Electromagnetic Interactions[1]

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Chapter 1

Maxwell's Equations and Electromagnetic Fields

1.1 Introduction

1.1.1 Maxwell's Equations (1865)

The governing equations of electromagnetism

$$\nabla.\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla.\mathbf{B} = 0$$

(Coulomb's Law)

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \wedge \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

(Faraday's Law)   (Ampere's Law)

$$(1.1)$$

$\mathbf{E}$ electric field, describes the force felt by a (stationary) charge q: $\mathbf{F} = q\mathbf{E}$
$\mathbf{B}$ magnetic field, describes the force felt by a current
i.e. a moving charge (velocity $\mathbf{v}$): $\mathbf{F} = q\mathbf{v} \wedge \mathbf{B}$

Thus the *Lorentz Force* (on charge q) is

$$\mathbf{F} = q\,(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \qquad (1.2)$$

ρ electric charge density (Coulombs/m^3). Total charge $Q = \int_V \rho d^3 x$
$\mathbf{j}$ electric current density (Coulombs/s/m^2)
Current crossing area element $\mathbf{dA}$ is $\mathbf{j} \,.\, \mathbf{dA}$ Coulomb/s = Amps.

1.1.2 Historical Note

Much scientific controversy in 2nd half of 19th century concerned question of whether $\mathbf{E}$, $\mathbf{B}$ were 'real' physical quantities of science or else mere mathematical conveniences for expressing the forces that charges exert on one another. English science (Faraday, Maxwell)
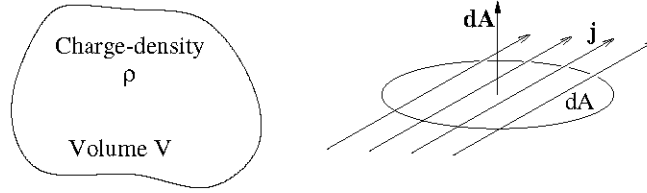
Figure 1.1: Charge density is local charge per unit volume. Current density is current per unit area.

emphasized the fields; German mostly the act-at-a-distance. Since ~ 1900 this question has been regarded as settled in favor of the fields. And modern physics, if anything, tends to regard the field as more fundamental than the particle.

1.1.3 Auxiliary Fields and Electromagnetic Media

Electromagnetic texts often discuss two additional "auxiliary" fields $\mathbf{D}$ the "electric displacement" and $\mathbf{H}$ the "magnetic intensity" which account for dielectric and magnetic properties of materials. These fields are not fundamental and introduce unnecessary complication and possible confusion for most of our topics. Therefore we will avoid them as much as possible. For the vacuum, $\epsilon_0 \mathbf{E} = \mathbf{D}$ and $\mathbf{B} = \mu_0 \mathbf{H}$.

1.1.4 Units

Historically there were two (or more!) <u>different</u> systems of units, one defining the quantity of charge in terms of the force between two stationary <u>charges</u> (the "Electrostatic" units) and one defining it in terms of forces between (chargeless) <u>currents</u> (the "Electromagnetic" system). Electrostatic units are based on Coulomb's law $\nabla.\mathbf{E} = \rho/\epsilon_0$ and electromagnetic units on the (steady-state version of) Ampere's law $\nabla \wedge \mathbf{B} = \mu_0 \mathbf{j}$. The quantities $1/\epsilon_0$ and μ_0 are therefore fundamentally calibration factors that determine the size of the unit charge. Choosing one or other of them to be 4π amounts to choosing electrostatic or electromagnetic units. However, with the unification of electromagnetism, and the subsequent realization that the speed of light is a fundamental constant, it became clear that the units of electromagnetism ought to be defined in terms of only one of these laws and the speed of light. Therefore the "System Internationale" SI (or sometimes MKSA) units adopts the electromagnetic definition because it can be measured most easily, but with a different μ_0, as follows. "One Ampere is that current which, when flowing in two infinitesimal parallel wires 1m apart produces a force of 2×10^{-7} Newtons per meter of their length." An Amp is one Coulomb per second. So this defines the unit of charge. We will show later that this definition amounts to defining

$$\mu_0 = 4\pi \times 10^{-7} \qquad \text{(Henry/meter)} \tag{1.3}$$

and that because the ratio of electromagnetic to electrostatic units is c^2

$$\epsilon_0 = \frac{1}{c^2 \mu_0} = 8.85 \times 10^{-12} \qquad \text{(Farad/meter)} \tag{1.4}$$

μ_0 is called the "permeability of free space". ϵ_0 is called the "permittivity of free space". [See J.D. Jackson 3rd Ed, Appendix for a detailed discussion.]

1.2 Vector Calculus and Notation

Electromagnetic quantities include vector fields $\mathbf{E}, \mathbf{B}$ etc. and so EM draws heavily on vector calculus. ∇ is shorthand for a vector operator (gradient)

$$\nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) = \frac{\partial \phi}{\partial x_i} \quad \text{(suffix notation)} \tag{1.5}$$

giving a vector gradient from a scalar field ϕ. ∇ can also operate on vector fields by scalar (.) or vector ($\wedge$) multiplication.

1.2.1 Divergence

$$\nabla . \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial x_z} = \frac{\partial E_i}{\partial x_i} \tag{1.6}$$

1.2.2 Curl

$$\nabla \wedge \mathbf{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) = \epsilon_{ijk} \frac{\partial E_k}{\partial x_j} \tag{1.7}$$

1.2.3 Volume Integration

$$\int_V d^3x \tag{1.8}$$

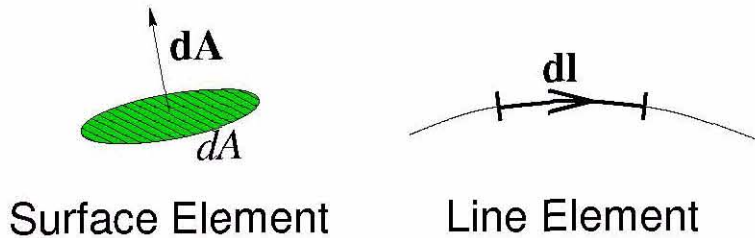d^3x is shorthand for $dxdydz = dV$, the volume element.

Figure 1.2: Elements for surface and line integrals.

1.2.4 Surface Integration

$$\int_S \mathbf{v}.\mathbf{dA} \quad . \tag{1.9}$$

The surface element $\mathbf{dA}$ or often $\mathbf{dS}$ is a vector normal to the element.

1.2.5 Line (Contour) Integration

$$\int_C \mathbf{v}.\mathbf{dl} \quad . \tag{1.10}$$

Line element $\mathbf{dl}$.

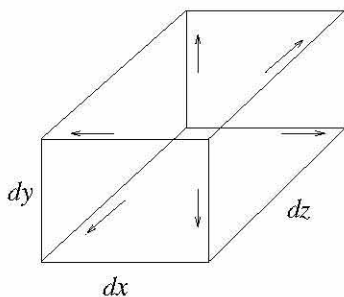## 1.2.6 The Meaning of divergence: ∇.

Figure 1.3: Cartesian volume element.

Consider a volume element. Evaluate the total flux of a vector field $\mathbf{v}$ out across the element's surface. It is the sum $\mathbf{v}.\mathbf{dA}$ over the six faces of the cuboid

$$
\begin{aligned}
[v_x (x + dx) - v_x (x)] \, dydz+ & \\
[v_y (y + dy) - v_y (y)] \, dzdx+ & \\
[v_z (z + dz) - v_z (z)] \, dxdy \;\; &= \\
dxdydz \left[\frac{dv_x}{dx} + \frac{dv_y}{dy} + \frac{dv_z}{dz} \right] \;\; &= \;\; d^3 x \nabla.\mathbf{v} = dV \nabla.\mathbf{v}
\end{aligned}
\tag{1.11}
$$

So for this elemental volume:

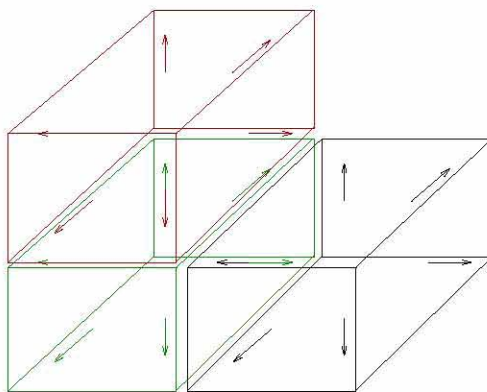$$\int_{dS} \mathbf{v}.\mathbf{dA} = \int_{dV} \nabla.\mathbf{v} \, d^3 x \tag{1.12}$$

Figure 1.4: Adjacent faces cancel out in the sum of divergence from many elements.

<u>But</u> any arbitrary finite volume can be considered to be the sum of many small cuboidal elements. Adjacent <u>internal</u> face contribution cancel out hence only the external surface contributions remain, so

$$\int_S \mathbf{v}.dA = \int_V \nabla.\mathbf{v}\, d^3x \tag{1.13}$$

for <u>any</u> volume V with surface S, and arbitrary vector field $\mathbf{v}$. This is <u>Gauss's Theorem</u>.

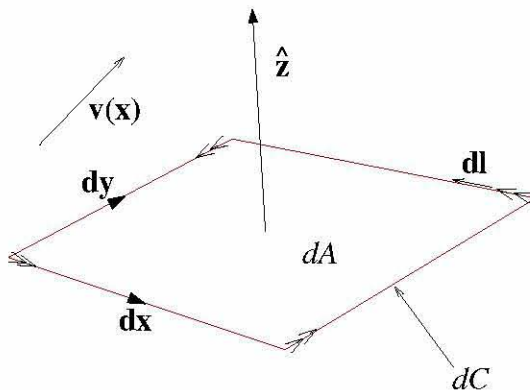## 1.2.7 The Meaning of Curl: $\nabla\wedge$

Figure 1.5: Rectangular surface element with axes chosen such that the normal is in the **z**-direction.

Consider an arbitrary rectangular <u>surface</u> element Choose axes such that normal is in z-direction and edges along $\mathbf{x}$ and $\mathbf{y}$. Arbitrary vector field $\mathbf{v}(\mathbf{x})$. Evaluate the contour integral of $\mathbf{v}$, clockwise around dC, round boundary of element.

$$\oint_{dC} \mathbf{v}.\mathbf{dl} = \mathbf{v}(\mathbf{x}).\mathbf{dx} + \mathbf{v}(x+dx,y).\mathbf{dy} + \mathbf{v}(x,y+dy).(-\mathbf{dx}) + \mathbf{v}(\mathbf{x}).(-\mathbf{dy})$$

$$= \left(\frac{dv_y}{dx} - \frac{dv_x}{dy}\right)dxdy = (\nabla\wedge\mathbf{v})_z dA = (\nabla\wedge\mathbf{v}).\mathbf{dA} \quad . \tag{1.14}$$

So integral $\mathbf{v}.\mathbf{dl}$ around element is equal to the curl scalar-product area element. Apply to arbitrary surface; divide surface up into many elements dA. All <u>internal</u> edge integrals cancel. Hence

$$\oint_C \mathbf{v}.\mathbf{dl} = \int_S (\nabla\wedge\mathbf{v}).\mathbf{dA} \tag{1.15}$$

This is <u>Stokes' Theorem</u>.

1.3 Electrostatics and Gauss' Theorem

Gauss's theorem is the key to understanding electrostatics in terms of Coulomb's Law $\nabla.E = \rho/\epsilon_0$.
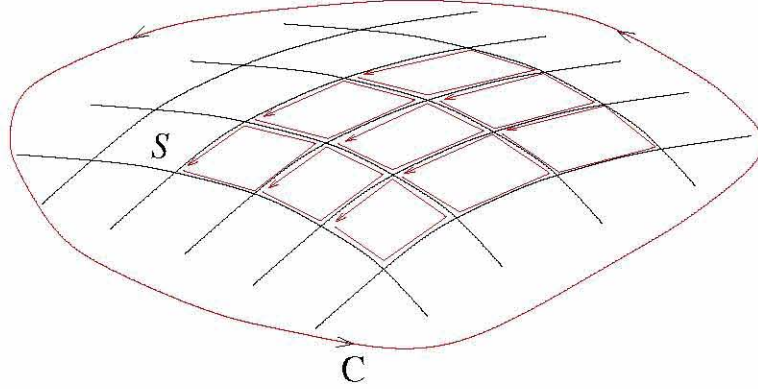
Figure 1.6: Arbitrary surface may be divided into the sum of many rectangular elements. Adjacent edge integral contributions cancel.

1.3.1 Point Charge q

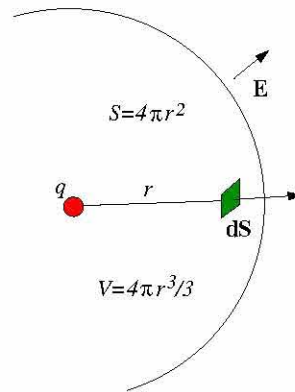Apply Gauss's Theorem to a sphere surrounding q

Figure 1.7: Spherical volume, V over which we perform an integral of Coulomb's law to deduce $\mathbf{E}$.

$$\int_S \mathbf{E}.\mathbf{dA} = \int_V \nabla.\mathbf{E}\,d^3x = \int_V \frac{\rho}{\epsilon_0}\,d^3x = \frac{q}{\epsilon_0} \quad . \tag{1.16}$$

But by spherical symmetry $\mathbf{E}$ must be in radial direction and E_r has magnitude constant over the sphere. Hence $\int_S \mathbf{E}.\mathbf{dA} = \int_S E_r dA = E_r \int_S dA = E_r 4\pi r^2$. Thus

$$E_r 4\pi r^2 = \frac{q}{\epsilon_0} \qquad \text{or} \tag{1.17}$$

$$E_r = \frac{q}{4\pi\epsilon_0 r^2} \qquad \text{i.e.} \qquad \mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3} \quad . \tag{1.18}$$

Consequently, force on a second charge at distance r is

$$F = \frac{q_1 q_2}{4\pi\epsilon_0 r^2} \tag{1.19}$$

Inverse-square-law of electrostatic force.

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1.3.2   Spherically Symmetric Charge $\rho(r)$

Notice that point-charge derivation depended only on symmetry. So for a distributed charge-density that is symmetric argument works just the same i.e.

$$\mathbf{E} = \hat{\mathbf{r}} E_r \qquad E_r = \frac{q}{4\pi\epsilon_0 r^2} \tag{1.20}$$

$$\text{where now} \quad q = \int_V \rho \, d^3x = \int_0^r \rho(r) 4\pi r^2 dr. \tag{1.21}$$

Electric field due to a spherically symmetric charge density is equal to that of a point charge of magnitude equal to the total charge within the radius, placed at the spherical center.

1.3.3   Arbitrary Charge Distribution

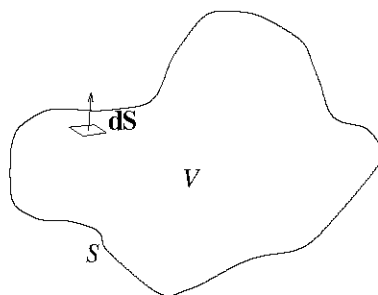If there is no specific symmetry Gauss's Theorem still applies:

Figure 1.8: Arbitrary volume for Gauss's Theorem.

$$\int_S \mathbf{E}.\mathbf{dA} = \int_V \nabla.\mathbf{E} \, d^3x = \int_V \frac{\rho}{\epsilon_0} \, d^3x = \frac{q}{\epsilon_0} \tag{1.22}$$

q is the total charge (integral of charge density) over the volume. $\int_S \mathbf{E}.\mathbf{dA}$ is the total flux of electric field across the surface S.

1.3.4   Intuitive Picture

Each $(+ve)$ charge is the origination point of an electric-field-line. [Each $-ve$ charge is termination ditto]. The total charge in volume V determines the number of field-lines that start in V. Field lines only start/end on charges (Coulomb's Law) so all must escape from the volume, crossing surface S (somewhere). [Field lines that start <u>and</u> end in V contribute neither to $\int \mathbf{E}.\mathbf{dA}$ nor to q, because of cancellation]. $\int_S \mathbf{E}.\mathbf{dA}$ can be thought of as counting the "number of field lines" crossing the surface. [Of course it is an arbitrary choice how big we consider the charge is that gives rise to one field-line.] Intuitive view of electric field "intensity": Strength of E is proportional to the number of field-lines per unit area. All these intuitive views are conceptually helpful but are not formally necessary. Electromagnetism is considered completely described by Maxwell's equations <u>without</u> need for these pictures.
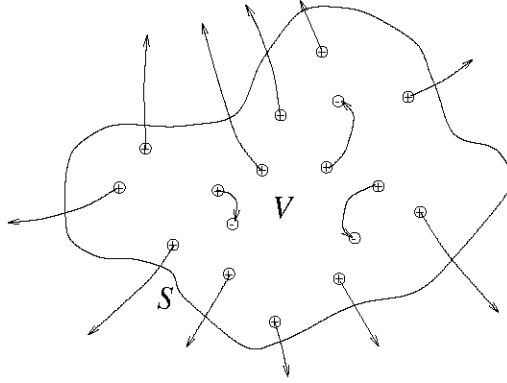
Figure 1.9: Intuitive picture of charges and field-lines.
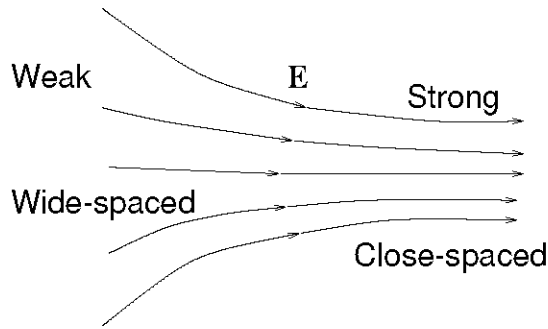
Field-Lines and Field Intensity.

Figure 1.10: Spacing of field-lines is inversely proportional to field-strength.

1.3.5 Electric Potential (for static problems $\frac{\partial}{\partial t} \to 0$)

In the static situation there is no induction and Faraday's law becomes $\nabla \wedge \mathbf{E} = 0$. By the way, this equation could also be derived from the inverse-square-law by noting that

$$\nabla \wedge \frac{\mathbf{r}}{r^3} = 0 \tag{1.23}$$

so by the linearity of the $\nabla \wedge$ operator the sum (integral) of all Electric field contributions from any charge distribution is curl-free "irrotational":

$$\nabla \wedge \int \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \, d^3\mathbf{r}' = 0 \tag{1.24}$$

[This shows that the spherical symmetry argument only works in the absence of induction $\dot{\mathbf{B}}$ would define a preferred direction; asymmetric!] For any vector field $\mathbf{E}$, $\nabla \wedge \mathbf{E} = 0$ is a necessary and sufficient condition that $\mathbf{E}$ can be written as the gradient of a scalar $\mathbf{E} = -\nabla \phi$.

Necessary

$$(\nabla \wedge \nabla \phi)_z = \frac{\partial}{\partial x} \frac{\partial \phi}{\partial y} - \frac{\partial}{\partial y} \frac{\partial \phi}{\partial x} = 0 \qquad (\text{et sim x, y}) \quad . \tag{1.25}$$
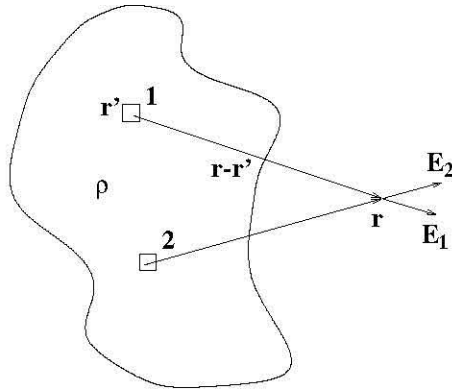
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Figure 1.11: Each element contributes an irrotational component to **E**. Therefore the total **E** is irrotational.

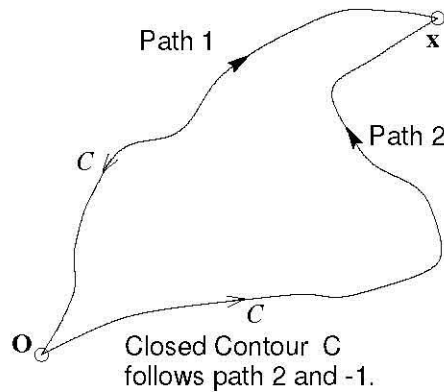Curl of a gradient is zero.

Sufficient (prove by construction)

Figure 1.12: Two different paths from 0 to **x** construct a closed contour when one is reversed.

Apply Stokes' theorem to a closed contour consisting of any 2 paths between points 0 and **x**.

$$\oint_C \mathbf{E}.\mathbf{dl} = \underbrace{\int_0^{\mathbf{x}} \mathbf{E}.\mathbf{dl}}_{\text{Path 2}} - \underbrace{\int_0^{\mathbf{x}} \mathbf{E}.\mathbf{dl}}_{\text{Path 1}} = \int_S \nabla \wedge \mathbf{E}.\mathbf{dS} \qquad \underbrace{= 0}_{\text{by hypothesis.}} \qquad (1.26)$$

So $\nabla \wedge \mathbf{E} = 0 \Rightarrow \int_o^{\mathbf{x}} \mathbf{E}.\mathbf{dl}$ is $\underline{\text{independent}}$ of chosen path, i.e. it defines a unique[1] quantity. Call it $-\phi(\mathbf{x})$. Consider $\nabla\phi$ defined as the limit of $\delta\phi$ between adjacent points.

$$-\nabla\phi = \nabla \left(\int^{\mathbf{x}} \mathbf{E}.\mathbf{dl} \right) = \mathbf{E} \qquad . \qquad (1.27)$$

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[1]The potential can always be altered by adding a constant, without changing **E**. This freedom can be considered equivalent to choosing the origin at which $\phi = 0$.

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Many electrostatic problems are most easily solved in terms of the electric potential ϕ because it is a scalar (so easier). Governing equation:

$$\nabla.\mathbf{E} \;=\; -\nabla.\nabla\phi = -\nabla^2\phi = \frac{\rho}{\epsilon_0} \tag{1.28}$$

$$\nabla^2 \;=\; \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} \qquad \text{is the ``Laplacian'' operator} \tag{1.29}$$

$$\nabla^2\phi = \frac{-\rho}{\epsilon_0} \qquad \text{``Poisson's Equation''.} \tag{1.30}$$

1.3.6 Potential of a Point Charge [General Potential Solution]

One can show by direct differentiation that

$$\nabla\frac{1}{r} = -\frac{\mathbf{r}}{r^3} \qquad . \tag{1.31}$$

So by our previous expression $\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3}$ we can identify

$$\phi = \frac{q}{4\pi\epsilon_0}\frac{1}{r} \tag{1.32}$$

as the potential of a charge q (at the origin $\mathbf{x} = 0$).

1.3.7 Green Function for the Laplacian

For a linear differential operator, $\mathcal{L}$, mathematicians define something called "Green's function" symbolically by the equation

$$\mathcal{L}G(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}') \qquad . \tag{1.33}$$

If we can solve this equation in general, then solutions to

$$\mathcal{L}\phi = \rho(\mathbf{x}) \tag{1.34}$$

can be constructed for arbitrary ρ as

$$\phi(\mathbf{x}) = \int G(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}')\,d^3x' \tag{1.35}$$

because of the (defining) property of the δ-function

$$\int f(\mathbf{x}')\delta(\mathbf{x} - \mathbf{x}')\,d^3x' = f(\mathbf{x}) \qquad . \tag{1.36}$$

When $\mathcal{L}$ is the Laplacian, ∇^2, the Green function is

$$\boxed{G(\mathbf{x},\mathbf{x}') = \frac{-1}{4\pi|\mathbf{x} - \mathbf{x}'|} \qquad . \tag{1.37}}$$

This fact may be derived directly from the solution for the potential for a point charge. Indeed, a point charge is exactly the delta-function situation whose solution is the Green function. In other words, the charge density for a point charge of magnitude q at position $\mathbf{x}'$ is

$$\rho(\mathbf{x}) = q\delta(\mathbf{x} - \mathbf{x}') \qquad ; \qquad (1.38)$$

so the point-charge potential, namely,

$$\phi = \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{x} - \mathbf{x}'|} \qquad , \qquad (1.39)$$

is the solution of the equation:

$$\nabla^2\phi = -\frac{q}{\epsilon_0}\delta(\mathbf{x} - \mathbf{x}') \qquad . \qquad (1.40)$$

Consequently, solution of Poisson's equation can be written as the integral of the Green function:

$$\boxed{\phi(\mathbf{x}) = \int \left(\frac{-\rho(\mathbf{x}')}{\epsilon_0}\right)\left(\frac{-1}{4\pi|\mathbf{x} - \mathbf{x}'|}\right)d^3x' = \int \frac{\rho(\mathbf{x}')}{4\pi\epsilon_0}\frac{d^3x'}{|\mathbf{x} - \mathbf{x}'|} \qquad .} \qquad (1.41)$$

Informally, the smooth charge distribution ρ can be approximated as the sum $(\to \int)$ of many point charges $\rho(x')\,d^3x'$, and the potential is the sum of their contributions.

1.3.8 Boundary Conditions

Strictly speaking, our solution of Poisson's equation is not unique. We can always add to ϕ a solution of the homogeneous (Laplace) equation $\nabla^2\phi = 0$. The solution only becomes unique when boundary conditions are specified. The solution

$$\phi(\mathbf{x}) = \int \frac{\rho(x')}{4\pi\epsilon_0}\frac{d^3x'}{|\mathbf{x} - \mathbf{x}'|} \qquad (1.42)$$

is correct when the boundary conditions are that

$$\phi \to 0 \qquad \text{as} \qquad |\mathbf{x}| \to \infty \qquad : \qquad (1.43)$$

no applied external field.

In practice most interesting electrostatic calculations involve specific boundaries. A big fraction of the work is solving Laplace's equation with appropriate boundary conditions. These are frequently the specification of ϕ on (conducting) surfaces. The charge density on the conductors is rarely specified initially.

1.3.9 Parallel Plate Capacitor

Idealize as 1-dimensional by ignoring the edge effects. 1-d Laplace equation in the vacuum gap (where $\rho = 0$) is

$$\frac{d^2\phi}{dz^2} = 0 \qquad (1.44)$$
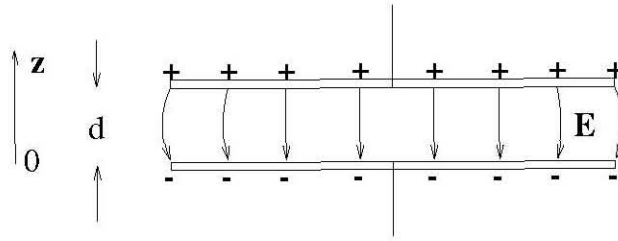
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Parallel Plate Capacitor.

Figure 1.13: The parallel-plate capacitor.

Solution $\phi = a - Ez$ a,E const.

Hence electric field is $\mathbf{E} = E\hat{z}$.

Notice how this arises purely from the translational invariance of $\phi \left(\frac{d}{dx} = \frac{d}{dy} = 0 \right)$.

Choose $z = 0$ as one plate of capacitor. Other at $z = d$.

Choose $\phi(z = 0) = 0$: reference potential, making $a = 0$.

Potential of other plate: $V = \phi(d) = -Ed$.

The question: how much charge per unit area is there on the plates when the field is E?
Answer by considering a flat elemental volume, with area A surrounding the $+ve$ plate.
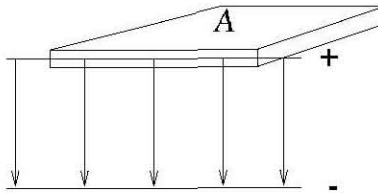

Figure 1.14: Elemental volume for calculating charge/field relationship.

Apply Gauss's law

$$
\begin{aligned}
\int_S \mathbf{E}.\mathbf{dS} &= \int_A O + (-E)ds = -EA \\
&= \int_V \frac{\rho}{\epsilon_0} d^3x = \frac{Q}{\epsilon_0} = \frac{1}{\epsilon_0}\sigma\, A
\end{aligned}
\tag{1.45}
$$

Hence

$$
\sigma = -\epsilon_0 E. = \epsilon_0 \frac{V}{d}
\tag{1.46}
$$

Therefore if the total area is A, the total charge Q, and the voltage V between plates are related by

$$
Q = \left(\frac{\epsilon_0 A}{d} \right) V
\tag{1.47}
$$

And the coefficient $\frac{\epsilon_0 A}{d}$ is called capacitance, C. Notice our approach:

- Solve Laplace's equation by choosing coordinates consistent with problem symmetry.

- Obtain charge using Gauss's law to an appropriate trial volume.

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1.3.10 Charge on an arbitrary conductor

Consider a conductor, electrostatically charged. Current is zero.
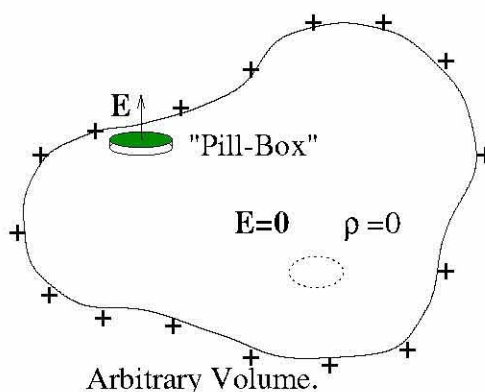

Figure 1.15: Arbitrary-shaped conductor possesses only surface charges related to the local normal field.

So $\mathbf{E}$ is zero, anywhere <u>inside</u> because of conductivity.

Choose any volume internally: $\mathbf{E} = 0 \Rightarrow \nabla.E = 0 \Rightarrow \rho = 0$. There is <u>no internal charge</u>. It all resides on <u>surface</u>. At the surface there is an $\mathbf{E}$ just outside. $\mathbf{E}$ is perpendicular to surface $\mathbf{ds}$ because surface is an equipotential (& $\mathbf{E} = -\nabla\phi$). Hence applying Gauss's law to a pill box

$$
\begin{aligned}
\int_V \nabla.\mathbf{E}\, d^3x = \int_S \mathbf{E}.\mathbf{dS} &= Eds \\
= \int \frac{\rho}{\epsilon_0} d^3x = \int \frac{\sigma}{\epsilon_0} ds &= \frac{\sigma}{\epsilon_0} ds
\end{aligned}
\tag{1.48}
$$

where $\sigma =$ surface charge density Hence $\sigma = \epsilon_0 E$. Of course, in this general case E ($= E_{\text{normal}}$) is <u>not</u> uniform on the surface but varies from place to place. Again procedure would be: solve ϕ externally from $\nabla^2\phi = 0$; then deduce σ; rather than the other way around.

1.3.11 Visualizing Electric Potential and Field

$$
\mathbf{E} = -\nabla\phi
\tag{1.49}
$$

Consider a (2-D) contour plot of ϕ. The value of ϕ can be thought of as the potential energy of a charge of 1 Coulomb. Thus there is a perfect analogy to gravitational potential energy and height contours. The force at any point on the hill is downward (on a $+ve$ charge), which is perpendicular to the contours of constant ϕ. The strength of the force ($\mathbf{E}$) is proportional to the steepness of the hill: i.e. how close together the <u>contours</u> are (of ϕ). When plotting field-lines, i.e. lines following the electric field direction, we generally also consider the electric field intensity to be the number of field-lines per unit area. So also the closeness of <u>field-lines</u> indicates field strength. In <u>charge-free</u> regions $\nabla.E = 0$ implies field-lines have no beginning or end. However if $\rho \neq 0$ then electric field lines <u>do</u> possibly have ends (on the charges). The potential contours never have ends.
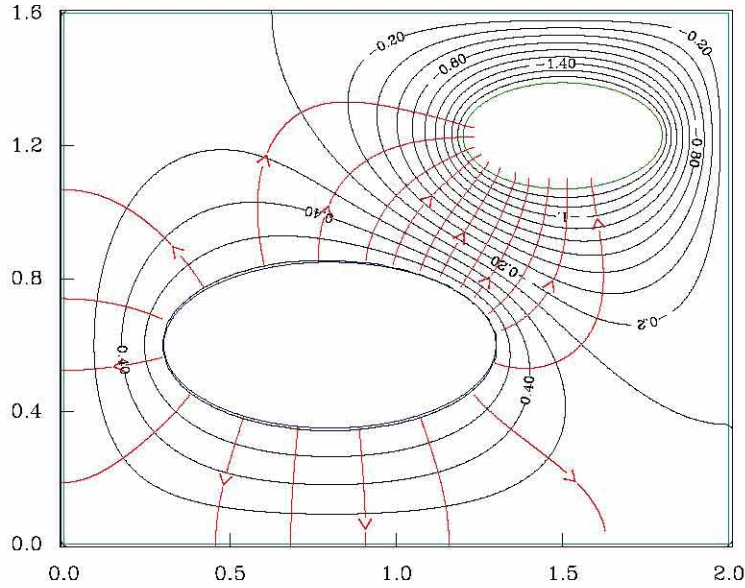
Figure 1.16: Contours of potential and corresponding field-lines (marked with arrows). Only the field-lines emanating from the larger elliptical conductor are drawn.

1.3.12 Complex Potential Representation 2-D

In charge-free region, $\nabla.\mathbf{E} = 0 \Rightarrow \nabla^2\phi = 0$. This causes there to be an intimate relationship between field-lines and ϕ-contours. In 2 dimensions this relationship allows complex analysis to be used to do powerful analysis of potential problems. Consider a complex function $f(z) = \phi(z) + i\psi(z)$ where $z = x + iy$ is the complex argument with real and imaginary parts x & y; and f has real and imaginary parts ϕ & ψ. f is "analytic" if there exists a well defined complex derivative $\frac{df}{dz}$ (which is also analytic), defined in the usual way as $\lim_{z' \to z}\left(\frac{f(z')-f(z)}{z'-z}\right)$. In order for this limit to be the same no matter what direction (x,y) it is taken in, f must satisfy the "Cauchy-Riemann relations"

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \qquad ; \qquad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \tag{1.50}$$

Which, by substitution imply $\nabla^2\phi = 0$, $\nabla^2\psi = 0$, and also

$$\nabla\phi.\nabla\psi = 0 \tag{1.51}$$

regarding x, y as 2-d coordinates. This shows that

1. The real part of an analytic function solves $\nabla^2\phi = 0$.

2. The contours of the corresponding imaginary part, ψ, then coincide with the electric field-lines.

Finding complex representations of potential problems is one of the most powerful analytic solution techniques. However, for practical calculations, numerical solution techniques are now predominant.

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1.4 Electric Current in Distributed Media

Ohms law, $V = IR$, relates voltage current and resistance for a circuit or discrete element. However we often care not just about the total current but about the current <u>density</u> in finite-sized conductors (e.g. electromagnets). This requires a local Ohm's law which is

$$\mathbf{E} = \eta\mathbf{j} \tag{1.52}$$

where η is the medium's electric <u>resistivity</u>. Often the <u>conductivity</u> $\sigma = 1/\eta$ is used. $\mathbf{j} = \sigma\mathbf{E}$, (but I'll try to avoid confusion with surface charge density σ). Such a linear relationship applies in most metals.

1.4.1 Steady State Conduction

Conservation of charge can be written

$$\nabla.\mathbf{j} = -\frac{\partial\rho}{\partial t} \tag{1.53}$$

so, in steady state, $\nabla.\mathbf{j} = 0$, i.e.

$$\nabla.\left(\frac{1}{\eta}\mathbf{E}\right) = (\mathbf{E}.\nabla)\frac{1}{\eta} + \frac{1}{\eta}\nabla.\mathbf{E} = 0 \tag{1.54}$$

If conductivity is uniform ($\nabla\frac{1}{\eta} = 0$) or invariant along $\mathbf{E}$, we therefore have $\nabla.\mathbf{E} = 0 \Rightarrow \rho = 0$. "Uniform conductivity conductors acquire zero volume charge density in steady state".

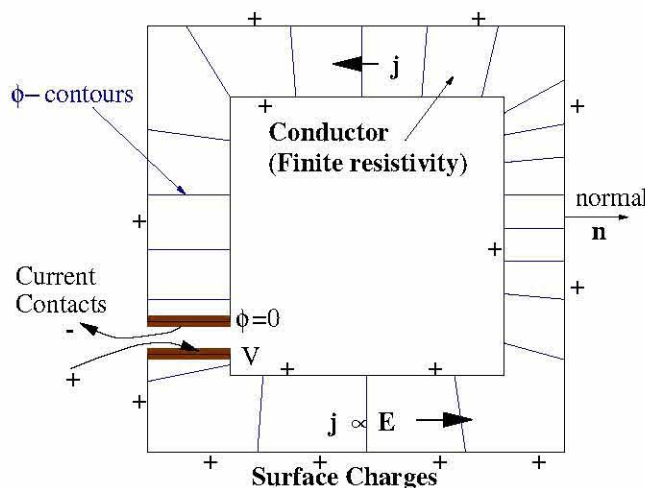### 1.4.2 Conductor Boundary Conditions (Steady Currents)

Figure 1.17: A distributed conductor of finite conductivity, carrying current.

If currents are flowing so that $\mathbf{E} \neq 0$ in the conductor then conductors are <u>not</u> equipotential surfaces for solutions of Laplace's equation outside. <u>Surface</u> charges (only) are present

on the conductor (uniform η). No current flows through the conductor surface (except at contacts) so

$$\mathbf{j.n} = 0 \Rightarrow \mathbf{E.n} = 0 \tag{1.55}$$

inside conductor, while outside we have

$$\mathbf{E.n} = \epsilon_0 \sigma \tag{1.56}$$
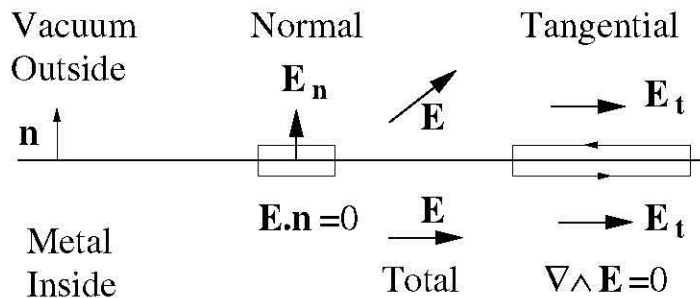
surface charge density. Normal components

Figure 1.18: Boundary conditions across a conductor/vacuum (or insulator) interface.

$$E_n|_{inside} = 0 \ [E_n]^{outside}_{inside} = \sigma/\epsilon_0 \tag{1.57}$$

Tangential components

$$[E_t]^{outside}_{inside} = 0 \tag{1.58}$$

In particular, for solving $\nabla^2 \phi = 0$ inside uniform η conductor, at conductor boundary:

$$\nabla\phi.\mathbf{n} = 0 \qquad \text{(Neumann B.C.)} \tag{1.59}$$

unlike the usual electrostatic B.C. $\phi =$ given. At electrical contacts ϕ given might be appropriate. A general approach to solving a distributed steady-current problem with uniform-η media:

1. Solve Laplace's equation $\nabla^2 \phi = 0$ <u>inside</u> conductors using Dirichlet (ϕ-given) or possibly inhomogeneous Neumann ($\nabla\phi|_n =$ given) BC's at contacts and $\nabla\phi.\mathbf{n} = 0$ at insulating boundaries.

2. Solve Laplace's equation $\nabla^2 \phi = 0$ <u>outside</u> conductors using $\phi =$ given (Dirichlet) B.C. with the ϕ taken from the internal solution.

1.5 Magnetic Potential

Magnetic field has zero divergence $\nabla.\mathbf{B} = 0$. For any vector field[2] $\mathbf{B}$, $\nabla.\mathbf{B} = 0$ is a necessary and sufficient condition that $\mathbf{B}$ can be written as the curl of a vector potential $\mathbf{B} = \nabla \wedge \mathbf{A}$.

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[2]satisfying $|\mathbf{B}| \to 0$ as $|\mathbf{x}| \to = \infty$ fast enough

1.5.1 $\nabla.\mathbf{B} = 0$ Necessary

$$\nabla.(\nabla \wedge \mathbf{A}) = \frac{\partial}{\partial x}\left(\frac{\partial A_z{}^{(z)}}{\partial y} - \frac{\partial A_y{}^{(y)}}{\partial z}\right) \tag{1.60}$$

$$+ \frac{\partial}{\partial y}\left(\frac{\partial A_x{}^{(x)}}{\partial z} - \frac{\partial A_z{}^{(z)}}{\partial x}\right) \tag{1.61}$$

$$+ \frac{\partial}{\partial z}\left(\frac{\partial A_y{}^{(y)}}{\partial x} - \frac{\partial A_x{}^{(x)}}{\partial y}\right) = 0 \tag{1.62}$$

So only divergenceless fields can be represented.

1.5.2 $\nabla.\mathbf{B} = 0$ Sufficient (outline proof by construction)

Consider the quantity

$$\mathbf{K}(\mathbf{x}) = \int \frac{\mathbf{B}(\mathbf{x}')}{4\pi|\mathbf{x} - \mathbf{x}'|}\, d^3x' \quad , \tag{1.63}$$

a vector constructed from the integral of each Cartesian component of $\mathbf{B}$. Applying our knowledge of the Green function solution of Poisson's equation, we know:

$$\nabla^2\mathbf{K} = -\mathbf{B} \quad . \tag{1.64}$$

Vector operator theorem (for any $\mathbf{v}$):

$$\nabla \wedge (\nabla \wedge \mathbf{v}) = \nabla(\nabla.\mathbf{v}) - \nabla^2\mathbf{v} \quad . \tag{1.65}$$

Hence

$$\mathbf{B} = -\nabla^2\mathbf{K} = \nabla \wedge (\nabla \wedge \mathbf{K}) - \nabla(\nabla.\mathbf{K}) \tag{1.66}$$

We have proved Helmholtz's theorem that any vector field can be represented as the sum of grad + curl.] When $\nabla.\mathbf{B} = 0$ and $|\mathbf{B}| \to 0$ (fast enough) as $|\mathbf{x}| \to \infty$, one can show that $\nabla.\mathbf{K} = 0$ and so we have constructed the required vector potential

$$\mathbf{A} = \nabla \wedge \mathbf{K} = \nabla \wedge \int \frac{\mathbf{B}(\mathbf{x}')}{4\pi}\frac{d^3x'}{|\mathbf{x} - \mathbf{x}'|} \quad . \tag{1.67}$$

Notice that we have constructed $\mathbf{A}$ such that $\nabla.\mathbf{A} = 0$. However $\mathbf{A}$ is undetermined from $\mathbf{B}$ because we can add to it the gradient of an arbitrary scalar without changing $\mathbf{B}$, since $\nabla \wedge \nabla\chi = 0$. So in effect we can make $\nabla.\mathbf{A}$ equal any desired quantity $\psi(\mathbf{x})$ by adding to $\mathbf{A}$ $\nabla\chi$ such that $\nabla^2\chi = \psi$. Choosing $\nabla.\mathbf{A}$ is known as choosing a "Gauge" $\nabla.\mathbf{A} = 0$ is the "Coulomb Gauge".

1.5.3 General Vector Potential Solution (Magnetostatic)

Static Ampere's law $\nabla \wedge \mathbf{B} = \mu_0\mathbf{j}$. Now

$$\mu_0\mathbf{j} = \nabla \wedge (\nabla \wedge \mathbf{A})$$
$$= \nabla(\nabla.\mathbf{A}) - \nabla^2\mathbf{A} = -\nabla^2\mathbf{A} \qquad \text{(Coulomb Gauge)}. \tag{1.68}$$

Hence Cartesian components of $\mathbf{A}$ are solutions of Poisson equation

$$\nabla^2 A_i = -\mu_0 j_i \qquad (1.69)$$

Using our general solution of Poisson's equation (see eq 1.37):

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' \qquad (1.70)$$

Resulting $\mathbf{B}$:

$$\begin{aligned} \mathbf{B} &= \nabla \wedge \mathbf{A} = \frac{\mu_0}{4\pi} \int \nabla \wedge \frac{\mathbf{j}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' \\ &= \frac{\mu_0}{4\pi} \int -\mathbf{j}(\mathbf{x}') \wedge \nabla \frac{1}{|\mathbf{x} - \mathbf{x}'|} d^3x' = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(x') \wedge (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d^3x' \end{aligned} \qquad (1.71)$$

This is the distributed-current version of the law of <u>Biot and Savart</u> (dating from ~ 1820). For a wire carrying current I the integral over volume $\mathbf{j}$ is replaced by the integral I $\mathbf{dl}$ i.e.

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int -\frac{(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \wedge \mathbf{j} d^3x = \frac{\mu_0}{4\pi} \int -\frac{(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \wedge I\mathbf{dl} \qquad . \qquad (1.72)$$

The Biot-Savart law gives us a direct means to calculate $\mathbf{B}$ by integrating over $\mathbf{j}(\mathbf{x}')$, numerically if necessary. However this integration brute-force method is excessively computationally intensive and if symmetries are present in the problem we can use them to simplify.
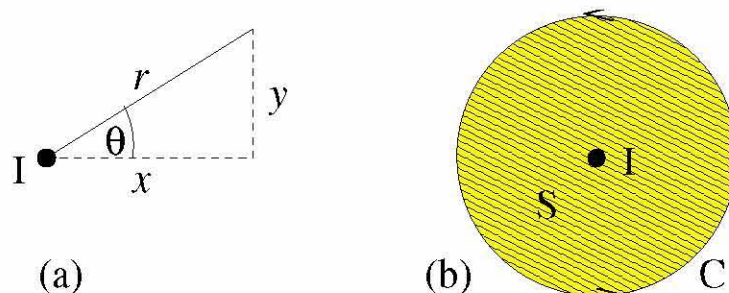
1.5.4 Cartesian Translational Symmetry (2-d x, y)

Figure 1.19: (a) The coordinates with respect to an infinite straight filament carrying current I, and (b) the contour and surface for use with Ampere's law.

If we consider a situation where $\partial/\partial z = 0$, corresponding to infinite straight parallel currents in z-direction $\mathbf{j} = j(x, y)\hat{\mathbf{z}}$. Our general vector potential solution shows us immediately that $\mathbf{A} = A\hat{\mathbf{z}}$, $A_x = A_y = 0$. (Assuming $\mathbf{A}, \mathbf{B} \to 0$ at ∞, i.e. no 'external' sources.) That fact tells us that $B_z = (\nabla \wedge \mathbf{A})_z = 0$. We can consider the elementary building block of this problem to be the single infinitesimal filament. Formally $j = I\delta(x)\delta(y)$. We could calculate

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$\mathbf{B}(\mathbf{x})$ by integrating over this filament. <u>Far</u> easier to use Ampere's Law directly

$$\int_S (\nabla \wedge \mathbf{B}) \cdot \mathbf{ds} = \int_C \mathbf{B} \cdot \mathbf{dl} = \int_S \mu_0 \mathbf{j} \cdot \mathbf{ds} = \mu_0 I \tag{1.73}$$

By symmetry $\oint_C \mathbf{B} \cdot \mathbf{dl} = 2\pi r B_\theta$
So

$$B_\theta = \frac{\mu_0 I}{2\pi r} \quad . \tag{1.74}$$

Also $B_r = 0$ by applying Gauss's Theorem to a volume (of unit length in z-dir)

$$0 = \int_V \nabla \cdot \mathbf{B} \, d^3 x = \int_S \mathbf{B} \cdot \mathbf{dS} = 2\pi r B_r \tag{1.75}$$

by symmetry.
Thus Maxwell's equations immediately show us what the 2-d Green function solving $\nabla \wedge \mathbf{B} = \mu_0 I \hat{\mathbf{z}} \delta(\mathbf{x} - \mathbf{x}')$ is

$$\mathbf{B} = \hat{\boldsymbol{\theta}} \frac{\mu_0 I}{2\pi} \frac{1}{\sqrt{(x - x')^2 + (y - y')^2}} \tag{1.76}$$

Any general $j(x, y)$ can be handled by 2-d integration using this function.

1.5.5 Cylindrical Symmetry (Circular Loops with common axis)

If there exist cylindrical coordinates (r, θ, z) such that $\partial/\partial\theta = 0$, $\mathbf{j} = j\hat{\theta}$. Then by symmetry $\mathbf{A} = A_\theta \hat{\theta}$, $B_\theta = 0$. This situation turns out to be soluble analytically but only in terms of the special functions known as Elliptic Integrals. If
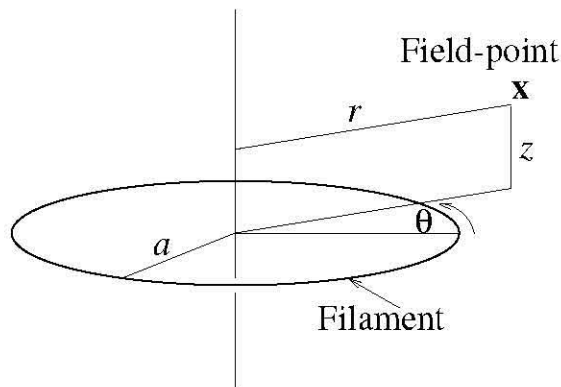

Figure 1.20: Cylindrical Coordinates near a circular current-carrying filament.

$$j_\theta = I \delta(r - a) \delta(z) \tag{1.77}$$

then

$$A_\theta(r, z) = \frac{\mu_0}{4\pi} 2 \sqrt{\frac{a}{r}} \left[\frac{(2 - k^2) K(k) - 2E(k)}{k} \right] \tag{1.78}$$

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where

$$k^2 \equiv \frac{4ra}{(r+a)^2 + z^2} \tag{1.79}$$

and K, E are the complete elliptical integrals of the first and second kind. This general form is so cumbersome that it does not make general analytic calculations tractable but it makes numerical evaluation easier by using canned routines for $K(k)$ & $E(k)$. On axis ($r = 0$) the field is much simpler

$$\mathbf{B} = B\hat{\mathbf{z}} = \frac{\mu_0 I}{4\pi} \frac{2\pi a^2}{(z^2 + a^2)^{\frac{3}{2}}} \hat{\mathbf{z}} \qquad . \tag{1.80}$$

1.5.6 General Property of Symmetry Situations: Flux Function

When there is a symmetry direction, the component of $\mathbf{B}$ perpendicular to that direction can be expressed in terms of a "flux function". The magnetic flux between two positions is defined as the $\mathbf{B}$-field flux crossing a surface spanning the gap (per unit length if translational). Since $\nabla.\mathbf{B} = 0$ it does not matter how the surface gets from the ref-point to P (provided it
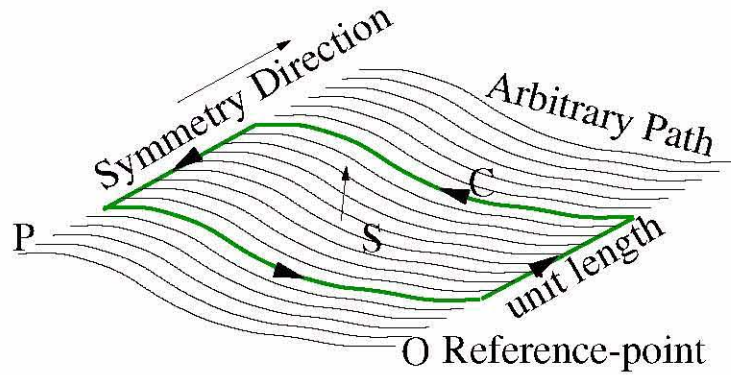

Figure 1.21: Path from a reference point to a field point defines a surface to which Stokes' theorem is applied, in a situation of translational symmetry.

stays symmetric). So the function

$$\psi \equiv \int_S \mathbf{B}.\mathbf{dS} \tag{1.81}$$

is well defined. For translational ($\hat{\mathbf{z}}$) symmetry, a consequence is

$$\mathbf{B}_\perp = -\hat{\mathbf{z}} \wedge \nabla\psi \tag{1.82}$$

This arises because

$$\psi = \int_S \mathbf{B}.\mathbf{dS} = \int \nabla \wedge \mathbf{A}.\mathbf{dS} = A_z(P) - A_z(0) \tag{1.83}$$

So really ψ is identical to the z-component of the vector potential and

$$\begin{aligned}
\mathbf{B} = \nabla \wedge \mathbf{A} &= B_z\hat{\mathbf{z}} + \nabla \wedge (A_z\hat{\mathbf{z}}) = B_z\hat{\mathbf{z}} + \mathbf{B}_\perp \\
\mathbf{B}_\perp &= A_z\nabla \wedge \hat{\mathbf{z}} + (\nabla A_z) \wedge \hat{\mathbf{z}} \\
&= -\hat{\mathbf{z}} \wedge \nabla A_z = -\hat{\mathbf{z}} \wedge \nabla\psi
\end{aligned} \tag{1.84}$$

$\mathbf{B}_\perp$ is the part of the field perpendicular to $\hat{\mathbf{z}}$. There could also be B_z.

For cylindrical symmetry some more variations arise from curvilinear coordinate system. There are even other symmetries, for example helical!

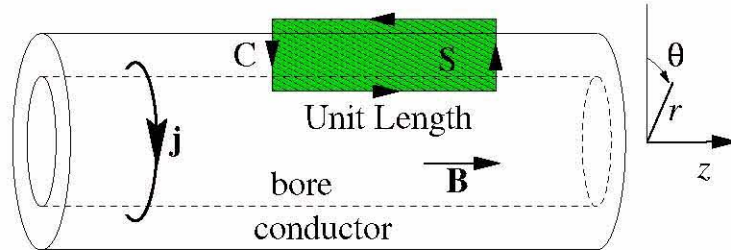# 1.6   Electromagnetism and Magnets

1.6.1   Simple Solenoid

Figure 1.22: Idealized long solenoid magnet coil.

A 'Long' solenoid has a translational symmetry so B is independent of z, as well as of θ. (Except near ends). So

$$0 = \nabla.\mathbf{B} = \frac{1}{r}\frac{\partial}{\partial r}rB_r + \underbrace{\frac{\partial}{r\partial\theta}B_\theta}_{=0} + \underbrace{\frac{\partial}{\partial z}B_z}_{=0} \tag{1.85}$$

So

$$rB_r = \text{const.} \qquad \text{and hence} \qquad B_r = 0 \quad . \tag{1.86}$$

Also

$$\nabla \wedge \mathbf{B} = \left(\frac{1}{r}\frac{\partial}{\partial r}rB_\theta\right)\hat{\mathbf{z}} + \left(-\frac{\partial}{\partial r}B_z\right)\hat{\boldsymbol{\theta}} = \mu_0\mathbf{j} \qquad \text{(steady)} \tag{1.87}$$

Inside the bore of the magnet, $\mathbf{j} = 0$ so

$$rB_\theta = \text{const.} \qquad \text{and hence} \qquad B_\theta = 0 \quad . \tag{1.88}$$

(actually if $j_z = 0$ everywhere then $B_\theta = 0$ everywhere, as may be seen immediately from the Biot-Savart law). Also

$$\frac{\partial B_z}{\partial r} = 0 \qquad \text{and hence} \qquad B_z = \text{const} \quad . \tag{1.89}$$

Use the surface and bounding curve shown and write

$$\int_S \mu_0\mathbf{j}.\mathbf{dS} = \int_S \nabla \wedge \mathbf{B}.\mathbf{ds} = \oint_C \mathbf{B}.\mathbf{dl} \tag{1.90}$$

So $\mu_0\times$ current <u>per unit length</u> (denoted J_θ) gives

$$\mu_0 J_\theta = B_{z\ inside} - B_{z\ outside} \tag{1.91}$$

But (by same approach) if $B = 0$ at infinity $B_{z\ outside} = 0$. So, inside

$$B_z = \mu_0 J_\theta \tag{1.92}$$

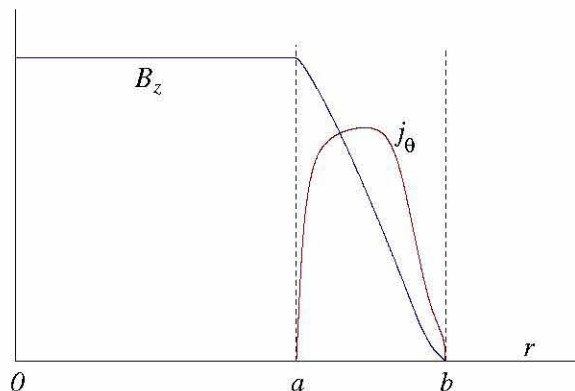Profile of field in coil: is determined by the current density in the coil:

Figure 1.23: The field profile within the conductor region of the coil depends on the current-density profile.

$$\frac{dB_z}{dr} = -\mu_0 j_\theta \tag{1.93}$$

$$B_{zb} - B_{za} = -\int_a^b \mu_0 j_\theta dr = -\mu_0 J_\theta \quad . \tag{1.94}$$

(as before). Notice that all this is independent of coil thickness $(b - a)$. Coils are usually multi-turn so

$$B_z = \mu_0 n I \tag{1.95}$$

where n is turns per unit length, I is current in each turn.

$$J_\theta = nI \quad . \tag{1.96}$$

1.6.2   Solenoid of Arbitrary Cross-Section

$$\frac{\partial}{\partial z} = 0 \tag{1.97}$$

Consider Biot-Savart Law, expressed as vector potential:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' \quad . \tag{1.98}$$

If currents all flow in azimuthal direction, i.e. $j_z = 0$, then $A_z = 0$.

$$\Rightarrow \quad B_x = B_y = 0 \quad \text{(everywhere.)} \tag{1.99}$$

Then integral form of Ampere's law is still

$$B_z(\text{inside}) = \mu_0 J_p \tag{1.100}$$

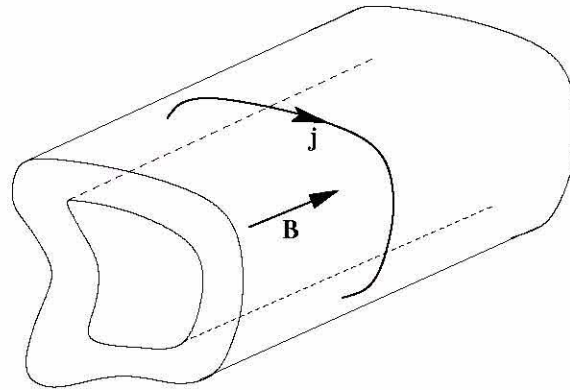where J_p is total current in azimuthal direction per unit length.

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Figure 1.24: Solenoid of arbitrary cross-section.

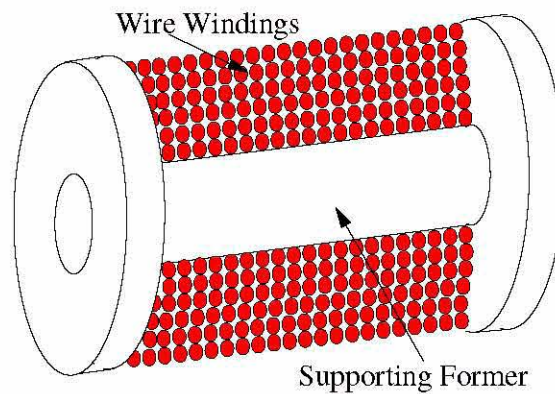## 1.6.3   Coil Types

(a) Wire (Filament):

Figure 1.25: Section through a wire-wound magnet coil.

Multiple layers wound on a former. Usually only for low-field low-current work.

(b) Tape wound:

Each coil consists of a spiral-wound tape, n_t turns. Many coils stacked to form a solenoid. Say n_c coils per unit length $n = n_t.n_c$.

(c) Pancake:

Similar to tape but using square or rectangular conductor. (Fewer turns/coil).

(d) Plate Coils:

Each turn is made of plate. Whole is a single helix (topologically). Plates may be spaced by air or solid insulator gap. $n = n_c$.
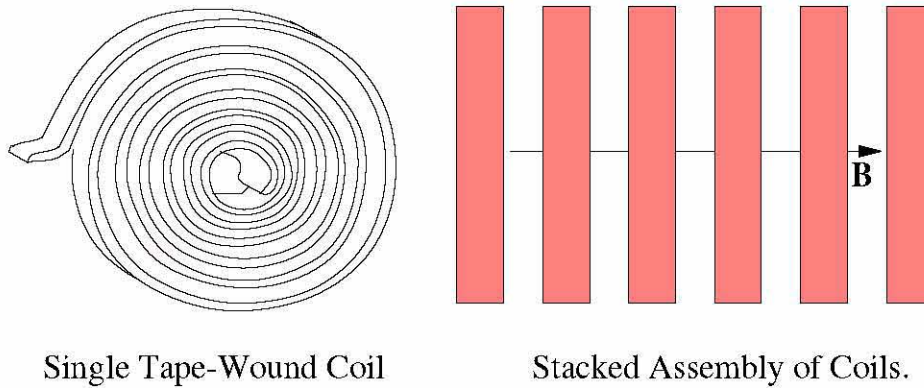
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Single Tape-Wound Coil   Stacked Assembly of Coils.

Figure 1.26: Tape-wound coils are stacked to produce a solenoid.
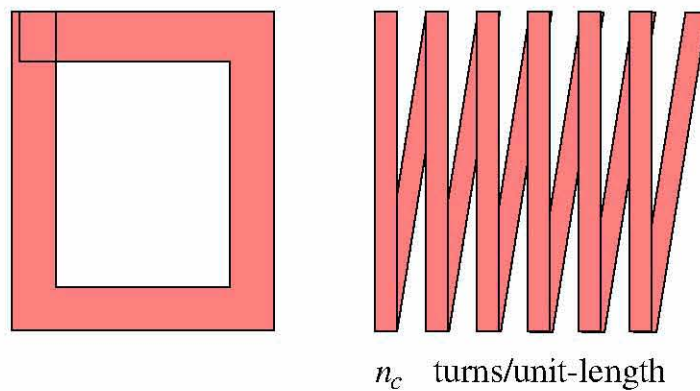

n_c turns/unit-length

Figure 1.27: A picture-frame type plate coil and the configuration of a solenoid.

There are many other configurations of electromagnet, designed for a tremendous variety of applications. Most require numerical computation to determine the field and its spatial variation.
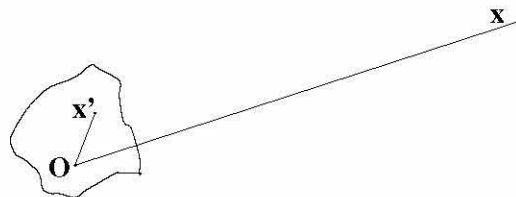
1.6.4   Magnetic Dipole

Figure 1.28: Currents localized to a small region close to the origin, with the field point far away.

The magnetic field from a "localized" current distribution. Suppose we want the field at a point $\mathbf{x}$ which is far from the currents, in the sense that for all points $\mathbf{x}'$ where $\mathbf{j}(\mathbf{x}')$ is non-negligible, $|\mathbf{x}'| << |\mathbf{x}|$, (relative to an origin near the currents). The general formula for

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A:

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3 x' \tag{1.101}$$

can be approximated by writing

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{(x^2 - 2\mathbf{x}.\mathbf{x}' + x'^2)^{1/2}} \approx \frac{1}{|\mathbf{x}|} \left(1 + \frac{\mathbf{x}.\mathbf{x}'}{|\mathbf{x}|^2} + ... \right) \tag{1.102}$$

so

$$\mathbf{A} \simeq \frac{\mu_0}{4\pi} \frac{1}{|\mathbf{x}|} \left[\int \mathbf{j}(\mathbf{x}') \, d^3 x' + \frac{1}{|\mathbf{x}|^2} \int \mathbf{x}.\mathbf{x}' \mathbf{j}(\mathbf{x}') \, d^3 x' \right] \quad . \tag{1.103}$$

Now we convert these integrals into more convenient expressions using $\nabla.\mathbf{j} = 0$. Actually the first one is zero. This follows immediately from the identity

$$\nabla.(\mathbf{j}\mathbf{x}) = \mathbf{x}(\nabla.\mathbf{j}) + (\mathbf{j}.\nabla)\mathbf{x} = \mathbf{j} \tag{1.104}$$

(which uses $\nabla \mathbf{x} = \mathbf{I}$ i.e. $\partial x_i / \partial x_j = \delta_{ij}$, and $\nabla.\mathbf{j} = 0$). So

$$\int \mathbf{j} \, d^3 x' = \int \nabla'.(\mathbf{j}\mathbf{x}') \, d^3 x' = \int_S \mathbf{x}' \mathbf{j}.\mathbf{dS} = 0 \quad , \tag{1.105}$$

for any surface S that encloses all currents so that $\mathbf{j} = 0$ on S. The second term is simplified using the same identity but being careful to distinguish between $\mathbf{x}$ and $\mathbf{x}'$, and using notation ∇' to denote the gradient operator that operates on $\mathbf{x}'$, $\mathbf{j}(\mathbf{x}')$, <u>not</u> on $\mathbf{x}$.

$$\begin{aligned}
\int (\mathbf{x}.\mathbf{x}') \mathbf{j}(\mathbf{x}') \, d^3 x' &= \int (\mathbf{x}.\mathbf{x}') \nabla'.(\mathbf{j}\mathbf{x}') \, d^3 x' \\
&= \int \nabla'.(\mathbf{j}\mathbf{x}'(\mathbf{x}.\mathbf{x}')) - \mathbf{x}'\mathbf{j}.\nabla'(\mathbf{x}'.\mathbf{x}) \, d^3 x' \\
&= -\int \mathbf{x}'\mathbf{j}.(\nabla\mathbf{x}').\mathbf{x} \, d^3 x' \\
&= -\int \mathbf{x}'\mathbf{j}.\mathbf{I}.\mathbf{x} \, d^3 x' = -\int \mathbf{x}'(\mathbf{j}.\mathbf{x}) \, d^3 x' \tag{1.106}
\end{aligned}$$

But

$$\mathbf{x} \wedge (\mathbf{x}' \wedge \mathbf{j}) = (\mathbf{j}.\mathbf{x})\mathbf{x}' - (\mathbf{x}.\mathbf{x}')\mathbf{j} \quad . \tag{1.107}$$

So

$$\int \mathbf{x} \wedge (\mathbf{x}' \wedge \mathbf{j}) \, d^3 x' = -2 \int \mathbf{x}.\mathbf{x}'\mathbf{j} \, d^3 x' \quad , \tag{1.108}$$

by the integral relation just proved. [This identity is true for <u>any</u> $\mathbf{x}$]. Therefore our approximation for $\mathbf{A}$ is

$$\mathbf{A}(\mathbf{x}) = -\frac{\mu_0}{4\pi} \frac{\mathbf{x}}{|\mathbf{x}|^3} \wedge \left(\frac{1}{2} \int \mathbf{x}' \wedge \mathbf{j}(\mathbf{x}') \, d^3 x' \right) \tag{1.109}$$

or

$$\boxed{\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \wedge \mathbf{x}}{|\mathbf{x}|^3} \quad ,} \tag{1.110}$$

where the <u>Magnetic Dipole Moment</u> of the localized curent distribution is

$$\mathbf{m} \equiv \frac{1}{2} \int \mathbf{x}' \wedge \mathbf{j} \, d^3 x' \quad . \tag{1.111}$$

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We have derived this expression for an arbitrary j distribution; but if the localized current is a loop current filament,
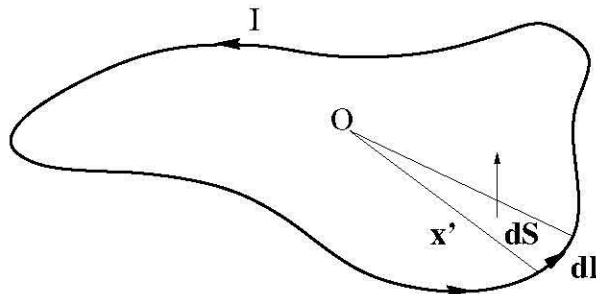

Figure 1.29: Current-carrying loop integration to give dipole moment.

$$\mathbf{m} = \frac{1}{2} \int \mathbf{x}' \wedge \mathbf{j} \; d^3x' = \frac{1}{2} \int \mathbf{x} \wedge I \mathbf{dl} \quad . \tag{1.112}$$

If the loop is planar,

$$\frac{1}{2} \mathbf{x} \wedge \mathbf{dl} = \mathbf{ds} \qquad , \tag{1.113}$$

where $\mathbf{ds}$ is the element of surface. So $\mathbf{m}$ is (current $\times$ area) for a planar filament. The magnetic field is obtained from $\mathbf{B} = \nabla \wedge \mathbf{A}$

$$\mathbf{B} = \frac{\mu_0}{4\pi} \left[3 \frac{\mathbf{x}}{|\mathbf{x}|} \left(\frac{\mathbf{x}}{|\mathbf{x}|} . \mathbf{m} \right) - \mathbf{m} \right] \frac{1}{|\mathbf{x}|^3} \tag{1.114}$$

1.6.5 Revisionist History of Electromagnetic Induction

Michael Faraday first showed the effect of induction: a transient current can be induced in one circuit by changes in another. This was ~ 1830. [Faraday knew no mathematics beyond the idea of proportionality EMF $\propto$ rate of change of B-flux]. Suppose history had been different and we knew only the Lorentz force law:

$$\mathbf{F} = q \left(\mathbf{E} + \mathbf{v} \wedge \mathbf{B} \right) \tag{1.115}$$

we could have "proved" the necessity of induction by "pure thought".

Assume Galilean Invariance: physical laws must be invariant under changes to moving coordinate systems $x' = x - vt$, $t' = t$. [Universally assumed in Faraday's time. Einstein doesn't come till 1905!] Consider a rigid (wire) circuit moved past a magnet: Each electron in the circuit (revisionist!) feels a Lorentz force

$$\mathbf{F} = q \left(\mathbf{v} \wedge \mathbf{B} \right) \tag{1.116}$$

as it is dragged through the magnetic field. The electric field in the rest frame of the magnet is zero. And the total electromotive force (integrated force per unit charge) round the entire circuit is

$$\frac{1}{q} \oint_C \mathbf{F}.\mathbf{dl} = \oint_C \mathbf{v} \wedge \mathbf{B}.\mathbf{dl} \tag{1.117}$$
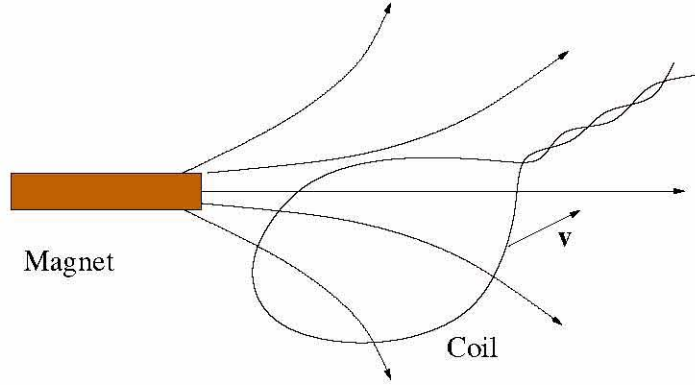
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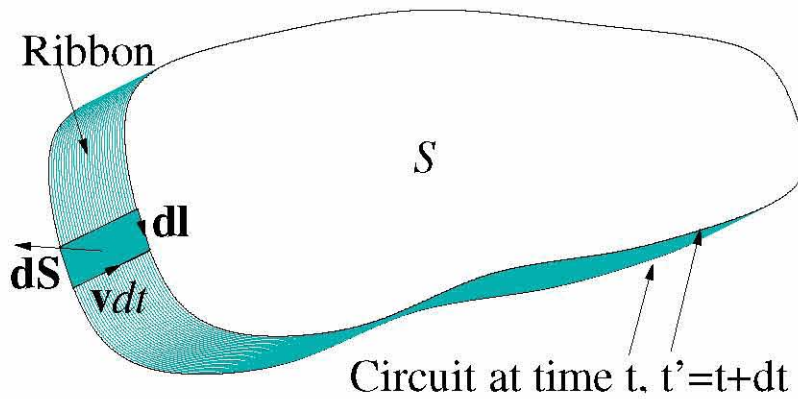Figure 1.30: A rigid coil moving past a steady magnet.

Figure 1.31: Surface elements in the application of Gauss's law to succeeding instants of time.

This is generally a non-zero quantity. In fact, this quantity can be transformed on the basis of purely geometrical considerations. Let's calculate the rate of change of total magnetic flux due to circuit motion, in a static B-field. Apply Gauss' law to volume

$$
\begin{aligned}
0 &= \int_V \nabla . \mathbf{B} \, d^3 x = \int_{S_{total}} \mathbf{B} . \mathbf{dS} = \int_{S'} - \int_S - \int_{ribbon} \mathbf{B} . \mathbf{ds} \\
&= \int_{S'} \mathbf{B} . \mathbf{dS} - \int_S \mathbf{B} . \mathbf{dS} - \int \mathbf{B} . (\mathbf{v} dt \wedge \mathbf{dl}) \\
&= d\Phi - dt \oint_C (\mathbf{B} \wedge \mathbf{v}) . \mathbf{dl}
\end{aligned}
\tag{1.118}
$$

[where $d\Phi$ is change in flux]. So

$$
\frac{d\Phi}{dt} = - \oint_C (\mathbf{v} \wedge \mathbf{B}) . \mathbf{dl} \quad .
\tag{1.119}
$$

(pure geometry when $\partial \mathbf{B} / \partial t = 0$).

This equation can alternatively be obtained algebraically by writing

$$
\frac{d\Phi}{dt} = \int \frac{d\mathbf{B}}{dt} . \mathbf{dS} = \int (\mathbf{v} . \nabla) \mathbf{B} . \mathbf{dS}
\tag{1.120}
$$

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and using

$$\nabla \wedge (\mathbf{B} \wedge \mathbf{v}) = (\mathbf{v}.\nabla)\,\mathbf{B} + (\nabla.\mathbf{v})\,\mathbf{B} - (\mathbf{B}.\nabla)\,\mathbf{v} - (\nabla.\mathbf{B})\,\mathbf{v} = (\mathbf{v}.\nabla)\,\mathbf{B} \quad . \tag{1.121}$$

So

$$\frac{d\Phi}{dt} = \int \nabla \wedge (\mathbf{B} \wedge \mathbf{v}).\mathbf{dS} = \oint_C (\mathbf{B} \wedge \mathbf{v}).\mathbf{dS} \tag{1.122}$$

Anyway EMF is

$$\frac{1}{q}\oint_C \mathbf{F}.\mathbf{dl} = \oint_C (\mathbf{v} \wedge \mathbf{B}).\mathbf{dS} = -\frac{d\Phi}{dt} \tag{1.123}$$

Now we consider the whole situation when the frame of reference is changed to one in which the circuit is stationary and the magnet is moving. By Galilean invariance the total EMF is the same, and

$$\frac{1}{q}\oint \mathbf{F}.\mathbf{dl} = -\frac{d\Phi}{dt} \tag{1.124}$$

But now $\mathbf{v} = 0$, and instead $\mathbf{B}$ is changing so

$$\frac{d\Phi}{dt} = \int \frac{\partial \mathbf{B}}{\partial t}.\mathbf{dS} \quad . \tag{1.125}$$

In this case also the Lorentz force on the charges is

$$\mathbf{F} = q\,(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) = q\mathbf{E} \qquad (\text{since } \mathbf{v} = 0) \tag{1.126}$$

There has to be an electric field in this frame of reference. And also

$$\frac{1}{q}\oint \mathbf{F}.\mathbf{dl} = \oint \mathbf{E}.\mathbf{dl} = -\frac{d\Phi}{dt} = -\int \frac{\partial \mathbf{B}}{\partial t}.\mathbf{dS} \tag{1.127}$$

Apply Stokes' theorem to the $\mathbf{E}.\mathbf{dl}$ integral:

$$\int_S \left[\nabla \wedge \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right].\mathbf{dS} = 0 \quad . \tag{1.128}$$

But this integral has to be zero for all S (and C) which can be true only if its integrand is everywhere zero:

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.129}$$

"Faraday's" Law (expressed in differential form) (which Faraday understood intuitively but could not have formulated in math)

1.6.6   Inductance

Suppose we have a set of circuits with currents $I_i (i = 1...N)$. These are inductively coupled if the current in one gives rise to flux linking the others. Because Ampere's law is linear ($\mathbf{B} \propto \mathbf{j}$), the flux linking circuit j from current I_i is proportional to I_i. Consequently, the total flux linking circuit j can be written

$$\Phi_j = \sum_i M_{ji} I_i \tag{1.130}$$

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(Summation over I_i) different currents. M is a matrix. The element M_{ij} is an <u>inductance</u> between currents i and j.. Its units are

$$\frac{flux}{current} \leftrightarrow \frac{\text{Wb}}{\text{A}} \leftrightarrow \text{Henrys} \quad . \tag{1.131}$$

The electromotive force or voltage V_j induced in the j'th circuit is then:

$$V_j = \frac{d}{dt}\phi_j = \sum_i M_{ji}\dot{I}_i \quad . \tag{1.132}$$

For the simplest case $N = 1$ circuit. $M_{ii} \to L$ the self inductance

$$V = L\dot{I} \quad . \tag{1.133}$$

It can be shown from Maxwell's equations that M_{ij} is <u>symmetric</u>.