theory we used origially to get the λ_D shielding comes right out of kinetic theory through taking moments!

2) Bump-on-tail Instability



→ find approximate ω/k location of unstable growth

→ We can find the approximate lower u bound for instability by finding where the beam contribution to the distribution function roughly equals the bulk's. (The upper bound is simply u_b)

Hence, find u for

$$\frac{(n_p + n_b)}{\pi^{3/2} v_e^3} \exp\left(\frac{-u^2}{v_e^2}\right) \sim \frac{n_b}{\pi^{3/2} v_b^3} \exp\left(\frac{-(u-u_b)^2}{v_b^2}\right)$$

w/ $v_e \sim v_b$, $u_b \gg v_e$ & v_b ← orderings

so, using $v_e \sim v_b$,

$$(n_p + n_b) \exp\left(\cancel{\frac{-u^2}{v_e^2}}\right) \sim n_b \exp\left(\frac{-\cancel{u^2} + 2uu_b - u_b^2}{v_b^2}\right)$$

2) Con't

$$(n_p + n_b) \sim n_b \exp\left(\frac{2uu_b - u_b^2}{v_b^2} \right)$$

$$\ln\left(\frac{n_p + n_b}{n_b}\right) \sim \frac{2uu_b - u_b^2}{v_b^2}$$

$$\frac{v_b^2}{2u_b}\ln\left(\frac{n_p+n_b}{n_b}\right) + \frac{u_b^{\cancel{2}}}{2} \sim \cancel{2}u\cancel{u_b}$$

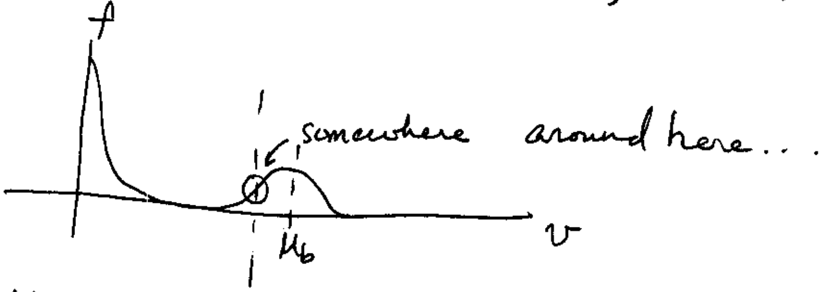
but $n_p \gg n_b$,

$$\underbrace{\frac{v_b^2}{2u_b}\ln\left(\frac{n_p}{n_b}\right)}_{\text{small number}} + \frac{u_b}{2} \sim u_{\text{lower bound}}.$$

so, the conditions for instability is roughly

$$\boxed{\frac{1}{2}\left[u_b + \frac{v_b^2}{u_b}\ln\left(\frac{n_p}{n_b}\right)\right] < \frac{\omega}{k} < u_b}$$

Neglecting the Bulk contribution because $u_b \gg v_e$, the maximum instability will occur when $\frac{df_b}{du}$ reaches a maximum; i.e:



Hence, we need to take two derivatives for $f_b := \frac{n_b}{\pi^{3/2} v_b^3} \exp\left(-(u-u_b)^2/v_b^2\right)$

2) con't

$$\frac{df_b}{du} = \frac{-2(u-u_b)}{v_b^2}\frac{n_b}{\pi^{3/2}v_b^3}\exp\left(-\frac{(u-u_b)^2}{v_b^2}\right)$$

$$\underbrace{0 = \frac{d^2 f_b}{du^2}}_{\text{set to zero for max } \frac{df_b}{du}} = \frac{d}{du}\left((u-u_b)\exp\left(-\frac{(u-u_b)^2}{v_b^2}\right)\right)$$

$$= \exp\left(-\frac{(u-u_b)^2}{v_b^2}\right) + u\left(-\frac{2(u-u_b)}{v_b^2}\right)\exp\left(-\frac{(u-u_b)^2}{v_b^2}\right)$$

$$- u_b\left(\frac{-2(u-u_b)}{v_b^2}\right)\exp\left(-\frac{(u-u_b)^2}{v_b^2}\right)$$

We then have (after dividing out $\exp\left(-\frac{(u-u_b)^2}{v_b^2}\right)$)

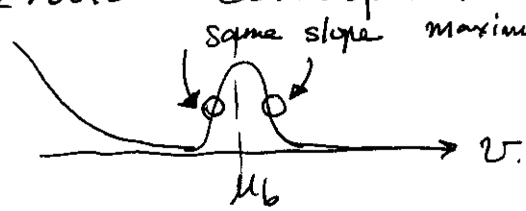
$$0 = 1 - \frac{2u(u-u_b)}{v_b^2} + \frac{2u_b(u-u_b)}{v_b^2}$$

$$0 = v_b^2 - 2u^2 + 2uu_b + 2u_bu - 2u_b^2$$

$$0 = (v_b^2 - 2u_b^2) + 4uu_b - 2u^2$$

$$u = \frac{-4u_b \pm \left(16u_b^2 + 8(v_b^2 - 2u_b^2)\right)^{\frac{1}{2}}}{-4}$$

→ the 2 roots correspond to



→ we want the smaller one....

2) Con't

$$\boxed{u_{mI} = u_b - \left(u_b^2 + \frac{1}{2}(v_b^2 - 2u_b^2)\right)^{\frac{1}{2}}}$$ is the velocity where maximum instability occurs.

→ From pg. 10-12 of the 12/03/02 notes, we've

$$\omega_r \approx \omega_{pe}\left(1 + \frac{3}{2}k^2\lambda_{De}^2\right)$$

$$\omega = \omega_r + i\gamma$$

$$\frac{\gamma}{\omega_r} = -\frac{\epsilon_I}{\omega_r\, \partial\epsilon_R/\partial\omega_r}$$

and

$$\epsilon(k,\omega_r) = 1 - \underbrace{\frac{\omega_{pe}^2}{k^2} P\left(\int_{-\infty}^{\infty} du \frac{1}{u - \omega_r/k}\frac{\partial F}{\partial u}\right)}_{} - i\pi \frac{\omega_{pe}^2}{k^2}\frac{k}{|k|}\frac{\partial F}{\partial u}\left(\frac{\omega_r}{k}\right)$$

$$= \frac{\omega_{pe}^2}{\omega_r^2}\left(1 + \frac{3}{2}\frac{k^2 v_e^2}{\omega_r^2}\right)$$ if we assume that the bulk contributes only to the principle value...

also, $$\frac{\partial F}{\partial u}(u_{mI}) = \frac{-2(u_{mI} - u_b)n_b}{v_b^5\, \pi^{3/2}} \exp\left(-\left(\frac{u_{mI} - u_b}{v_b}\right)^2\right)^2$$

(we neglect the bulk here because $\frac{\omega_r}{k} = u_{mI} \gg v_e$)

So now, we've

$$\epsilon(k,\omega_r) = 1 - \frac{\omega_{pe}^2}{\omega_r^2}\left(1 + \frac{3}{2}k^2\frac{v_e^2}{\omega_r^2}\right) - i\pi\frac{\omega_{pe}^2}{k^2}\frac{k}{|k|}\frac{\partial F}{\partial u}(u_{mI})$$

w/

$$\omega_r = \omega_{pe}\left(1 + \frac{3}{2}k^2\lambda_{De}^2\right)$$

Now, we have to eavaluate $\gamma = -\frac{\epsilon_I \omega_r}{\omega_r\, \partial\epsilon_R/\partial\omega_r}$ to determine the growth rate.

from the notes,

$$\omega_r \frac{\partial \epsilon_R}{\partial \omega_r} \cong 2$$

$$\frac{\gamma}{\omega_r} = \frac{-\epsilon_I}{2}$$

$$\frac{\gamma}{\omega_r} = \frac{+i\pi \frac{\omega_{pe}^2}{k^2}\frac{k}{|k|}\left(\frac{-\cancel{2}(u_{mI}-u_b)n_b}{v_b^5\,\pi^{3/2}}\right)e^{\left(-\frac{(u_{mI}-u_b)^2}{v_b}\right)^2}}{\cancel{2}}$$

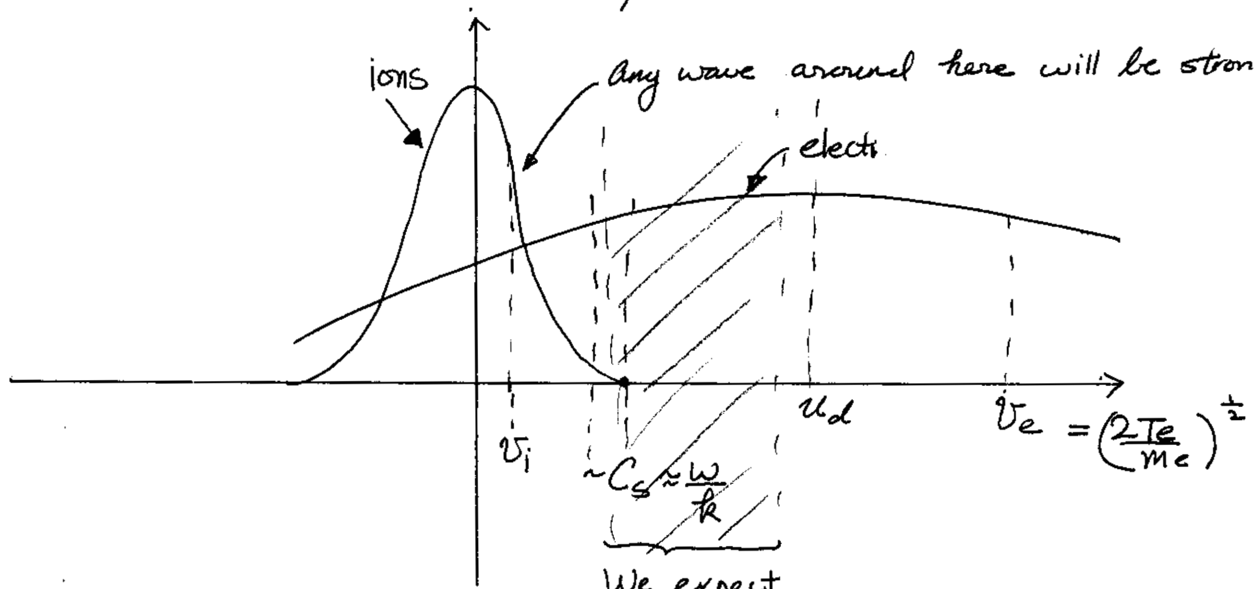
for most unstable growth.

$$\boxed{\begin{aligned}\gamma &\cong -i\pi\frac{\omega_{pe}^3(1+\frac{3}{2}k^2\lambda_{De}^2)}{k|k|}\left(\frac{(u_{mI}-u_b)n_b}{v_b^5\,\pi^{3/2}}\right)e^{\left(\frac{-(u_{mI}-u_b)^2}{v_b}\right)^2}\\ &\text{w/}\quad u_{mI} = u_b-\left(u_b^2+\tfrac{1}{2}(v_b^2-2u_b^2)\right)^{\frac{1}{2}}\end{aligned}}$$

$\therefore \operatorname{Im}(\gamma)$ is possitive since $u_{mI}-u_b<0$ hence unstable.

→ remember that ω has to be $\sim \omega_{pe}$ for these modes.... hence we have a choice of k, but not ω.

3) e^- current instability



We expect unstable waves to occur around here ... (little ion damping & positive e^- slope)

→ Our ordering is: $v_i < C_s < u_d < v_e$

$$v_i < \frac{\omega_r}{k} << v_e$$

→ We can get $\frac{\gamma}{\omega_r}$ by using the Z functions:

$$\epsilon = 1 + \chi_e + \chi_i$$

where

$$\chi_e = \frac{1}{k^2\lambda_e^2}\left[1 + \frac{(\omega - ku_d)}{kv_e} Z\left(\frac{\omega - ku_d}{kv_e}\right)\right]$$

since $\omega \to \omega - ku_d$ for a shifted Maxwellian

$$\chi_e = \frac{1}{k^2\lambda_e^2}\left[1 + \left(\cancelto{0}{\frac{\omega}{kv_e}} - \frac{\cancel{k}u_d}{\cancel{k}v_e}\right) Z\left(\cancelto{0}{\frac{\omega}{kv_e}} - \frac{u_d}{v_e}\right)\right]$$

to leading order

$$\chi_e = \frac{1}{k^2\lambda_e^2}\left[1 - \frac{\cancel{k}u_d}{v_e} Z\left(-\frac{u_d}{v_e}\right)\right]$$

Using small Z expansion, and keeping only the 1st term,

$$\chi_e = \frac{1}{k^2\lambda_e^2}\left[1 - \frac{u_d}{v_e} i\sqrt{\pi}\, e^{\left(\frac{u_d}{v_e}\right)^2}\right]$$

3) Con't

Now, determine χ_i

$$\chi_i = \frac{1}{k^2\lambda_i^2}\left[1 + \frac{\omega}{kv_i} Z\left(\frac{\omega}{kv_i}\right)\right]$$

$$\cong \frac{1}{k^2\lambda_i^2}\left[1 + \frac{C_s}{v_i} Z\left(\frac{C_s}{v_i}\right)\right] \; ; \quad \frac{C_s}{v_i} \gg 1, \text{ using large expansion}$$

$$\chi_i \cong \frac{1}{k^2\lambda_i^2}\left[1 + \frac{C_s}{v_i} \underbrace{2i\sqrt{\pi}}_{} e^{-\left(\frac{C_s}{v_i}\right)^2}\right]$$

2 here because $Im\left(\frac{\omega}{kv_i}\right) < 0$ for damping

So now, we've

$$\epsilon = \left(1 + \frac{1}{k^2\lambda_i^2} + \frac{1}{k^2\lambda_e^2}\right) + i\sqrt{\pi}\left[\frac{2C_s}{v_i}\frac{e^{-\left(\frac{C_s}{v_i}\right)^2}}{k^2\lambda_i^2} - \frac{u_d}{v_e}\frac{e^{\left(\frac{u_d}{v_e}\right)^2}}{k^2\lambda_e^2}\right]$$

As before,

$$\frac{\gamma}{\omega_r} = \frac{-\epsilon_I}{\omega_r \frac{\partial \epsilon_R}{\partial \omega_r}} \quad \& \text{ since } \quad \frac{\omega_r}{k} = C_s = \left(\frac{T_e}{m_i}\right)^{\frac{1}{2}}$$

$$\frac{\gamma}{\omega_r} \approx -\frac{k^2\lambda_e^2}{2}\epsilon_I$$

$$\frac{\gamma}{\omega_r} = -\frac{i\sqrt{\pi}\,\cancel{k^2}\lambda_e^2}{2}\left[\frac{2C_s}{v_i}\frac{e^{-\left(\frac{C_s}{v_i}\right)^2}}{\cancel{k^2}\lambda_i^2} - \frac{u_d}{v_e}\frac{e^{\left(\frac{u_d}{v_e}\right)^2}}{\cancel{k^2}\lambda_e^2}\right]$$

$$\boxed{\frac{\gamma}{\omega_r} = \frac{i\sqrt{\pi}}{2}\left[-\frac{2C_s}{v_i}\frac{\lambda_e^2}{\lambda_i^2}e^{-\left(\frac{C_s}{v_i}\right)^2} + \frac{u_d}{v_e}e^{\left(\frac{u_d}{v_e}\right)^2}\right]}$$

ion term (damping) — e^- term (destablilizing)

so, if $\frac{\gamma}{\omega_r} > 0$, we get instability.

3) Con't

Sub. the Temperature dependences in:

$$\frac{\gamma}{\omega_r} = \frac{i\sqrt{\pi}}{2}\left[-\frac{2\,T_e^{1/2}\,\cancel{m_i^{1/2}}}{\cancel{m_i^{1/2}}(2T_i)^{1/2}}\frac{T_e}{T_i}\,e^{-\left(\frac{T_e\,\cancel{m_i}}{\cancel{m_i}\cdot 2T_i}\right)} + \frac{u_d\,m_e^{1/2}}{(2T_e)^{1/2}}\,e^{\left(\frac{u_d^2 m_e}{2T_e}\right)}\right]$$

$$\frac{\gamma}{\omega_r} = \frac{i\sqrt{\pi}}{2}\left[-2\left(\frac{T_e}{T_i}\right)^{3/2} e^{-\left(\frac{T_e}{2T_i}\right)} + u_d\left(\frac{m_e}{(2T_e)}\right)^{1/2} e^{\left(\frac{u_d^2 m_e}{2T_e}\right)}\right]$$

hence, we get $\frac{\gamma}{\omega_r} > 0$ for $T_e \gg T_i$ since the ion damping term drops out due to the exponential dependence.

In the limit $T_e \gg T_i$,

$$\frac{\gamma}{\omega_r} = \frac{i\sqrt{\pi}}{2}\left[u_d\left(\frac{m_e}{2T_e}\right)^{1/2} \cancel{e^{\left(\frac{u_d^2 m_e}{2T_e}\right)}}^{\,1}\right]$$

$$\boxed{\frac{\gamma}{\omega_r} = \frac{i\sqrt{\pi}}{2}\frac{u_d}{v_e}}$$ for $T_e \gg T_i$, leading order $\omega_r \sim kC_s$

→ What range of real frequencies and wavenumbers do we expect?

∴ from our picture and discussion above, we know that $\frac{\omega}{k} \sim C_s$ .... anything above it would result in non-(ion-acoustic) waves (that're less & less affected by f_i as $\frac{\omega}{k} \to$ large) and anything below would result in strongly damped ion waves.

→ We can make an estimate of the range based on the ω_r relationship for ion-acoustic waves

→

3)

Rewriting ω_r,

$$\omega_r \Rightarrow 1 + \frac{1}{k^2\lambda_e^2} - \frac{\omega_{pi}^2}{\omega^2} = 0$$

$$\omega_r = \frac{\omega_{pi} k \lambda_e}{(1+k^2\lambda_e^2)^{\frac{1}{2}}}$$

two limits :

$k\lambda_e \ll 1$, $\omega_r = \omega_{pi} k \lambda_e = k C_s$

$k\lambda_e \gg 1$ $\omega_r = \omega_{pi}$

Hence, we've limits on the range of ω_r

But, if $k\lambda_e \gg 1$, $v_\phi \rightarrow \frac{\omega_{pi}}{k}$

resulting in strong damping (since $v_\phi \sim v_i$)

Therefore, another limit on k is

$$k \lesssim \frac{\omega_{pi}}{v_i} \sim \frac{1}{\lambda_i}$$

→ Overall, the range of $\frac{\omega}{k}$ is roughly

$C_s \lesssim \frac{\omega}{k} \ll u_d$

→ to get real limits on both ω & k would require specifying either ω_r or k, or solving for all the waves $\epsilon(k,\omega)$'s and matching γ_{growth} & γ_{decay}.