Final Exam

3 Hours. Closed Book. No written or electronic aids.

Finish as many questions as possible in the time.

21 Oct 2013, 9am to 12noon. NW14-1112.

14% 1. Reduce the following ordinary differential equation to a first-order vector differential equation, which you should write out completely, in vector format.

$$\left(\frac{d^3y}{dx^3}\right)^2 - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} - y^3 = 0.$$

18% 2. Consider an approximate discrete step in x and y, starting at x_n, y_n of the ODE $dy/dx = f(y,x)$. The Taylor expansion of the derivative function along the solution in terms of $\delta x = x - x_n$ is

$$f(y(x),x) = f_n + \frac{df_n}{dx}\delta x + \frac{d^2 f_n}{dx^2}\frac{\delta x^2}{2!} + \ldots. \tag{1}$$

Subscript n on f and its derivatives denotes evaluated at x_n, y_n. The approximate scheme is the following for the step from x_n to $x_{n+1} = x_n + \Delta x$:

"Evaluate $y^{(1)} = y_n + f_n\,\frac{\Delta x}{2}$, then take the step to be $y_{n+1} = y_n + f(y^{(1)}, x_n + \frac{\Delta x}{2})\,\Delta x$."
Document the accuracy of this scheme, using the notation $x_n + \frac{\Delta x}{2} = x_{n+\frac{1}{2}}$ as follows.

(a) Express the exact solution for $y(x)$ as a Taylor expansion.

(b) Express the quantity $y^{(1)} - y(x_n + \Delta x/2)$ in terms of the Taylor expansion.

(c) Express $f(y^{(1)}, x_{n+\frac{1}{2}}) - f(y(x_{n+\frac{1}{2}}), x_{n+\frac{1}{2}})$ to lowest order in $y^{(1)} - y(x_{n+\frac{1}{2}})$ using $\frac{\partial f}{\partial y}$.

(d) Hence find an expression for y_{n+1} correct to third order in Δx, and state the order to which this scheme is accurate.

18% 3. A diffusion equation in 2 dimensions with suitably normalized time units is

$$\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2},$$

on a finite domain with fixed ψ on the boundary. It is to be advanced in time using an *explicit* scheme.

$$\psi_{j,k}^{(n+1)} - \psi_{j,k}^{(n)} = \Delta t\,\mathbf{D}\psi^{(n)}.$$

where $\psi^{(n)}$ denotes the value at the nth time step. The matrix $\mathbf{D}$ represents the finite difference form of the spatial differential operator ∇^2 on a uniform grid with spacing Δx and Δy in the x and y directions, whose indices are j,k.

(a) Write out the right-hand-side ($\Delta t\mathbf{D}\psi^{(n)}$) of the above discrete difference equation in terms of a stencil of coefficients (whose values you should specify) times values $\psi_{j,k}$ at adjacent j,k positions, to complete the formulation of the difference scheme.

(b) Consider a particular Fourier mode $\propto \exp(ik_x x)\exp(ik_y y)$. Substitute it into the difference equation, and rearrange the resultant into the form $\psi^{(n+1)} = A\psi^{(n)}$. Hence find the amplification factor, A.

(c) Deduce the condition that Δt must satisfy to make this mode stable.

(d) By deciding which k_x and k_y are the most unstable, deduce the requirement on Δt for the whole scheme to be stable.

18% 4. Consider the partial differential system in time t and one spatial coordinate x

$$\frac{\partial}{\partial t}\boldsymbol{u} + \frac{\partial}{\partial x}\boldsymbol{f} = 0$$

where in terms of the components of $\boldsymbol{u}$ (which, incidentally, is not a velocity):

$$\boldsymbol{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \qquad \boldsymbol{f} = \begin{pmatrix} v \\ v^2/u + w \\ -kv \end{pmatrix},$$

with k a constant. Use the chain rule of spatial differentiation of $\boldsymbol{f}(\boldsymbol{u})$ to write the equations as

$$\frac{\partial}{\partial t}\boldsymbol{u} = -\mathbf{J}\frac{\partial}{\partial x}\boldsymbol{u}.$$

(a) Find the entire 3×3 matrix $\mathbf{J}$ and write it out in tabular form.

(b) Find the eigenvalues of $\mathbf{J}$.

(c) Under what conditions is this system hyperbolic?

(d) Assuming these conditions are satisfied, what are the characteristic speeds of propagation of disturbances?

(e) If a suitable explicit discrete finite difference scheme is used to solve this system numerically, then it is stable provided that the Courant-Friedrichs-Lewy (CFL) condition is satisfied. Unless you have lots of unused time, don't derive this condition for any particular scheme. Instead, just state how it relates Δt, Δx and the characteristic speeds of propagation.

14% 5. A random variable is required, distributed on the interval $0 \le x \le 1$ with probability distribution $p(x) = 2(1 - x)$. A library routine is available that returns a uniform random variate y (i.e. with uniform probability $0 \le y \le 1$). Give formulas and an algorithm to obtain the required randomly distributed x value from the returned y value.

18% 6. (a) Write out Boltzmann's equation governing the velocity distribution function $f(t, x, v)$ in time, t, and one-dimension in space x, and velocity v, for particles subject to a positive uniform constant acceleration a, which collide with a uniform background of stationary targets of density n_2 that do nothing but absorb the particles with a cross-section, σ, independent of velocity.

(b) Sketch in phase space (x, v) the paths of the trajectories ("orbits") of the particles.

(c) Obtain the equation of the trajectories in the form $v_0 = g(x, v)$, where v_0 is the velocity on the orbit at position $x = 0$, and $g(v, x)$ is a (relatively simple) function of x and v, which you must find.

(d) Prove that

$$f(x, v) = f_0(g(x, v)) \exp(-n_2 \sigma x)$$

is a solution of the steady-state ($\partial/\partial t = 0$) Boltzmann equation. The function $f_0(v_0)$ is the distribution function at $x = 0$.

(c) If $f_0(v_0) = 1/(1+v_0^2)$ for $v_0 > 0$, then find the distribution function $f(x, v)$ at position $x > 0$ and velocity v such that v_0 is real, in steady state

(d) If there are no particle sources in the positive half-plane $x > 0$, what is the value of $f(x, v)$ in steady state for $x > 0$, when v is such that v_0 is imaginary? Why?

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