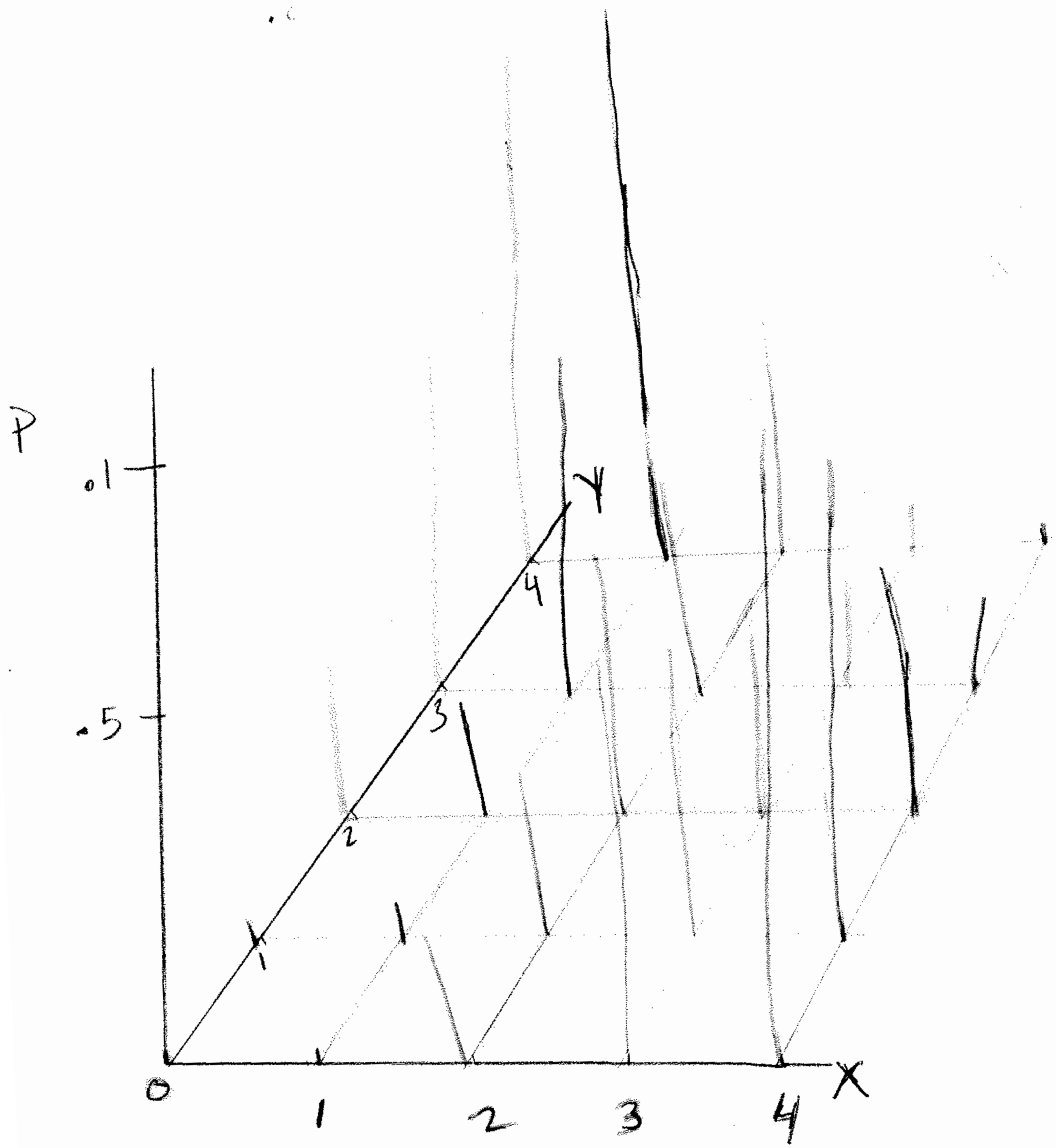
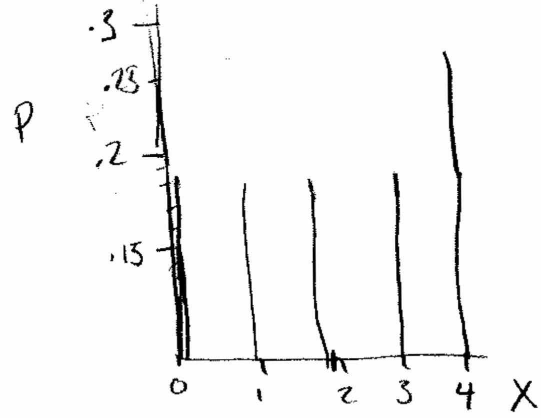
22.38 - PS # 5 - Solutions

3-61) a) joint PMF (X, Y):



3-61) b) marginal PMF of X:

$$P_X(x) = \sum_{\text{all } y_i} P_{X,Y}(X, y_i)$$



$P_X(0) = .18045$
$P_X(1) = .1789$
$P_X(2) = .1819$
$P_X(3) = .19697$
$P_X(4) = .2677$

Marginal PMF_X

c) $P(A|B) = \frac{P(AB)}{P(B)} \Rightarrow X=3, \ P(Y)=?$

$$P_X(3) = \sum_{i=0}^{4} P_{X,Y}(3, y_i) = .19097$$

$$P_Y(Y|X=3) = \frac{P_{X,Y}(3,Y)}{.19097}$$

$\Longrightarrow$ PMF_Y given $X=3$ = $P_Y(0) = .354$, $P_Y(1) = .252$, $P_Y(2) = .197$, $P_Y(3) = .118$, $P_Y(4) = .079$

d) $P(X|Y=4) = \frac{P_{XY}(X,4)}{P_Y(Y=4)} \Rightarrow P_Y(Y=4) = .24812$

$$P(X=4|Y=4) = \frac{.00752}{.24812} = \boxed{.0303}$$

e) $\text{cov}(X,Y) = E(XY) - E(X)E(Y)$

$E(X) = \sum_{x=i} P_X(x_i|y_j)$ for all y_j $= 2.186$

$E(XY) = \sum_i \sum_j x_i y_j P_{XY}(x_i, y_j) = 3.344$; $E(Y) = \sum_{y=i} P_Y(x_j|y_i)$ for all x_j $= 2.13$

$\text{cov}(X,Y) = 3.344 - 2.186(2.13) = \underline{-1.3199} \Rightarrow$ X & Y are not statistically independent.

$\rho = \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y} \Rightarrow \sigma_A = \sqrt{\sum_{\text{all } A_i} (A_i - \mu_A)^2 P_A(A_i)} \Rightarrow \sigma_X = \sqrt{2.1276}$

$\sigma_Y = \sqrt{2.1438}$

X & Y exhibits a fairly strong linear correlation, and, As expected, the correlation is negative.

$$\Rightarrow \rho = \frac{-1.3199}{\sqrt{2.13(2.14)}} = \boxed{-.618}$$

3-62) $f_{X,Y}(x,y) = 2ye^{-y(2+x)}$

a) $P_{XY}(X \le \$100k,\ Y \le \$200k) = ?$

$F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(u,v)\,du\,dv = \int_0^x\int_0^y 2v\,e^{-v(2+u)}\,du\,dv$

$= \int_0^y -2\left(e^{-v(2+x)} - e^{-2v}\right)dv = \frac{+2e^{-v(2+x)}}{(2+x)}\Big|_0^y - e^{-2v}\Big|_0^y$

$F_{XY}(x,y) = \frac{2}{2+x}\left(e^{-y(2+x)} - 1\right) - \left(e^{-2y} - 1\right)$

$F(1,2) = .31667$

b) $PDF_X:\ f_X(x) = \int_0^\infty f_{XY}(x,y)\,dy = \int_0^\infty 2ye^{-y(2+x)}\,dy$

$f_X(x) = \frac{2ye^{-y(2+x)}}{-(2+x)} - \frac{2}{(2+x)^2}e^{-y(2+x)}\Big|_{0}^{\infty}$

$f_X(x) = \frac{2}{(2+x)^2}$

c) $PDF_Y:\ f_Y(y) = \int_0^\infty f_{XY}(x,y)\,dx = 2e^{-2y}$

d) $E(XY) = \int_0^\infty\int_0^\infty dx\,dy\, xy\left(\frac{2}{2+x}\left(e^{-y(2+x)} - 1\right) - e^{-2y} - 1\right) \neq \int_0^\infty x\left(\frac{2}{(2+x)}\right)dx \int_0^\infty y\left(2e^{-2y}\right)dy = E(X)E(Y)$

$\therefore$ They are not independent.

e) $P_Y(y > 2 \mid X = 2) = 1 - P_Y(y \le 2 \mid x = 2) = 1 - \frac{P_{xy}(2,2)}{P_X(2)}$

$P(2,2) = F(2,2) = .4819$

$P_X(2) = \int_0^2 \frac{2}{(2+x)^2} = \frac{2}{(2+x)}\Big|_2^0 = 1 - \frac{1}{2} = .5$

$P_Y(y > 2 \mid x = 2) = 1 - (.4819/.5) = .0363$

R & H:

2.2) a) $X_{50} = 100 \Rightarrow \int_0^{100} \lambda e^{-\lambda t} dt = .5$

$\Rightarrow \underline{\lambda = 6.9 \times 10^{-3}}$

b) $P(t=500) = R(500) = e^{-\lambda(500)} = \underline{.032}$

c) $P(t=1000 \mid \text{surving at } 500) = P(t=500) = \underline{.032}$

$\Rightarrow$ Note: for an exponential distribution the failure rate is constant, and so the function is "memoryless"

2.8) a) $P(T>200) = R(200) = \exp\left(-\int_0^{200} Kt\,dt\right) = \underline{.961}$

b) $MTTF = \int_0^{\infty} R(t)\,dt = \int_0^{\infty} e^{-Kt^2/2}\,dt = \underline{886.2 \text{ hours}}$

c) $R(x \mid t) = P(T > x+t \mid T>t) = \frac{P(T>x+t)}{P(T>t)} = \frac{R(x+t)}{R(t)}$

where $R(t) = e^{-Kt^2/2}$

$R(200 \mid 200) = \frac{R(400)}{R(200)} = \underline{.8868}$

d) $R(t) = e^{-Kt^2/2} = e^{-\left(\sqrt{\frac{K}{2}}\,t\right)^2}$

it looks like a weibull distribution where $\lambda = \sqrt{\frac{K}{2}}$, $\alpha = 2$
(specifically it looks like a Rayliegh distribution)

where $MTTF = \frac{1}{\lambda}\Gamma\left(\frac{1}{\alpha}+1\right)$

$= \sqrt{\frac{K}{2}}\,\Gamma(1.5)$

$= \underline{886.2 \text{ hr}}$ which is what we expect from (b)

2.9 (R&H)

a) $R(t) = \exp\left(-\int_0^t z(u)\,du\right) = \exp \int_0^t -(\lambda_0 + \alpha t)\,dt$

$= \exp\left[-\lambda_0 t - \frac{\alpha t^2}{2}\right] \Rightarrow e^{-\left(\lambda_0 t + \frac{\alpha t^2}{2}\right)} = R(t)$

b) $MTTF = \int_0^\infty R(t)\,dt = \int_0^\infty e^{-(\lambda_0 t + \alpha t^2/2)}\,dt$

$= e^{\frac{\lambda^2}{4\alpha}} \sqrt{\pi}\, \mathrm{Erf}\left[\frac{1+2\alpha t}{2\sqrt{\alpha}}\right] / 2\alpha$

c) $z(t)$ is linear, so this models fatigue of parts over time (the end of the bath tub curve). The distribution looks like a product of an exponential and weibull distribution.

2.27) $f(t) = \frac{1}{b-a}$ for $a < t \leq b$

- $R(t) = \int_t^b \frac{1}{b-a}\,du = \frac{b}{b-a} - \frac{t}{b-a} = \frac{b-t}{b-a}$ $a < t \leq b$

- $z(t) = f(t)/R(t) = \frac{1}{b-t}$ for $a < t < b$