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Lecture 7: Fault Tolerance and the threshold.

① Analog vs Digital

② concept computational complexity

③ the Threshold thm

④ FTQC the threshold.

I/ Analog comp

$\Rightarrow$ QC + Analog comp

Def/ Analog Comp

Input $\longrightarrow$ output

$\vec{X}(0) = \{x_0, \dots x_n\}$ $\quad \vec{X}(T)$

$$\frac{d\vec{X}}{dt} = F(\vec{X})$$

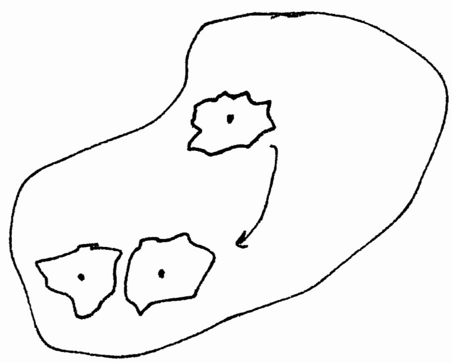
Assumptions

① NO noise in evolution

② perfect measurment.

Thm/ Analog comp $\xrightarrow[\text{to}]{\text{Reduces}}$ Digital comp.
in the presence of finite noise
or meas. error.

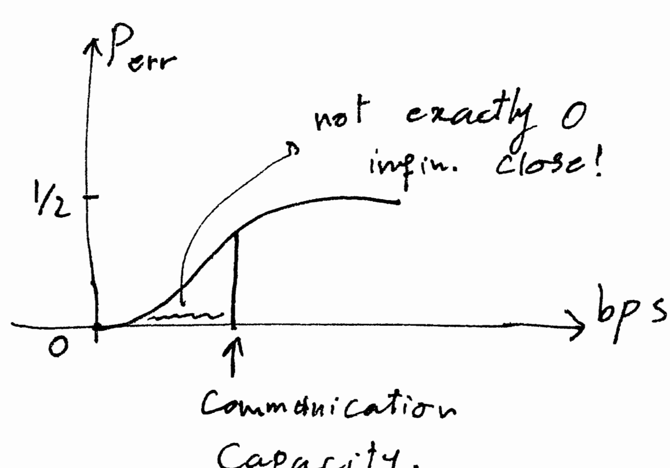
Idea



QC + noise $\overset{?}{\Longrightarrow}$ Digital comp

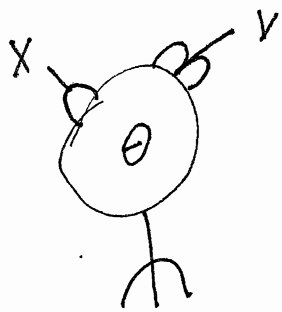
II / Computational capacity

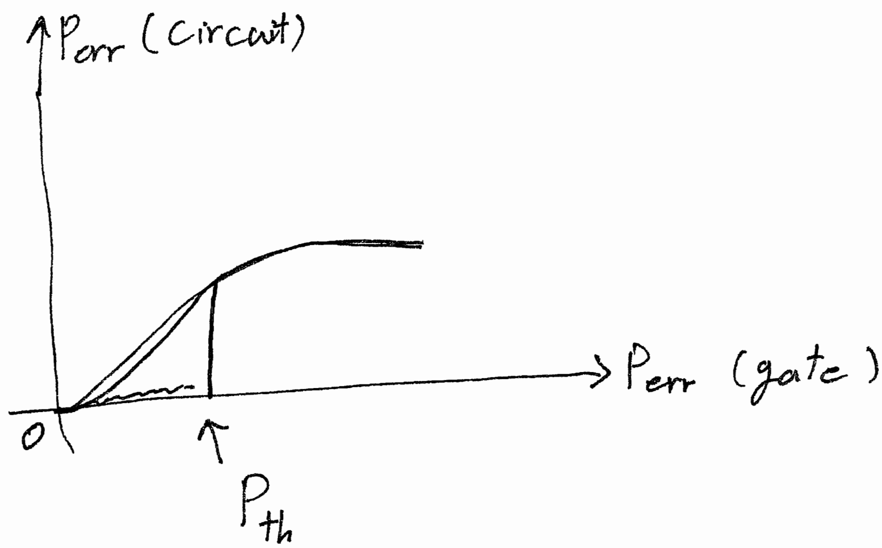
$\Rightarrow$ Communication (1940's Shannon)



⟹ 1956 Von Neamann

"probabilistic logics and the synthesis of reliable organisms from unreliable components"

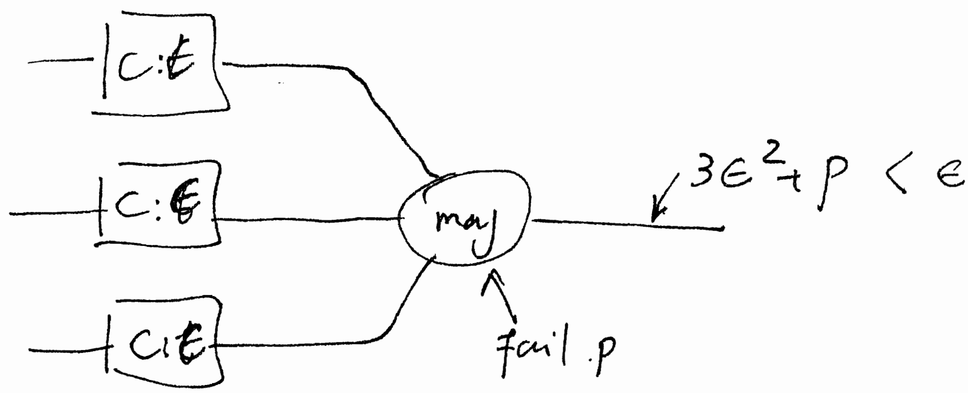




Observation

A circuit containing N error-free gates can be simmulated w.p. error $< \epsilon$ using $O(N$[illegible]$)$ gates which fail w.p. error $p < P_{th}$

Choose $P_{th} \sim \epsilon/N$



problems

- Inefficient # gates $\sim 1/\epsilon$
- not a threshold. $P_{th} \sim$ indep of N, ϵ
- avoid single points of failure

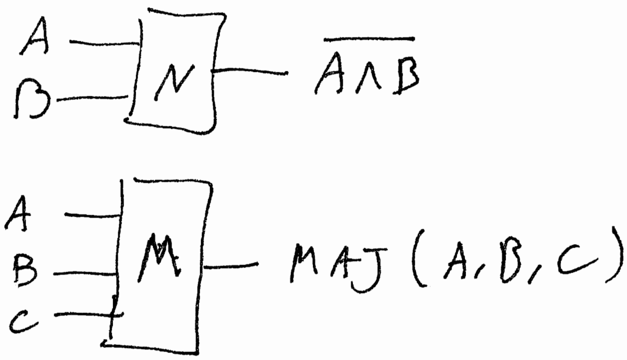
III/ The threshold thm.

A circuit with N error-free gates can be simulated w.p. error $< \epsilon$ using $O(\text{poly}(\log N/\epsilon) \cdot N)$ gates which fail w.p. p as long $p < P_{th}$, where P_{th} is <u>indep</u> of N, ϵ

Proof ~~Sk~~ Sketch

Idea: Compute on encoded data, never decode.

Ex/

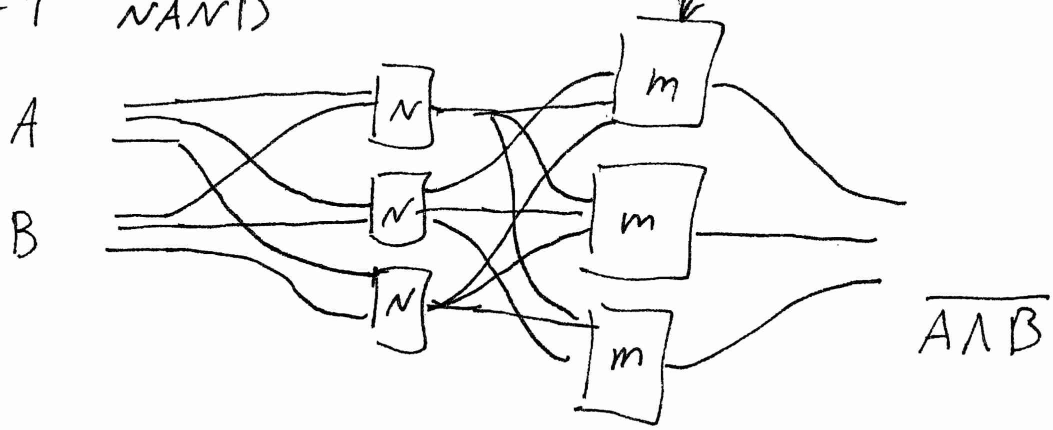


each fails w.p. $\sim p$

Encode: $0 \longrightarrow 000 \qquad 1 \longrightarrow 111$

"Level 1 encoding"

FT NAND



output is wrong only if there are two or more failures

$$\sim \binom{6}{2} = 15 \text{ possibilities}$$

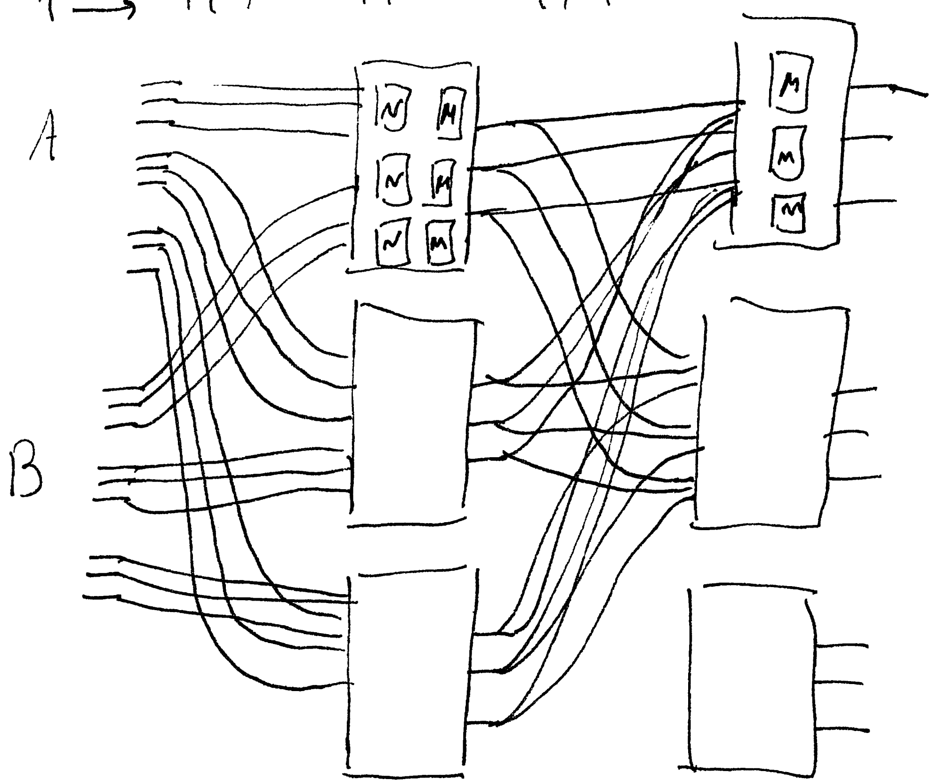
$$P_{fail} \simeq 15 p^2$$

Good if $p < 1/15$

level 2 encoding

$0 \to 000 \quad 000 \quad 000$

$1 \to 111 \quad 111 \quad 111$



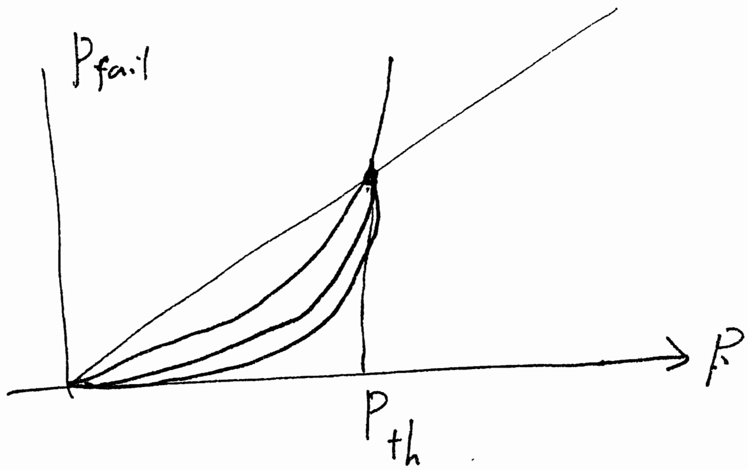
$P_{fail} \sim \binom{6}{2} (15p^2)^2$

In general if c fault paths

level encoding	
1	$cP_{fail} = (cp)^2$
2	$cP_{fail} = (cp)^4$
K	$cP_{fail} = (cp)^{2^k}$

$P_{th} \equiv 1/c \leftarrow$ indep. of N, ϵ

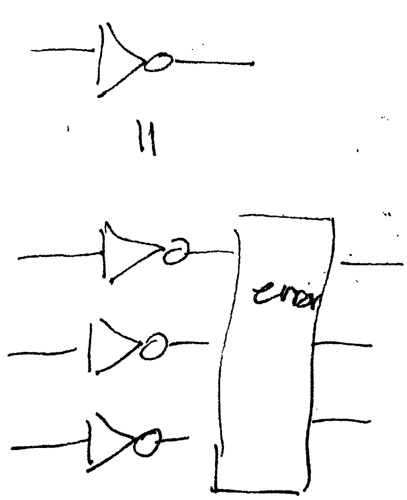
$$\boxed{\frac{P_{fail}}{P_{th}} = \left(\frac{p}{P_{th}}\right)^{2^k}}$$



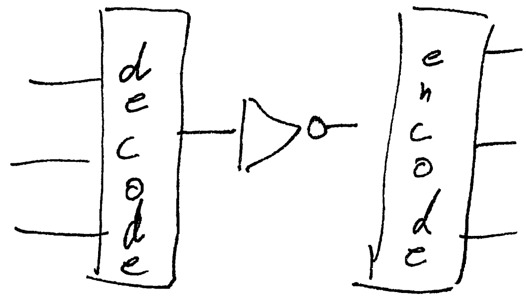
IV/ Fault Tolerance Criteria

def/ A procedure is FT if a single ~~error~~ component failure causes at most one error in each encoded block of bits in the output.

good



bad



cost of FT

circuit size $\sim d^k \times$ original circuit

$d \equiv$ size of FT procedure

what's k?

$$\left(\frac{p}{p_{th}}\right)^{2^k} < \frac{\epsilon}{N p_{th}}$$

$$2^k \log\left(p/p_{th}\right) < \log\left(\epsilon/N p_{th}\right)$$

$$2^k < \frac{\log\left(\epsilon/N p_{th}\right)}{\log\left(p/p_{th}\right)}$$

$\Rightarrow$ circuit size is

$$N d^k \cong \underbrace{\left(\frac{\log \epsilon / N p_{th}}{\log p / p_{th}} \right)^{\log d} \cdot N}_{\text{poly}\left(\log N/\epsilon\right) \cdot N} \checkmark$$

V/ FTQC and threshold

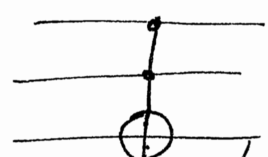
Two principles

1) compute on encoded data
2) control / limit error propagation

→ universal gate set?

CSS codes: $\{H, S, CNOT\}$

↙ clifford gates $\neq$ univ. QC.

Note: the Toffoli:  $\notin$ clifford

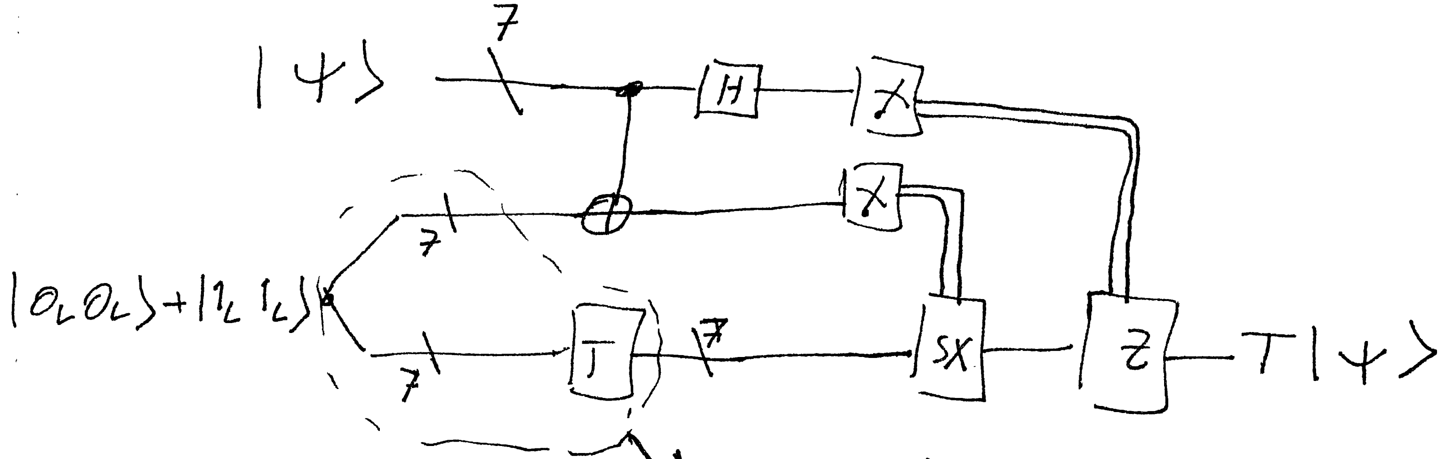
$$|0_L\rangle + e^{2\pi i/3}|1_L\rangle$$

↳ univ. QC.

claim —$\boxed{T}$— "$\pi/8$" gate

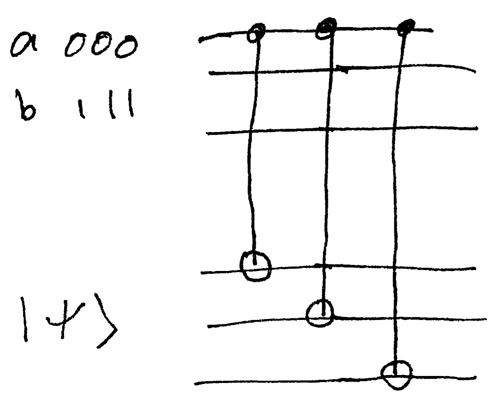
$$T = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{i} \end{bmatrix} \sim \sqrt{S} \sim \sqrt[4]{Z}$$

→ can be implemented on a QECC using clifford gates meas. (Z) and pre-prepared entangelment.

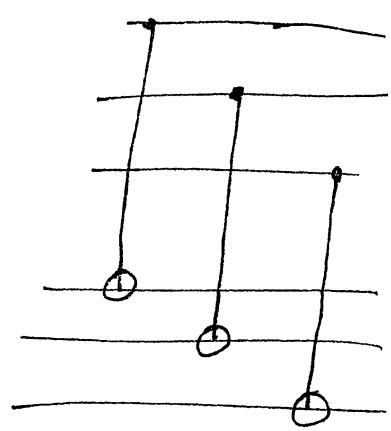


verified / checked

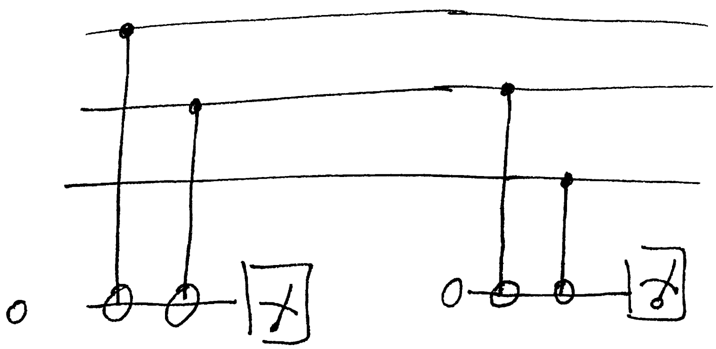
$\Rightarrow$ error propagation

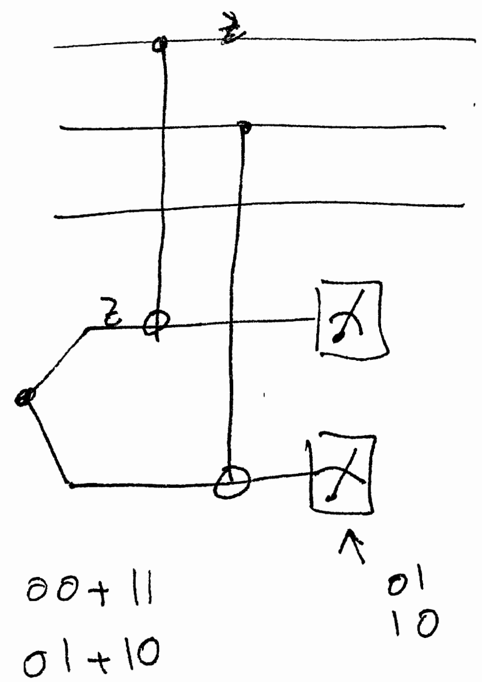


non FT



Syndrome meas





$\Rightarrow$ Threshold estimate

steane 7-qubit code

$$\begin{array}{rl} 6 & \text{syndrome op} \\ \times\ 4 & \text{gates each} \\ \times\ 3 & \text{times repeat meas.} \\ \hline 72 & \text{gates} \\ +\ 7 & \text{CNOT} \\ \hline 79 & \text{gates} \end{array}$$

$$\sim \binom{79}{2} = 3081 \text{ fault paths}$$

$$P_{th} \sim \frac{1}{\#\text{ fault paths}} \sim 3 \times 10^{-4}$$