1. **Give the functional form and sketch both the real and imaginary components of the Fourier transformation (with respect to x) of the following functions. Identify the spatial frequencies, wave-numbers, and features in the spectra.**

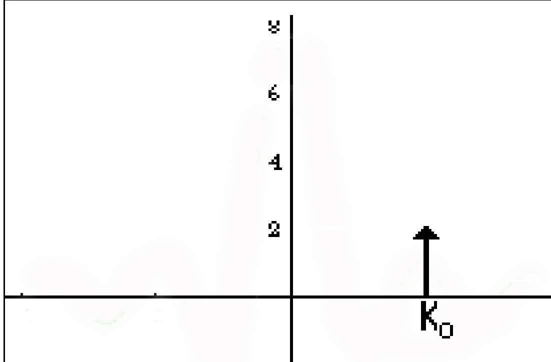
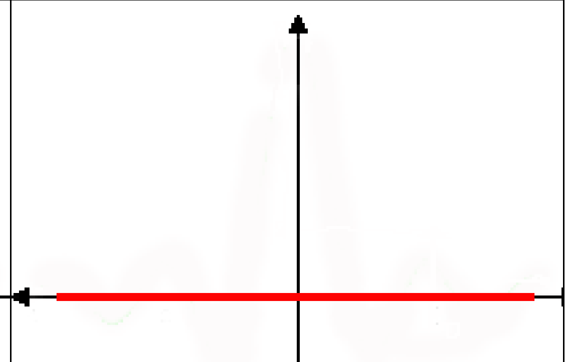
   k_o **is a constant real number**

   a. $\cos(k_o x) + i\sin(k_o x)$ **(5 points)**

   Functional Form:

   $$F\{\cos(k_o x)\} = \pi[\delta(k - k_o) + \delta(k + k_o)]$$
   $$F\{\sin(k_o x)\} = i\pi[\delta(k + k_o) - \delta(k - k_o)]$$
   $$F\{\cos(k_o x) + i\sin(k_o x)\} = 2\pi\delta(k - k_o)$$

   Sketches:

Real:	Imaginary:
	

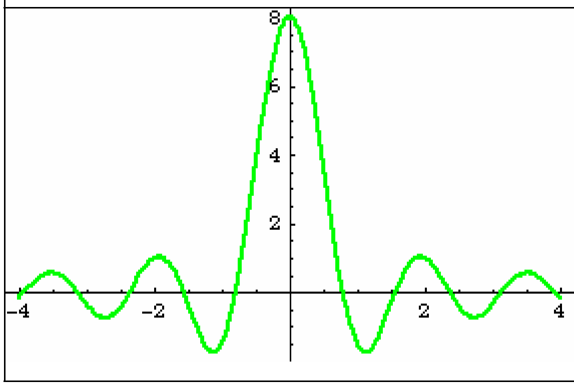
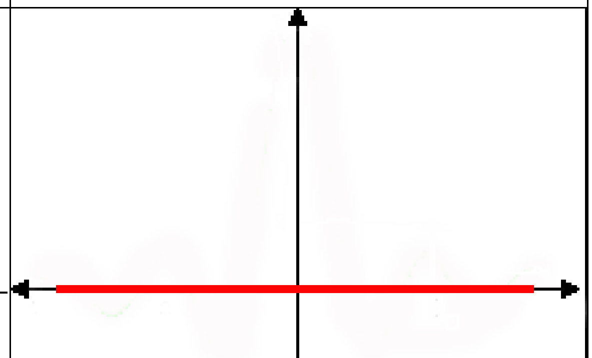
b. $TopHat\left(\frac{x}{4}\right)$ **(5 Points)**

Functional Form:

$$F\{TopHat(x)\} = 2\,\text{sinc}(k)$$

$$\therefore F\left\{TopHat\left(\frac{x}{4}\right)\right\} = 8\,\text{sinc}(4k)$$

Sketches:

Real:	Imaginary:
	

c. $\sin(8k_o x)\cos(k_o x)$ **(5 points)**

Functional Form:

$$F\{\cos(k_o x)\} = \pi[\delta(k-k_o)+\delta(k+k_o)]$$

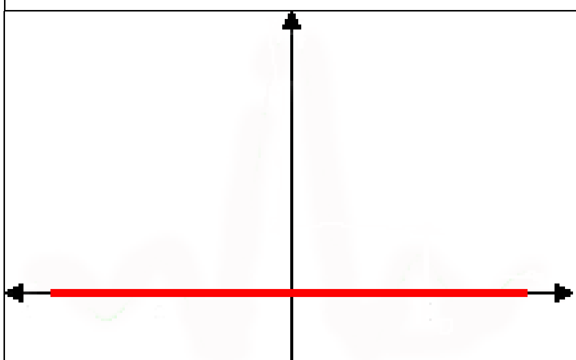
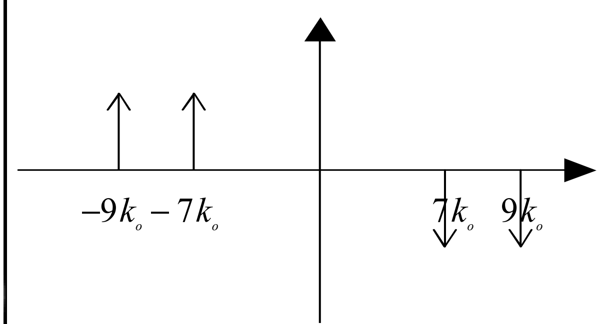
$$F\{\sin(8k_o x)\} = i\pi[\delta(k+8k_o)-\delta(k-8k_o)]$$

$$F\{\sin(8k_o x)\cos(k_o x)\}$$

$$= i\pi^2 [[\delta(k+8k_o)-\delta(k-8k_o)] \otimes [\delta(k-k_o)+\delta(k+k_o)]]$$

$$= i\pi^2 [\delta(k+9k_o)+\delta(k+7k_o)-\delta(k-9k_o)-\delta(k-7k_o)]$$

Sketches:

Real:	Imaginary:
	

d. $\sum_{n=-2}^{2} \delta(x-n) = \sum_{n=-\infty}^{\infty} \delta(x-n) \bullet TopHat\left(\frac{x}{2}\right)$ **(5 Points)**

Functional Form:

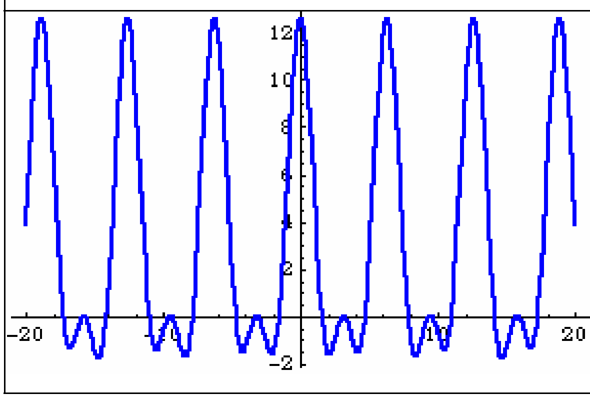
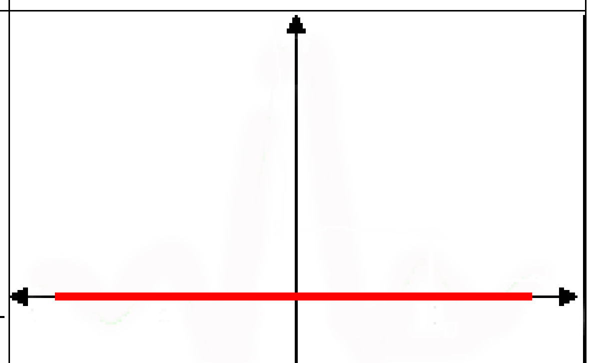
$$F\left\{\sum_{n=-\infty}^{\infty} \delta(x-n)\right\} = \pi \sum_{n=-\infty}^{\infty} \delta(x-n)$$

$$F\left\{TopHat\left(\frac{x}{2}\right)\right\} = 4\,\text{sinc}(2k)$$

$$\therefore F\left\{\sum_{n=-2}^{2} \delta(x-n)\right\} = \pi \sum_{n=-\infty}^{\infty} \delta(x-n) \otimes 4\,\text{sinc}(2k)$$

$$= 4\pi \sum_{n=-\infty}^{\infty} \text{sinc}(2k-n)$$

Skteches:

Real:	Imaginary:
	

e. $TopHat(x-4)$ **(5 Points)**

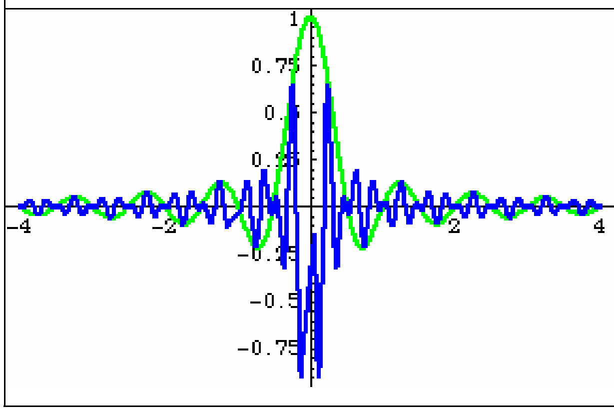
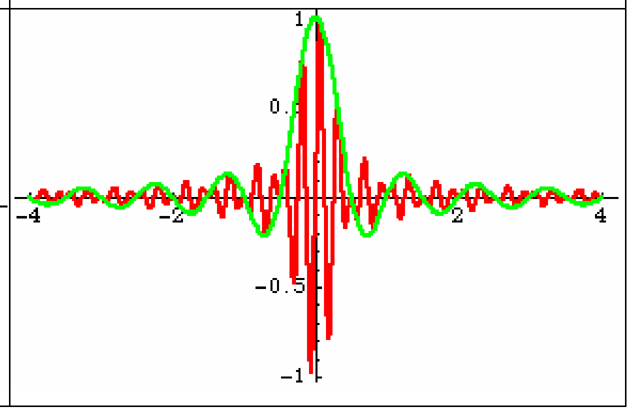
Functional Form:

$$F\{TopHat(x)\} = 2\,\text{sinc}(k)$$

$$F\{g(x-a)\} = e^{ika}G(k)$$

$$F\{TopHat(x-4)\} = 2\underbrace{e^{ik4}}_{\substack{period \\ is\ \pi/2}}\ \text{sinc}(k)$$

Sketeches:

Real:	Imaginary:
	

2. **Show that the k=0 point of F (k) is equal to the area f(x), where f(x) ⇔ F(k). (10 points)**

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\ e^{-ikx} dx$$

$$F\{f(x)\}\big|_{k=0} = F(0) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx\big|_{k=0}$$

$$= \int_{-\infty}^{\infty} f(x) dx = \ area\ \ of\ \ f(x)$$

3. **Show that the** $k_x = 0$ **point of** $F(k_x, k_y)$ **is equal to the projection of** $f(x,y)$ **onto the y-axis where** $f(x,y) \Leftrightarrow F(k_x, k_y)$**. (5 points)**

$$F\{f(x,y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\ e^{-ik_x x} e^{-ik_y y} dxdy$$

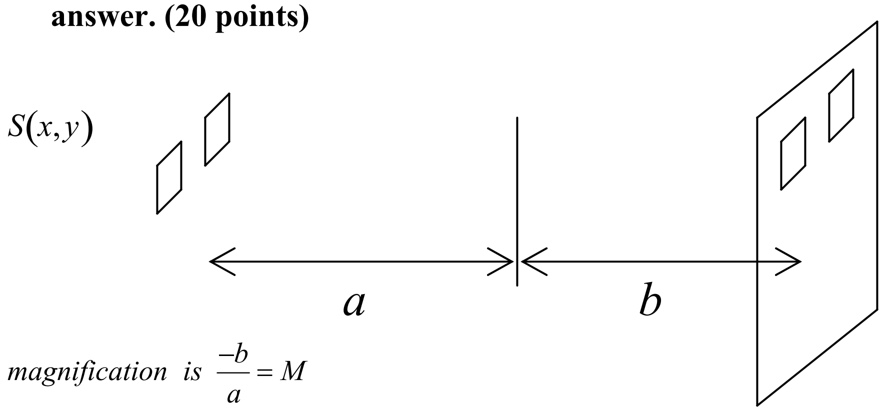
$$F\{f(x,y)\}\big|_{k_x=0} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\ e^{-ik_x x} e^{-ik_y y} dxdy\big|_{k_o=0}$$

$$= \int_{-\infty}^{\infty} dy\ \ e^{-ik_y y} \underbrace{\int_{-\infty}^{\infty} f(x,y) dx}_{\substack{P(y) = projection\ \ of \\ f(x,y)\ \ onto\ \ the\ \ y-axis}} = \int_{-\infty}^{\infty} P(y)\ e^{-ik_y y} dy$$

$$= F\{P(y)\}$$

4. **A simple way of characterizing the spatial distribution of a radiation source is to image it with a pin-hole imager.**

   a. **Draw the experimental geometry, and explain why this is a useful experiment. What is the form of the image, $I(x,y)$ in terms of the source distribution, $S(x,y)$ assuming a perfect pin-hole camera? Place the a distance, a from the source and the screen (detectors) and a distance, b from the pin-hole. Include the magnification in your answer. (20 points)**



$$magnification\ \ is\ \ \frac{-b}{a} = M$$

$$\text{Image}(x,y) = \frac{1}{M^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S\left(\frac{x'}{M},\frac{y'}{M}\right) IRF\left(x,y\middle|x',y'\right)dx'dy'$$

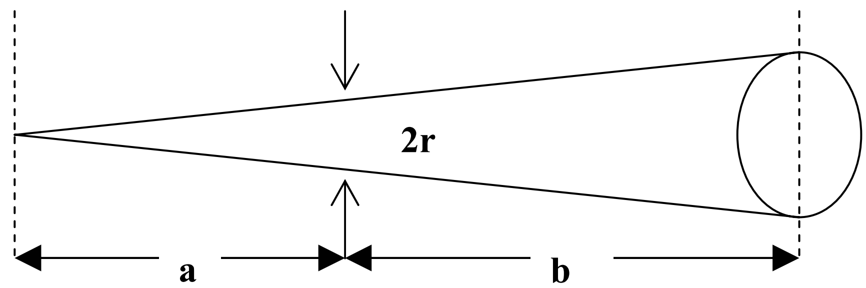
with a perfect pinhole, then

$$IRF\left(x,y\middle|x',y'\right) = \delta(x-x_o)\delta(y-y_o)$$

$$and\ \ \text{Image}(x,y) = \frac{S\left(\frac{x}{M},\frac{y}{M}\right)}{M^2}$$

So, in this ideal case, a pinhole camera provides a magnified image of the source function.

b. **Assume the pin-hole is of a finite size (radius = r), and therefore a true representation of the source is not observed. In linear imaging terms, explicitly describe the detected signal, $I(x,y)$ in both the source distribution and the pin-hole size. (10 points)**

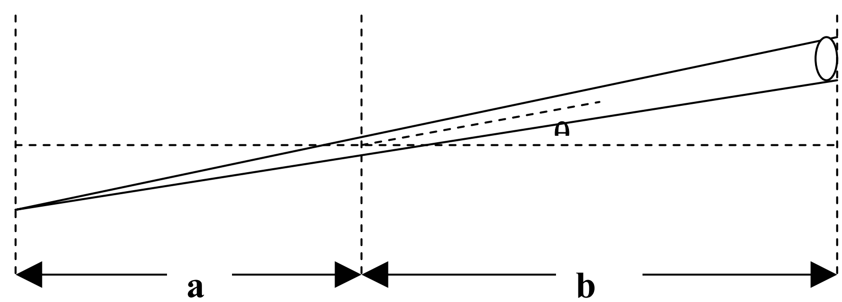


A point gets mapped to a circle of radius $\left(\frac{a+b}{a}\right)r$.

$$\therefore I(x,y) = S(x,y) \otimes Circ\left(\frac{xa}{(a+b)r}, \frac{ya}{(a+b)r}\right)$$

$$where \quad Circ(x,y) = \begin{cases} 1, & x^2 + y^2 \le 1 \\ 0. & otherwise \end{cases}$$

c. **There is an oblique effect if the source extends far from the central line through the pin-hole. Without calculating the geometry of this effect, describe how it arises. (10 points)**



The solid angle is proportional to $\cos^2(\theta)$. The angle between rays and detector is proportional to $\cos(\theta)$. The effective spot size of the hole is proportional to $\cos(\theta)$. Therefore, I is proportional to $\cos^4(\theta)$.

5. The Nyquist condition states that to correctly measure a frequency the signal must be sampled at least twice a period.

   a. Let f be the Nyquist frequency, show that the signals, $\cos[2\pi(f+\Delta f)t]$ *and* $\cos[2\pi(f-\Delta f)t]$, lead to the exact same data points when sampled at times $t(n) = n/2f$.

$$\{P_+\} = \cos\left[2\pi(f+\Delta f)\frac{n}{(2f)}\right],\quad n = 0 \to N$$

$$= \cos\left[\pi n + n\pi\frac{\Delta f}{f}\right] = \cos(\pi n)\cos\left(n\pi\frac{\Delta f}{8}\right) - \sin(\pi n)\sin\left(n\pi\frac{\Delta f}{8}\right)$$

$$\{P_-\} = \cos\left[2\pi(f-\Delta f)\frac{n}{(2f)}\right],\quad n = 0 \to N$$

$$= \cos\left[\pi n - n\pi\frac{\Delta f}{f}\right] = \cos(\pi n)\cos\left(n\pi\frac{\Delta f}{8}\right) + \sin(\pi n)\sin\left(n\pi\frac{\Delta f}{8}\right)$$

   b. Explain aliasing in terms of the above result. (10 points)

From the above, we see that if the sampled frequency is less than f it is correctly measured, otherwise if it is aliased, it appears as $f - \Delta f$.