22.01 Fall 2016, Problem Set 5 Solutions

November 13, 2016

Complete all the assigned problems, and do make sure to show your intermediate work.

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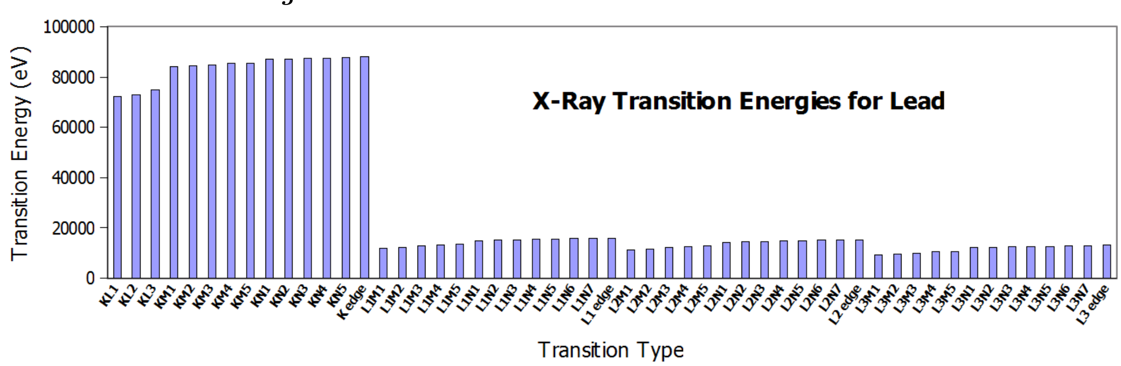
1 (50 points) Skill Building Problems

1.1 (16 points) Consider the three methods by which photons can interact with electrons in matter: Compton scattering, the photoelectric effect, and pair production. Use the Yip reading on photon interactions to help write your answers.

1. At which photon energies are each of these effects the most prominent? In other words, which of these effects can be neglected at which energies?
The photoelectric effect is most dominant at low energies, especially in the realm of 0-500keV, depending on the Z of the scattering material. Pair production cannot occur at energies below 1.022 MeV (twice the rest mass of the electron), and it doesn't really pick up in a huge way until 2-5 MeV, depending on the Z of the scattering material. Compton scattering fills in the remaining space. Both the photoelectric effect and pair production are more dominant for higher-Z materials, so their dominant energies are higher and lower, respectively, for higher-Z elements.

2. In figure 10.17, which electron energy shell transitions (give the numbers of the levels involved) are responsible for the discontinuities in the attenuation coefficient? Which of the three photon interaction methods is responsible for these discontinuities? Confirm that your estimates of the shell levels are correct, by looking up their energy transitions in the NIST x-ray energy transition database.
The K-transitions (from any shell to shell level 1) and the L-transitions (from any shell above 2 to level 2) are responsible. The simplified diagram does not show all the fine detail of the L-transitions, some of the K-transitions, and the energy scale cuts off before the M-***transitions (from levels <3 to level 3) are shown. Using the NIST table, many, many more transition energies can be seen:***

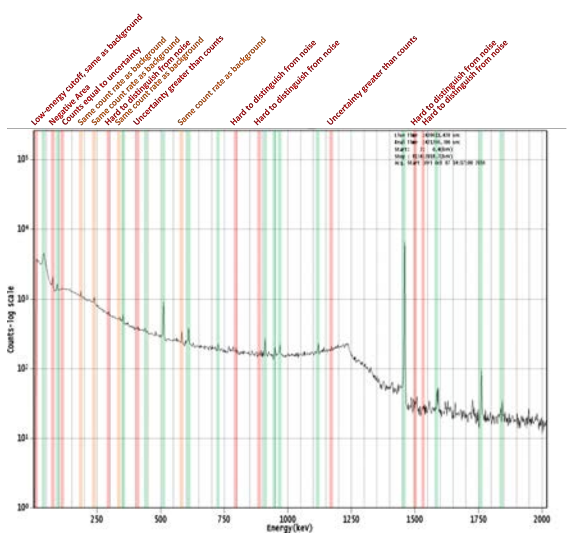


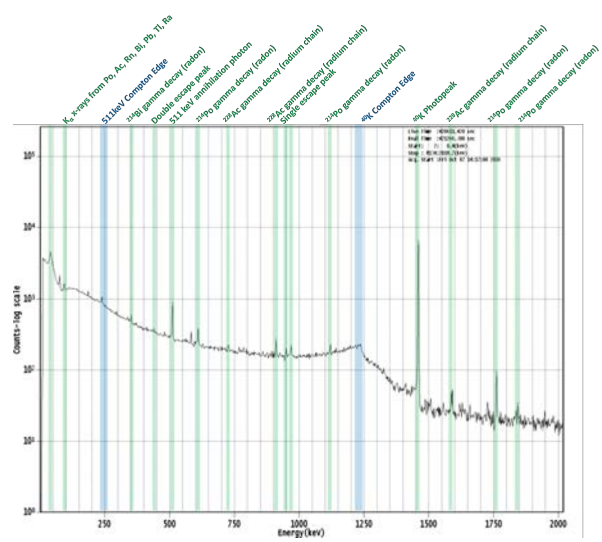
3. Why does the pair production cross section become non-zero abruptly at energies above $\frac{\hbar\omega}{2m_e c^2}$?
Because pair production cannot proceed unless there is enough energy to create an electron-positron pair. This occurs when an incoming photon interacts with/near the nucleus, creating an oppositely charged pair of electrons from pure energy. Therefore, both need to be created to conserve total charge, and the relation $\mathrm{E}_\gamma \geq (2\mathrm{m_e})\,\mathrm{c}^2$ ***must be satisfied for this to happen.***

4. Explain why it is more likely to see a single-escape pair production peak from a NaI detector (figure 10.18), while it is more likely to see a double-escape pair production peak from a semiconductor detector (figure 10.19).
NaI detectors tend to be quite large, meaning that the 511 keV photons originating from a pair production event inside the detector are quite likely to also be absorbed by the detector. They would then go on to make their own photoelectric effect or Compton scatter, leaving that energy behind almost instantaneously. If only one photon escapes, while the other is absorbed by the detector, the detector sees it as an event producing enough electrons to count as $(E_\gamma - 511\,keV)$ ***worth of energy as a single event. Semiconductor detectors tend to be very small, greatly increasing the chances that both 511 keV photons escape the detector and are not counted.***

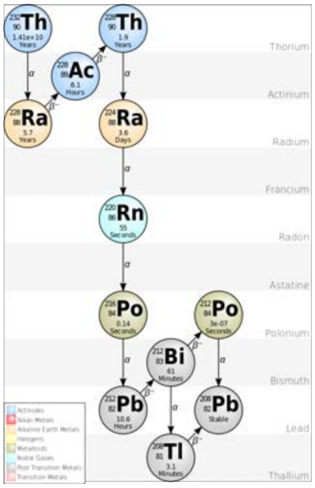
1.2 (20 points) Banana Spectral Identification

1. (8 points) Using our banana counting data, identify every single peak and major feature that you can in the spectrum. Label them with the process that created them, and calculate their true expected energies. There may be more than one isotope found in bananas!
First, we look at the spectrum itself and the tabulated uncertainties to see which <u>software identified</u> peaks don't look real to us. We are intelligently deciding which peaks, identified by a pre-programmed software algorithm, are really noise. Below is an image of the spectrum, with what I believe to be real peaks highlighted in green, and noise highlighted in red. Those whose count rates equaled those from the background spectrum are highlighted in orange, while peaks which should have physically existed are shown in blue. Note that I used this handy >>>[gamma radiation search database]<<< to find out which isotopes may have been responsible for each observed energy.





All peaks can either be attributed to various interactions of the $^{40}\mathrm{K}$ ***gamma rays interacting with the detector or its surrounding materials, or to various radium/radon decay products. A quick search yielded >>>[this paper on radon decay gamma ray energies]<<<, which greatly simplified things. For the remaining unidentified peaks, this image was used to identify all potential elements to search:***



These elements (Th, Ac, Ra, Rn, Po, Pb, Tl, Bi) were used to search for the stated peak energies on >>>this SUPER handy alpha/gamma energy searching tool<<<, identify-ing all remaining peaks. The identified, tabulated peaks are calculated using the following formulas:

$$E_\tau = E_\gamma - E_{Binding,Ge} = E_\gamma - 5\,eV \approx E_\gamma \qquad (Photopeaks) \tag{1}$$

$$E_{single-escape} = E_\tau - 511\,keV \qquad (Single\ Escape) \tag{2}$$

$$E_{double-escape} = E_\gamma - 1,022\,keV \qquad (Double\ Escape) \tag{3}$$

Using these formulas, here is a table of identified and predicted peak energies:

Peak (keV)	Nuclide	Effect	Calc. (keV)	Ref.
39.43	???	Who knows?	???	???
92.70	All Ra daughters	K_α x-rays (can't separate)	Po - 93.11	NIST
351.66	^{214}Bi	Photopeak	351.93	Chiste et al.
438.99	^{40}K	Pair prod., double escape	438.00	KAERI
510.90	Any	Pair prod. outside detector	511.00	Yip/Turner
609.13	^{214}Po	Photopeak	609.31	Chiste et al.
727.82	^{228}Ac	Photopeak	726.86	Gamma Search
911.15	^{228}Ac	Photopeak	911.20	Gamma Search
949.53	^{40}K	Pair prod., single escape	949.00	KAERI
968.95	^{228}Ac	Photopeak	968.97	Gamma Search
1,120.33	^{214}Po	Photopeak	1,120.29	Chiste et al.
1,459.78	^{40}K	Photopeak	1,460.00	KAERI
1,590.96	^{228}Ac	Photopeak	1,588.19	Gamma Search
1,761.70	^{214}Po	Photopeak	1,764.49	Chiste et al.
1,843.97	^{214}Po	Photopeak	1,847.42	Chiste et al.

2. (8 points) Given your answers for 1.2.1, show the locations, energies, and physical processes for any peaks that *should* exist on our spectrum, but were for some reason not detected.
All peaks were detected on the actual spectrum, though not all were identified by the software. The only two peaks missing from the software identification were the Compton edges for the 1,459 keV ^{40}K ***photopeak, and for the 511 keV annihilation photon's photopeak. These energies are calculated using the Compton energy shift formula:***

$$T = E_\gamma - E'_\gamma = E_\gamma \left[\frac{\alpha\,(1 - cos\,(\pi))}{1 + \alpha\,(1 - cos\,(\pi))}\right] = E_\gamma \left[\frac{2\alpha}{1 + 2\alpha}\right]; \quad \alpha = \frac{E_\gamma}{m_e c^2} \tag{4}$$

where we note that the Compton edge physically corresponds to backscattering, or scattering at an angle of $\theta = \pi$***. These Compton edges should exist at the following energies:***

Incident Photon	E_γ *(MeV)*	*Predicted, Eq. 1 (MeV)*	*Observed (MeV)*
^{40}K ***photopeak***	***1.460***	***1.243***	***˜1.24***
511 keV photopeak	***0.511***	***0.341***	Not observed

It is likely that the Compton edge for the 511 keV photon resulting from pair production outside the detector is just too weak to be seen by our detector.

3. (4 points) Why were the peaks in 1.2.2 not detected?
Probably just the software algorithm for identifying peaks, since Compton edges aren't sharp like photopeaks.

1.3 (14 points) Mass Attenuation Coefficients: For these problems, refer to the NIST table of x-ray attenuation coefficients.

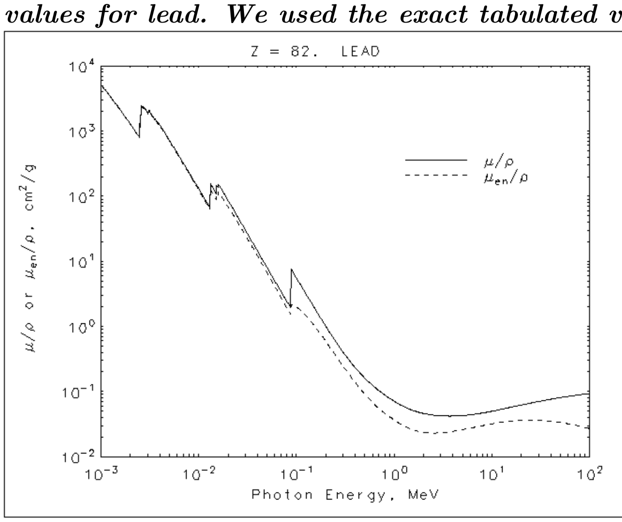
1. (8 points) Choose a lead apron thickness to shield dental patients from 90% of the x-rays emitted from a pure 40kV x-ray source. Do you have to consider any additional x-rays produced in the lead, and if so, account for this in your calculations.
Here we want to set up an exponential attenuation problem of the following form:

$$\frac{I(x)}{I_0} = 0.1 = e^{-\left(\frac{\mu}{\rho}\right)\rho x} \tag{5}$$

where $\left(\frac{\mu}{\rho}\right)$ ***is the mass attenuation coefficient in*** $\frac{cm^2}{g}$***,*** ρ ***is the density of lead in*** $\frac{g}{cm^3}$ ***(11.34*** $\frac{g}{cm^3}$***), and x is the thickness of the apron that we're solving for. We rearrange Equation 5 as follows:***

$$ln(0.1) = ln\left(e^{-\left(\frac{\mu}{\rho}\right)\rho x}\right) \qquad x = \frac{ln(0.1)}{\left(\frac{\mu}{\rho}\right)\rho} = \frac{-2.3}{-(14.36)(11.34)} = 0.014\,cm \tag{6}$$

where we have used the NIST Table of X-Ray Mass Attenuation Coefficients to find the values for lead. We used the exact tabulated value beneath the graph.



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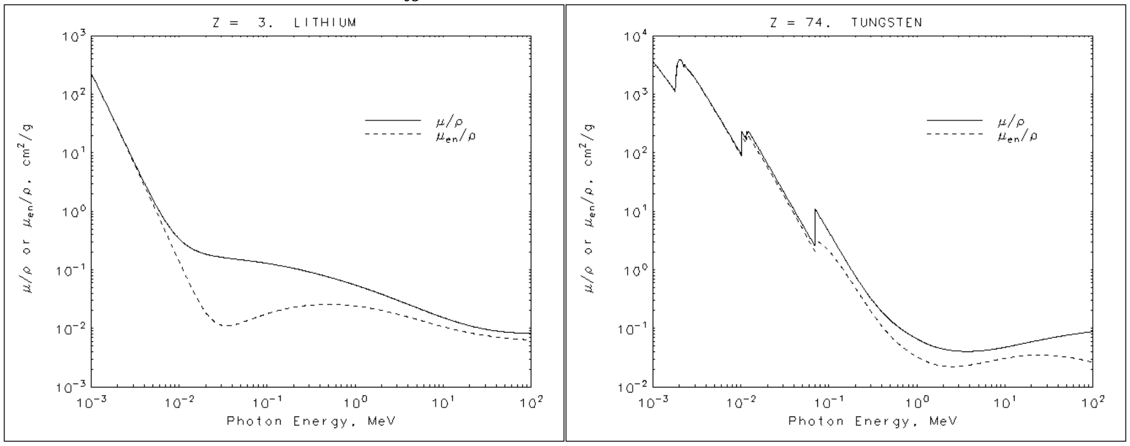
The only additional x-rays that we'd have to worry about come from the photoelectric effect knocking out electrons, and those electrons giving off L- or M-lines at around 20 keV and 3 keV, respectively. Therefore, we don't really have to take the extra x-rays into account.

2. (2 points) Explain the qualitative differences in the attenuation coefficients of lithium and tungsten in a quantitative manner, at the following energies: $E_\gamma < 100$keV, $E_\gamma = 1$MeV, $E_\gamma = 100$MeV. ***By this, we mean compare relative values of the relevant scattering cross sections, and explain any discrepancies between these and the relative values of the attenuation coefficients.***

The two mass attenuation coefficients can be seen below:



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The main differences with explanations are:

(a) ***The K- L- and M-line transitions from the photoelectric effect in tungsten being enabled, or "turning on," once the incoming photon has enough energy to remove each shell's electrons by ionization. Lithium has (1) far lower electron energy shell levels and (2) no M-shell.***

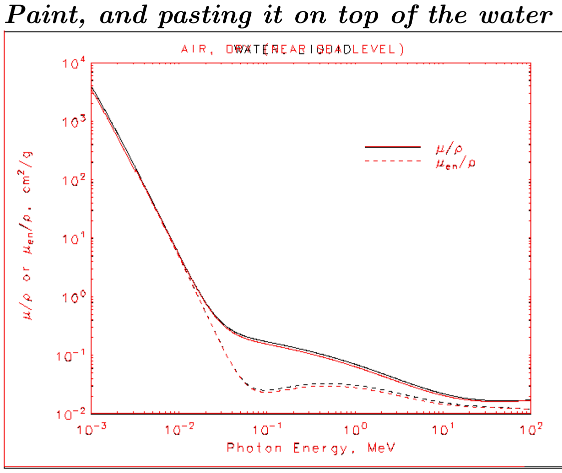
(b) ***The overall higher photoelectric effect cross section for tungsten, due to the cross section being proportional to*** Z^5

(c) ***The earlier takeoff in pair production, as evidenced by the rise in the curve around 3 MeV in tungsten, due to the*** Z^2 ***dependence of its cross section.***

(d) ***The slightly higher Compton scattering probability, evidenced by a higher value for tungsten than lead around the 1 MeV level, due to its Z-dependent cross section.***

3. (2 points) What is the origin of the discontinuities in the attenuation coefficient for tungsten? Why is there more than one step change within close proximity at some places?
The discontinuities are due to the incoming photon's energy being sufficient to ionize more tightly bound electrons from more inner shells, as the photon energy increases. The ones visible on tungsten are, in order of increasing energy, the M-lines, the L-lines, and the K-lines.

4. (2 points) For which energies is the *attenuation coefficient* in water higher than that in air? What about the *mass attenuation coefficient*?
By plotting the mass attenuation coefficients on top of each other, we can easily see where one or the other is larger. This was done by taking the air graph, making it red in MS Paint, and pasting it on top of the water curve:



Here we can clearly see that the water curve is marginally higher than the air curve at all values, particularly in the 100 keV to 1 MeV region where Compton scattering is dominant. However, multiplying by the densities of air and water would greatly bring the total attenuation coefficient for water greater by the ratio of their densities.

2 (50 points) Noodle-Scratchers

1. (20 points) Using conservation of energy and momentum, derive formulas relating the incoming photon energy (E), its outgoing energy (E'), the electron's outgoing energy (T), and the scattering angle of the photon (θ) in Compton scattering.

   (a) ***Answer:*** $T = \hbar\omega - \hbar\omega' = \hbar\omega\frac{\alpha(1-cos(\theta))}{1+\alpha(1-cos(\theta))}; \quad \alpha = \frac{\hbar\omega}{m_e c^2}$
   Note that the general strategy for this problem is given in the Turner reading on p. 180 (Equations 8.18 to 8.19). Here we start by conserving energy and work our way through:

$$T = \hbar ck - \hbar ck' = \hbar\omega - \hbar\omega' \tag{7}$$

   We substitute the following relation for ω', which is a slight modification of Turner Equation 8.12 (as the reading suggests to do):

$$\frac{\omega'}{\omega} = \frac{1}{1 + \alpha\,(1 - cos\theta)} \tag{8}$$

$$T = \hbar\omega - \hbar\omega' = \hbar\omega - \hbar\frac{\omega}{1 + \alpha\,(1 - cos\theta)} \tag{9}$$

   We multiply both sides of the first term on the right by $\frac{1+\alpha(1-cos\theta)}{1+\alpha(1-cos\theta)}$:

$$T = \hbar\omega\frac{1 + \alpha\,(1 - cos\theta)}{1 + \alpha\,(1 - cos\theta)} - \hbar\omega\frac{1}{1 + \alpha\,(1 - cos\theta)} \tag{10}$$

$$T = \hbar\omega\left[\frac{1 + \alpha\,(1 - cos\theta)}{1 + \alpha\,(1 - cos\theta)} - \frac{1}{1 + \alpha\,(1 - cos\theta)}\right] \tag{11}$$

$$T = \hbar\omega\frac{\alpha\,(1 - cos\theta)}{1 + \alpha\,(1 - cos\theta)} \tag{12}$$

2. (10 points) Using photon energy and momentum relations, derive the formula for the Compton wavelength shift as a function of θ.

   (a) ***Answer:*** $\Delta\lambda = \frac{h}{m_e c}\,(1 - cos\,(\theta))$
   First, please note that this derivation is given in the Turner reading, on pp. 177-179 (in Turner Equations 8.9 to 8.13). In case you used energy and total momentum, here we present another way: We start with the momentum and energy conservation equations:

$$\hbar k = \hbar k' + p \tag{13}$$

$$\hbar ck = \hbar ck' + T \tag{14}$$

   where k is the initial wave vector of the photon, and p is the ending momentum of the electron, given by the formula $cp = \sqrt{T\,(T + 2m_e c^2)}$. We start by conserving total momentum, and isolating the electron momentum on one side (which we have an expression for):

$$p_e = \hbar k - \hbar k' \tag{15}$$

   Squaring both sides we obtain:

$$p_e^2 = \hbar^2 k^2 + \hbar^2 k'^2 - 2\hbar^2 kk' cos\theta \tag{16}$$

by recognizing that momenta are vector quantities. Now we take energy conservation into account. We can rearrange the energy T of the electron from the electron momentum-energy relation:

$$p_e^2 = \frac{T\left(T + 2m_e c^2\right)}{c^2} \tag{17}$$

And we can use the energy conservation equation to substitute in for T:

$$T = \hbar ck - \hbar ck' \tag{18}$$

to yield:

$$p_e^2 = \frac{(\hbar ck - \hbar ck')\left(\hbar ck - \hbar ck' + 2m_e c^2\right)}{c^2} = \frac{\hbar^2 \cancel{c^2} k^2 - \hbar^2 \cancel{c^2} kk' + 2\hbar ckm_e \cancel{c^2} - \hbar^2 \cancel{c^2} kk' + \hbar^2 \cancel{c^2} k'^2 - 2\hbar ck' m_e \cancel{c^2}}{\cancel{c^2}} \tag{19}$$

$$p_e^2 = \hbar^2 k^2 - 2\hbar^2 kk' + 2\hbar ckm_e + \hbar^2 k'^2 - 2\hbar ck' m_e \tag{20}$$

Now we equate the two electron momentum equations (10 and 14) to eliminate the momentum of the electron:

$$\cancel{\hbar^2 k^2} + \cancel{\hbar^2 k'^2} - 2\hbar^2 kk' cos\theta = \cancel{\hbar^2 k^2} - 2\hbar^2 kk' + 2\hbar ckm_e + \cancel{\hbar^2 k'^2} - 2\hbar ck' m_e \tag{21}$$

$$-2\hbar^2 kk' cos\theta = -2\hbar^2 kk' + 2\hbar ckm_e - 2\hbar ck' m_e \tag{22}$$

$$\cancel{2}\hbar^{\cancel{2}} kk' \left(1 - cos\theta\right) = \cancel{2}\cancel{\hbar} cm_e \left(k - k'\right) \tag{23}$$

$$\hbar kk' \left(1 - cos\theta\right) = m_e c \left(k - k'\right) \tag{24}$$

$$\frac{\hbar}{m_e c} kk' \left(1 - cos\theta\right) = k - k' \tag{25}$$

$$\frac{\hbar}{m_e c} \left(1 - cos\theta\right) = \frac{1}{k'} - \frac{1}{k} \tag{26}$$

Finally we multiply each side of the equation by 2π***:***

$$\frac{h}{m_e c} \left(1 - cos\theta\right) = \frac{2\pi}{k'} - \frac{2\pi}{k} \tag{27}$$

and we recognize that $k = \frac{2\pi}{\lambda}$***:***

$$\frac{h}{m_e c} \left(1 - cos\theta\right) = \lambda' - \lambda = \Delta\lambda \tag{28}$$

3. (20 points, open ended) Find the minimum and maximum values of the Klein-Nishina cross section as a function of incoming photon energy. Graph the angle of the angle of minimum scattering probability as a function of incoming photon energy, and intuitively explain the features of the graph to check your answer. The angle of maximum scattering probability is always the same... with one exception. What is it, and what physical process does that process represent?
For this problem, we will use the unpolarized cross section, as we didn't discuss any implications of polarization in class. This is Equation 10.16 in Yip, or Equation 8.28 in Turner:

$$\frac{d\sigma_C}{d\Omega} = \frac{r_e^2}{2}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - sin^2\theta\right) \tag{29}$$

We know that the maximum value of the cross section is always at $\theta = 0$***, since*** $\sin^2\theta$ ***is always positive, and this term is always subtracted, and*** $\sin^2 0 = 0$***. Now we just have to find an expression for the minimum value of the cross section. Our strategy is as follows:***
1) Substitute $\frac{\omega'}{\omega}$ ***into Equation 29 to get something purely in terms of*** α ***and*** θ
2) Take the derivative of this function with respect to θ
3) Set this derivative equal to zero, and solve for θ_{min} ***in terms of*** α***, which is a measure***

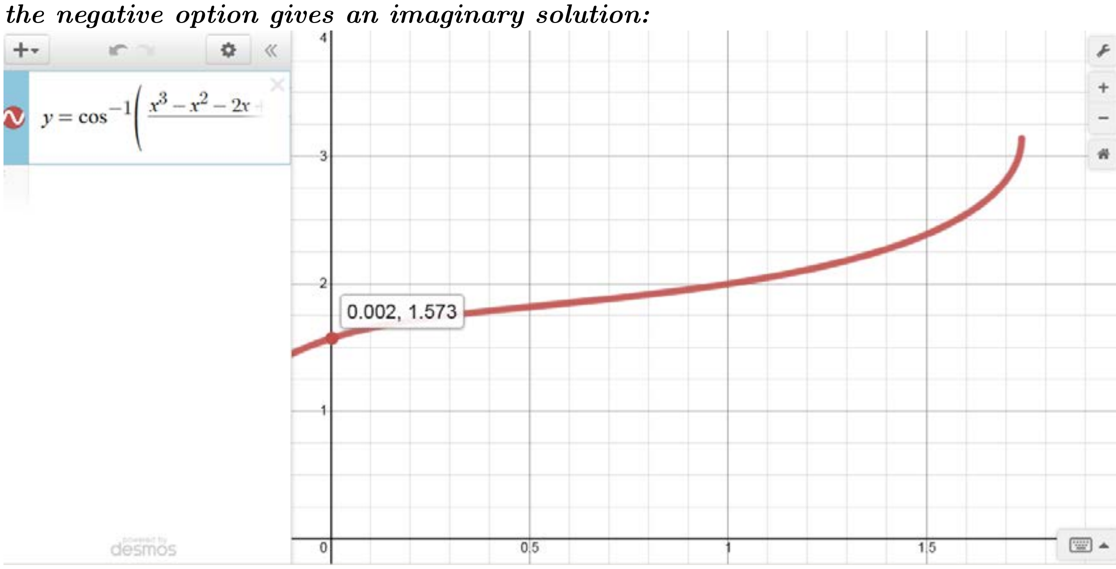
of the photon energy
4) Plus this value of θ_{min} into Equation 7 to get the value of $\frac{d\sigma_C}{d\Omega}_{min}$
We can use $\frac{\omega'}{\omega} = \frac{1}{1+\alpha(1-cos\theta)}$ to substitute for $\frac{\omega'}{\omega}$, using Equation 10.19 in Yip:

$$\frac{d\sigma_C}{d\Omega} = \frac{r_e^2}{2}\left(1+cos^2\theta\right)\left(\frac{1}{1+\alpha\left(1-cos\theta\right)}\right)^2\left[1+\frac{\alpha^2\left(1-cos\theta\right)^2}{\left(1+cos^2\theta\right)\left(1+\alpha\left(1-cos\theta\right)\right)}\right] \tag{30}$$

Next we differentiate this cross section with respect to angle and set it equal to zero, to find the angle where the cross section reaches a minimum. This was done numerically, using Maple 15 on Athena:

$$\frac{\left(\frac{d\sigma_C}{d\Omega}\right)}{d\theta} = 0 \Longrightarrow \theta_{min} = cos^{-1}\left[\frac{\alpha^3-\alpha^2-2\alpha\pm\sqrt{-3\alpha^4+6\alpha^3+10\alpha^2+4\alpha+1}-1}{\alpha^3-2\alpha^2-2\alpha}\right] \tag{31}$$

Graphing this function, we obtain a plot of α vs. θ_{min} using the positive square root, as the negative option gives an imaginary solution:



As a sanity check, this actually makes sense. The value of the angle of $\frac{d\sigma_C}{d\Omega}_{min}$ at $\alpha = 0$ is $\frac{\pi}{2}$, which can be seen in Figure 10.4 of Yip. This also follows mathematically, as in the limit of very low photon energy, the cross section reduces to:

$$\frac{d\sigma_C}{d\Omega} \approx \frac{r_e^2}{2}\left(1+cos^2\theta\right) \tag{32}$$

which is symmetric about $\theta = \frac{\pi}{2}$. The angle continues to increase with increasing photon energy, as Figure 10.4 shows, until one hits approximately $\alpha = 1.74$, where the function is undefined. This means that there is no minimum, and the cross section is continuously decreasing. This can also be seen in Figure 10.4. When $\alpha = 1$ $(\hbar\omega = 0.51\,MeV)$, there is a defined, yet shallow minimum. Once we reach $\hbar\omega = 2.56\,MeV$ $(\alpha = 5)$, the function is continuously decreasing.