22.02 – Introduction to Applied Nuclear Physics

Problem set # 2

Issued on Wednesday Feb. 22, 2012. Due on Wednesday Feb. 29, 2012

Problem 1: Gaussian Integral (solved problem)

The Gaussian function $g(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$ is often used to describe the shape of a wave packet. Also, it represents the probability density function (p.d.f.) of the Gaussian distribution.

a) Calculate the integral $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$

Solution

Here I will give the calculation for the simpler function: $G(x) = e^{-x^2}$. The integral $I = \int_{-\infty}^{\infty} e^{-x^2}$ can be squared as:

$$I^2 = \left(\int_{-\infty}^{\infty} dx\, e^{-x^2}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dy\, e^{-x^2} e^{-y^2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dy\, e^{-(x^2+y^2)}$$

This corresponds to making an integral over a 2D plane, defined by the cartesian coordinates x and y. We can perform the same integral by a change of variables to polar coordinates:

$$\begin{cases} x = r\cos\vartheta \\ y = r\sin\vartheta \end{cases}$$

Then $dxdy = rdrd\vartheta$ and the integral is:

$$I^2 = \int_0^{2\pi} d\vartheta \int_0^{\infty} dr\, r\, e^{-r^2} = 2\pi \int_0^{\infty} dr\, r\, e^{-r^2}$$

Now with another change of variables: $s = r^2$, $2rdr = ds$, we have:

$$I^2 = \pi \int_0^{\infty} ds\, e^{-s} = \pi$$

Thus we obtained $I = \int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}$ and going back to the function $g(x)$ we see that its integral just gives $\int_{-\infty}^{\infty} g(x) = 1$ (as needed for a p.d.f).

Note: we can generalize this result to $\int_{-\infty}^{\infty} ae^{-(x+b)^2/c^2}dx = ac\sqrt{\pi}$

Problem 2: Fourier Transform

Give the Fourier transform of :

(a – solved problem) The sine function $\sin(ax)$

Solution

The Fourier Transform is given by: $\mathcal{F}[f(x)][k] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ikx} f(x)$. The sine function $\sin(ax)$ can be written as $\sin(ax) = \frac{e^{iax} - e^{-iax}}{2i}$. By using a property of the Dirac delta function that we saw in Pset 1, we have

$$\mathcal{F}[\sin(ax)] = \sqrt{\frac{\pi}{2}} i\left(\delta(k-a) - \delta(k+a)\right)$$

b) The Gaussian function $g(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$.

[Note: this result can be found in any math book, together with the proof. Make sure you understand how it is derived and how the integral is made by reproducing the proof here.]

b) The rectangular function $r(x)$

$$r(x) = \begin{cases} \frac{1}{2a} & -a \le x \le a \\ 0 & |x| > a \end{cases}$$

Problem 3: Kinetic energy measurement - Method I

An electron is prepared in a state such that its wavefunction (in a 1D space) is $\psi(x) = \frac{1}{\sqrt{\pi}} \cos(\sqrt{21}x)$.

a) Following the "recipe" given in class, what do you expect to obtain when measuring the electron **kinetic** energy?
[You should answer the questions: What are the possible results of a kinetic energy measurement? What are the probabilities of each possible kinetic energy measurement result?]

Problem 4: Kinetic energy measurement - Method II

A neutron is in the quantum state $|\psi\rangle$ such that its wavefunction is $\psi(x) = \frac{1}{\sqrt{\pi}} \sin(\sqrt{13}x)$.

a) Write the state $|\psi\rangle$ in the kinetic energy basis.

b) Given your answer to question a) what do you expect to obtain when measuring the neutron kinetic energy?

Problem 5: Expectation values

Consider a wavefunction described by

$$\psi(x) = \frac{1}{(2\pi a^2)^{1/4}} e^{-x^2/(4a^2)} e^{-ik_0 x}$$

where k_0 is the wave number associated with a momentum $p_0 = \hbar k_0$. (Note that this state describes a traveling wavepacket.)

a) What is the expectation value of the position?

b) What is the uncertainty of the position $\Delta x^2 = \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2$?
[Here $\langle \hat{x}^2 \rangle$ is the expectation value of the operator $\hat{x}^2$.]

c) Write the wavefunction $\psi(x)$ in the *momentum* basis. What are the possible outcomes of a momentum measurement? What are their probabilities?
[Hint: remember the property of the Fourier transform: if $\mathcal{F}[f(x)] = F(k)$, we have that $\mathcal{F}[f(x)e^{ik_0 x}] = F(k+k_0)$]

d) What is the expectation value of the momentum operator $\langle \hat{p} \rangle$?
Solve this questions in two ways – making sure they give the same answer!
i) Using the results in question c, calculate the average from a formula such as $\langle \hat{p} \rangle = \int pP(p)dp$.
ii) Use the definition of expectation value for the momentum operator: $\langle \hat{p} \rangle = \int \psi(x)^* \hat{p}[\psi(x)]\, dx$.

e) What is its uncertainty $\Delta p^2 = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2$? (Complete the following partial answer)

Solution:

To find Δp we first need $\langle p^2 \rangle$.

$$\langle \hat{p}^2 \rangle = \ldots \text{fill in the missing steps} \cdots = \frac{-\hbar^2}{\sqrt{2\pi a^2}} \int_{-\infty}^{\infty} dx \left[-\frac{1}{2a^2} + \frac{1}{4a^4} x^2 - \frac{ik_0}{a^2} x - k_0^2 \right] e^{-x^2/2a^2}$$

The term linear in x is zero, since it is an odd function that will integrate to zero. Using the Gaussian integral identity given in Problem 1 and the second, following identity:

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{a^3}}$$

we obtain:

$$\langle p^2 \rangle = \frac{-\hbar^2}{\sqrt{2\pi a^2}} \left[-\frac{1}{2a^2}\sqrt{\pi 2a^2} + \frac{1}{4a^2}\frac{1}{2}\sqrt{\pi(2a^2)^3} - k_0^2\sqrt{\pi 2a^2} \right] = \hbar^2 k_0^2 + \frac{\hbar^2}{4a^2}$$

The uncertainty is then : ...

c) What is the relationship between the product of the position and momentum uncertainties? Assume that the wave-function is very localized (that is, its spread a is very small, $a \to 0$). What happens to the momentum uncertainty?

Problem 6: Parity operator

The parity operation flips the sign of the spatial coordinates. This can be formalized by defining the parity operator (in one dimension), $\widehat{P}$,

$$\widehat{P}[f(x)] = f(-x)$$

a) Now consider a free particle in one dimension with Hamiltonian, $\widehat{H} = \widehat{p}^2/2m$. Prove that parity commutes with this Hamiltonian, $\left[\widehat{H}, \widehat{P}\right] = 0$.

b) If the two operators commute, there must be a set of eigenfunctions common to both energy and parity. Starting from the two independent energy eigenfunctions,

$$\begin{aligned} \varphi_1(x) &= e^{ikx} \\ \varphi_2(x) &= e^{-ikx} \end{aligned}$$

construct two linear combinations of $\varphi_1(x)$ and $\varphi_2(x)$,

$$\begin{aligned} \psi_1(x) &= T_{11}\varphi_1(x) + T_{12}\varphi_2(x) \\ \psi_2(x) &= T_{21}\varphi_1(x) + T_{22}\varphi_2(x) \end{aligned}$$

that are common eigenfunctions to both energy and parity and give the corresponding eigenvalues for both operators. (Hint: See lecture notes, 2.6.2.B)

Problem 7: Commutation, common eigenfunctions and conserved quantities

Consider a quantum particle moving in a 3D potential, $V(\mathbf{x}) = V(x, y, z)$. Which directions (if any) of the particle's momentum commute with the Hamiltonian, if the potential takes the following form: (a, b, and c are constants)

a) $V(x, y, z) = e^{-x}(y^3 + 3z)$

b) $V(x, y, z) = a\frac{\sin(bx)}{bx}$

c) $V(x, y, z) = be^{-(x^2+y^2)/c^2}$

d) For which of the above potentials do the energy and parity (in 3D) observables have common eigenfunctions?

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