2.830J / 6.780J / ESD.63J Control of Manufacturing Processes (SMA 6303)
Spring 2008

Control of Manufacturing Processes

Subject 2.830/6.780/ESD.63

Spring 2008

Lecture #12

Full Factorial Models

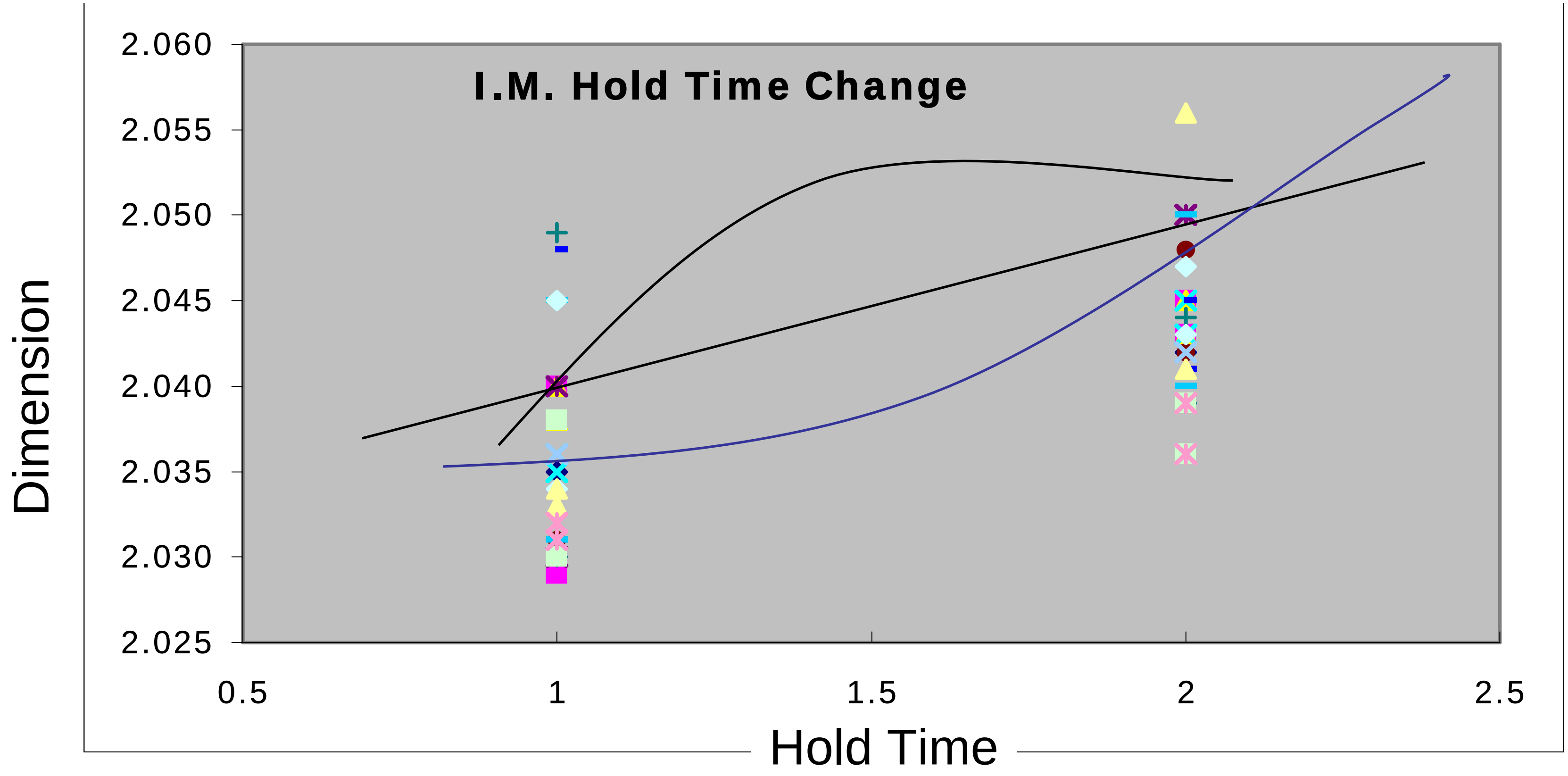
March 20, 2008

Outline

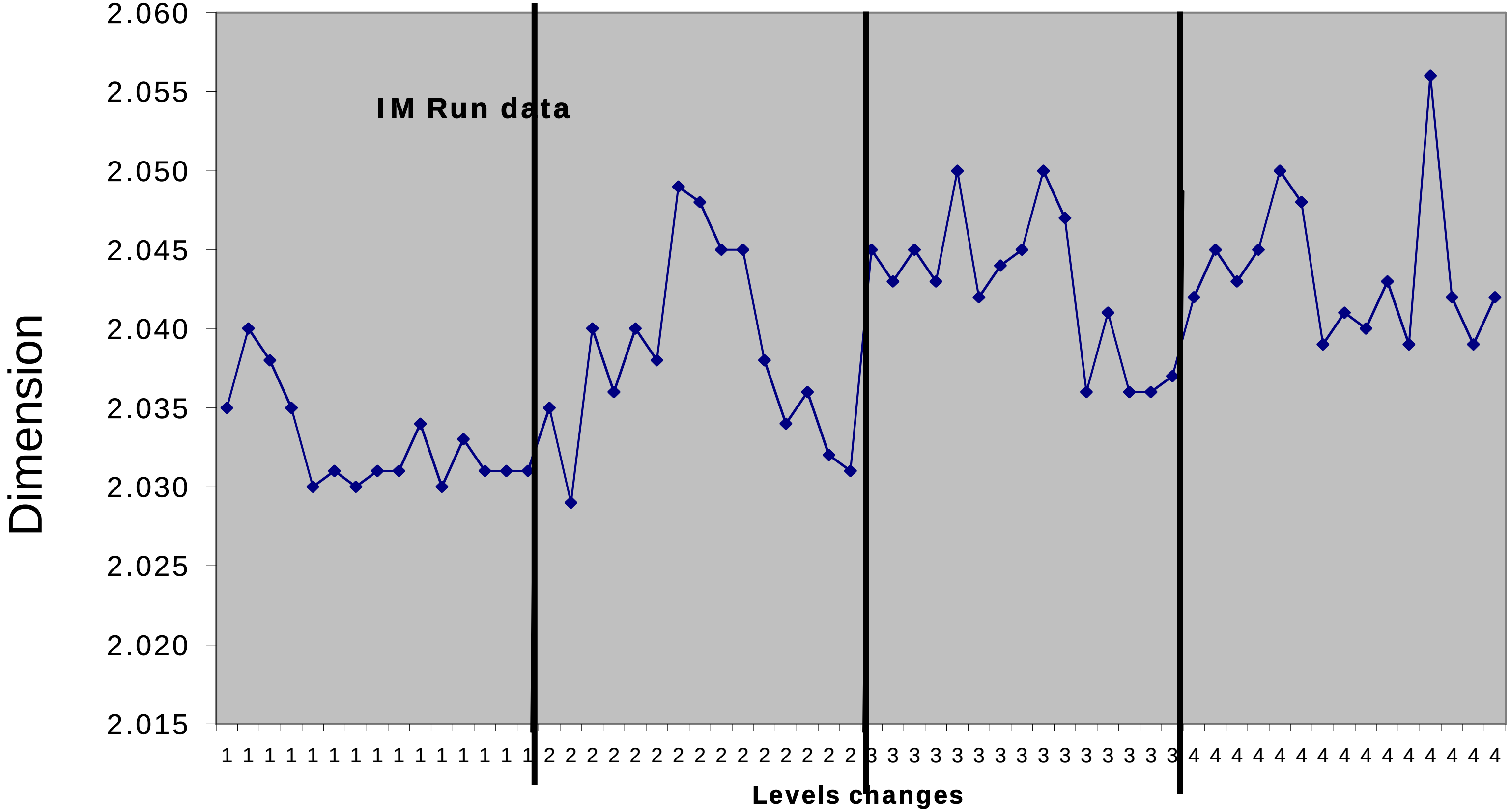
- Modelling "Effects" from Multiple Inputs
- ANOVA on Effects
- Linear and Quadratic Models
- Model Coefficient Calculation
  - Regression (General Approach)
  - Contrasts (for Factorial Designs)

What Is the Effect?

- What is the relationship between Hold Time and Dimension?



But Wait… There's More!



Velocity	low	high	low	high
Hold time	low	low	high	high

Do the two inputs interact?

Model Form

- Linear
- Quadratic
- Exponential
- General Polynomial?
- Interactions

What data needed to decide and/or estimate parameters of different model forms?

Multiple Input/Treatment Models

- In general *k* inputs
  - If 2 levels for each ….. 2^k combinations
  - If 3 levels for each …… 3^k combinations
- Why use more than one input?
  - More than one output
  - Change mean and variance
    - Process Robustness
    - Optimization of Quality Loss

A General Linear Model for k inputs

$$\eta = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{\substack{j=1 \\ j<i}}^{k} \sum_{i=1}^{k} \beta_{ij} x_i x_j + h.o.t. + \varepsilon$$

mean

linear term

interaction term

higher order terms (model form error)

residual error

i = input index

k = total number of inputs

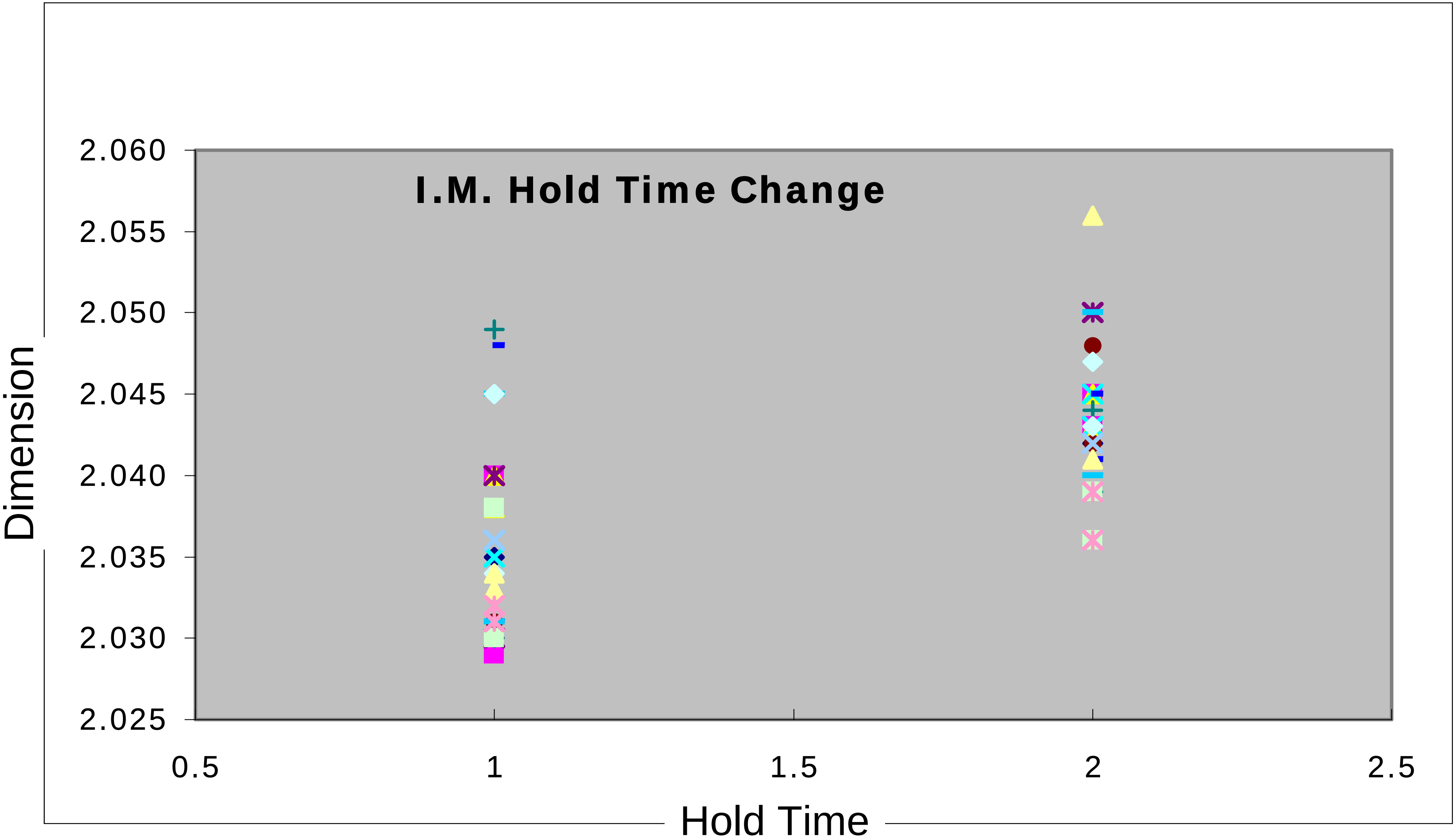
Two Input Model

$$\eta = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + h.o.t. + \varepsilon$$

4 coefficients to determine

How many data points (factors, levels)
are needed to uniquely identify?

Consider a One Input Case



Linear One Input Example

Linear model $\eta = \beta_0 + \beta_1 x$

Assume 2 levels x_-, x_+

With 1 trial at each level we get:

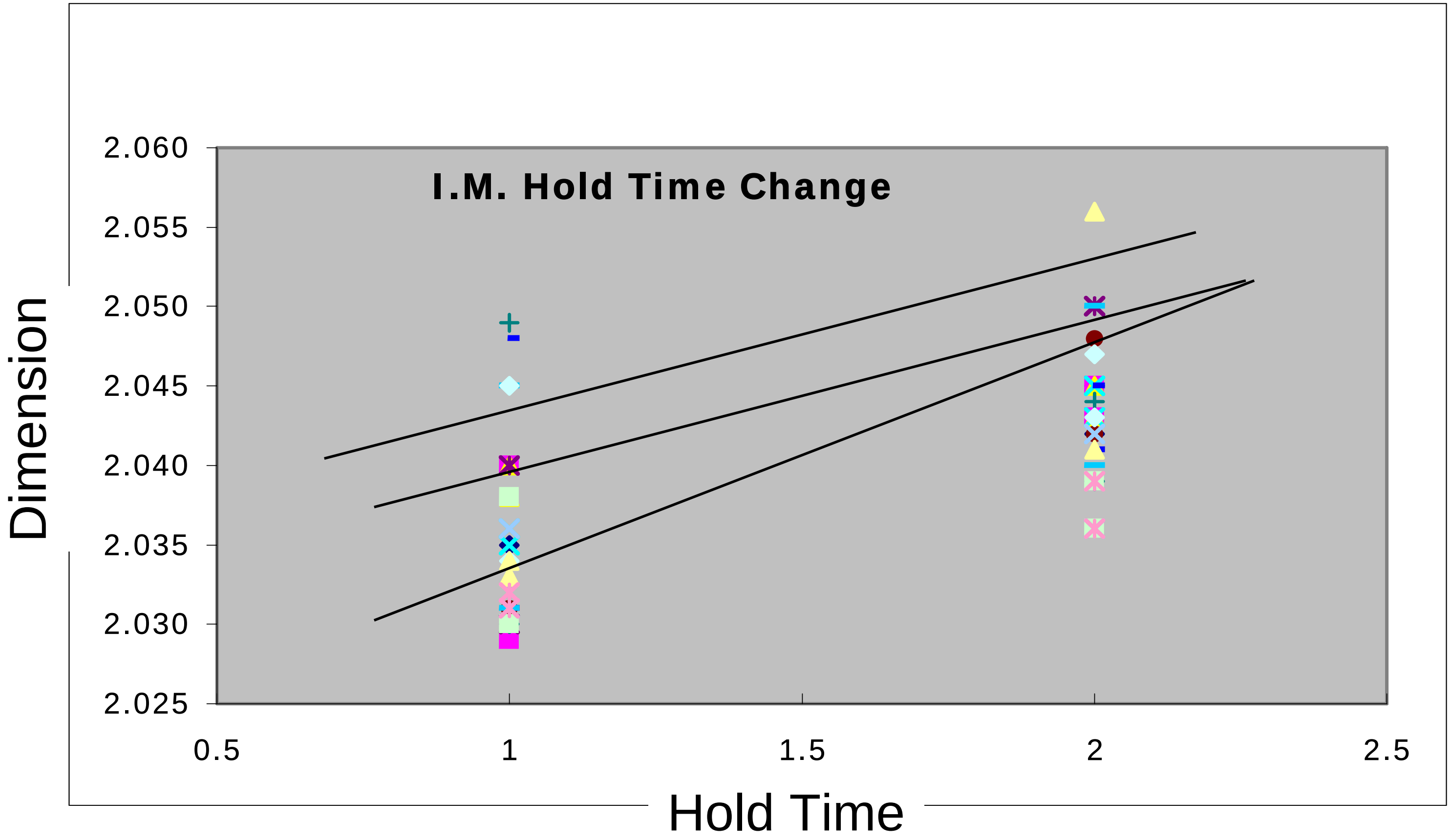
$$\underline{\eta} = \begin{vmatrix} \eta_1 \\ \eta_2 \end{vmatrix} \qquad \mathbf{X} = \begin{vmatrix} 1 & x_- \\ 1 & x_+ \end{vmatrix} \qquad \underline{\beta} = \begin{vmatrix} \beta_0 \\ \beta_1 \end{vmatrix} \qquad \underline{\varepsilon} = 0 \ \text{(for means)}$$

$$\underline{\eta} = \mathbf{X}\underline{\beta} + \underline{\varepsilon}$$

Since X is square and $\varepsilon = 0$ $\quad \underline{\beta} = \mathbf{X}^{-1}\eta$

Linear Model with Replicates

- Line will no longer intersect specific points
- What is "best fit?



Minimum Error Line Fits

- Define squared error for data for a given β_o and β_1
- Find β_o and β_1 that lead to minimum of the sum of all e^2

$$e^2 = (\eta - (\beta_0 - \beta_1 x))^2$$

- OR - Solve the matrix equation to get

$$\underline{\beta} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \underline{\eta}$$

where $\mathbf{X}$ is a non-square matrix of all inputs for all replicates and η is the vector of all trial outputs

Aside

$$\underline{\eta} = \mathbf{X}\underline{\beta} + \underline{\varepsilon}$$

$$\underline{\varepsilon} = \underline{\eta} - \mathbf{X}\underline{\beta}$$

squared error $\quad J = \underline{\varepsilon}^T \underline{\varepsilon} = \left(\underline{\eta} - \mathbf{X}\underline{\beta}\right)^T \left(\underline{\eta} - \mathbf{X}\underline{\beta}\right)$

The minimum value of *J* is then found by the vector partial derivative:

$$\frac{\partial J}{\partial \underline{\beta}} = 0 = -2\mathbf{X}^T \underline{\eta} + 2\mathbf{X}^T \mathbf{X}\underline{\beta}$$

$$\underline{\beta} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \underline{\eta}$$

General result for any ***X*** matrix

Solution with Replicates

$$\underline{\eta} = \begin{vmatrix} \eta_1 \\ \eta_2 \\ \cdots \\ \eta_{2n-1} \\ \eta_{2n} \end{vmatrix} \qquad \mathbf{X} = \begin{vmatrix} 1 & x_- \\ 1 & x_+ \\ 1 & \cdots \\ 1 & x_- \\ 1 & x_+ \end{vmatrix} \qquad \underline{\beta} = \begin{vmatrix} \beta_0 \\ \beta_1 \end{vmatrix}$$

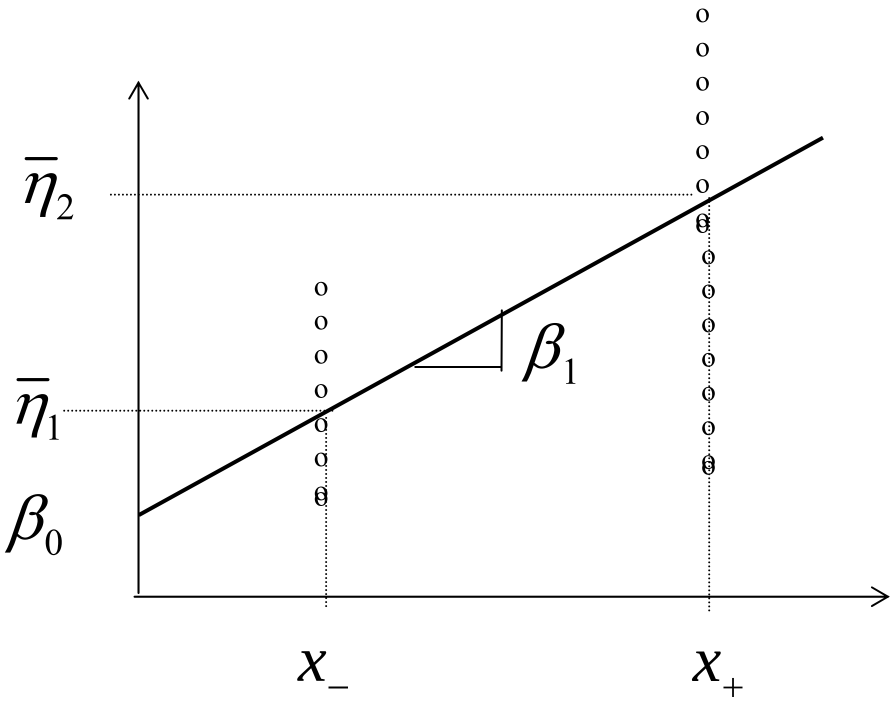
For n trials at two levels for x: x_- and x_+

$$\underline{\eta} = \mathbf{X}\underline{\beta} + \underline{\varepsilon} \qquad \underline{\beta} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\underline{\eta}$$

Or…

- Notice that for only 2 levels, the minimum squared error line must pass through the mean at each level

Linear Curve Fit



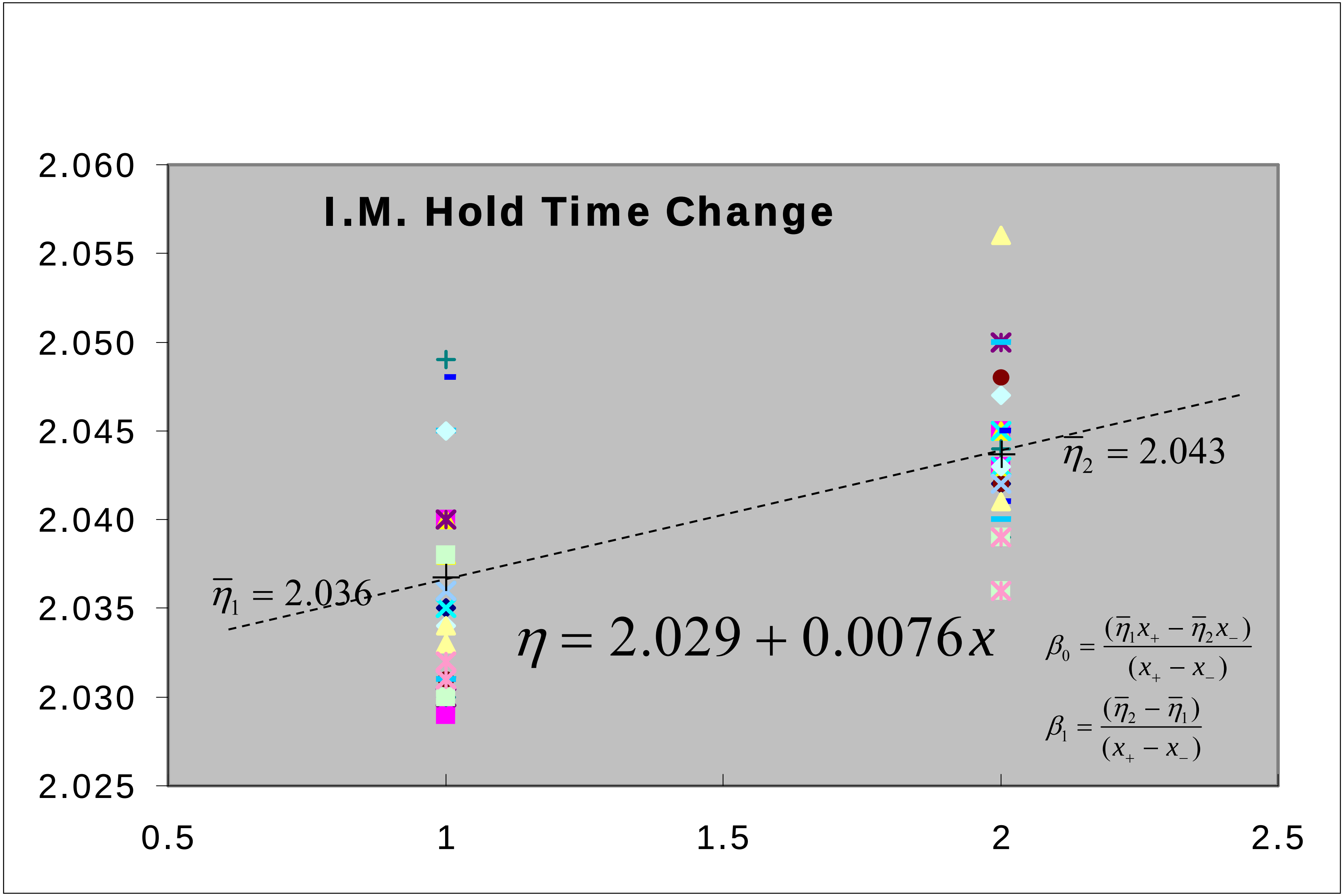
$$\begin{vmatrix}\beta_0\\ \beta_1\end{vmatrix} = \frac{1}{(x_+ - x_-)}\begin{vmatrix}x_+ & x_-\\ -1 & 1\end{vmatrix}^{-1}\begin{vmatrix}\bar{\eta}_1\\ \bar{\eta}_2\end{vmatrix}$$

or

$$\beta_0 = \frac{(\bar{\eta}_1 x_+ - \bar{\eta}_2 x_-)}{(x_+ - x_-)}$$

$$\beta_1 = \frac{(\bar{\eta}_2 - \bar{\eta}_1)}{(x_+ - x_-)}$$

For "Real" Data



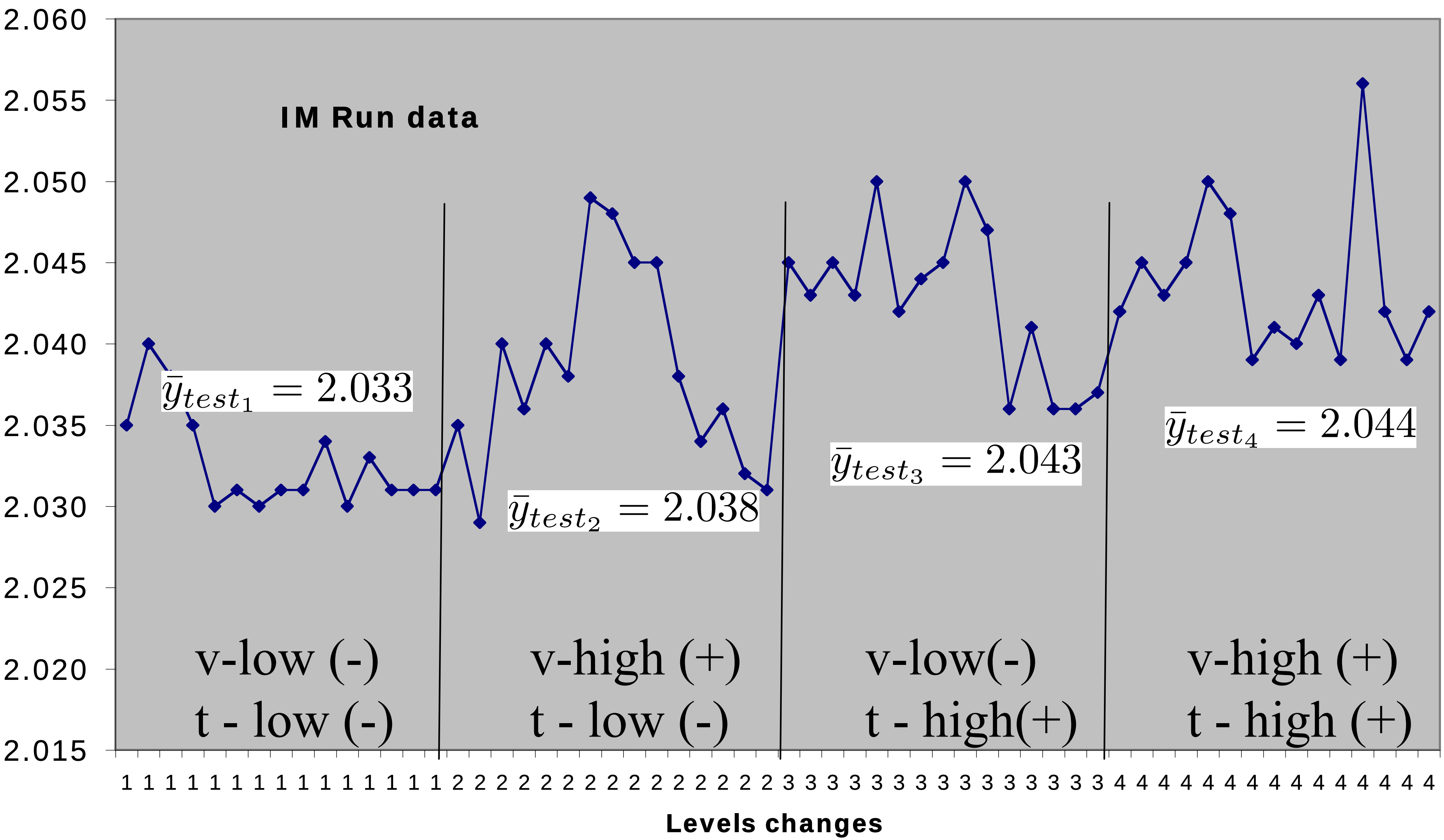
Outline, cont'd

- Multiple Effects (Inputs)

- ANOVA Test for Multiple Effects
  - Are effects due to different factors significant?

- Linear Models for "k" inputs
  - Visualization
  - Coefficient Estimation (Model Calibration) Using Contrasts
  - Significance Test

Reconsider the Injection Molding Problem

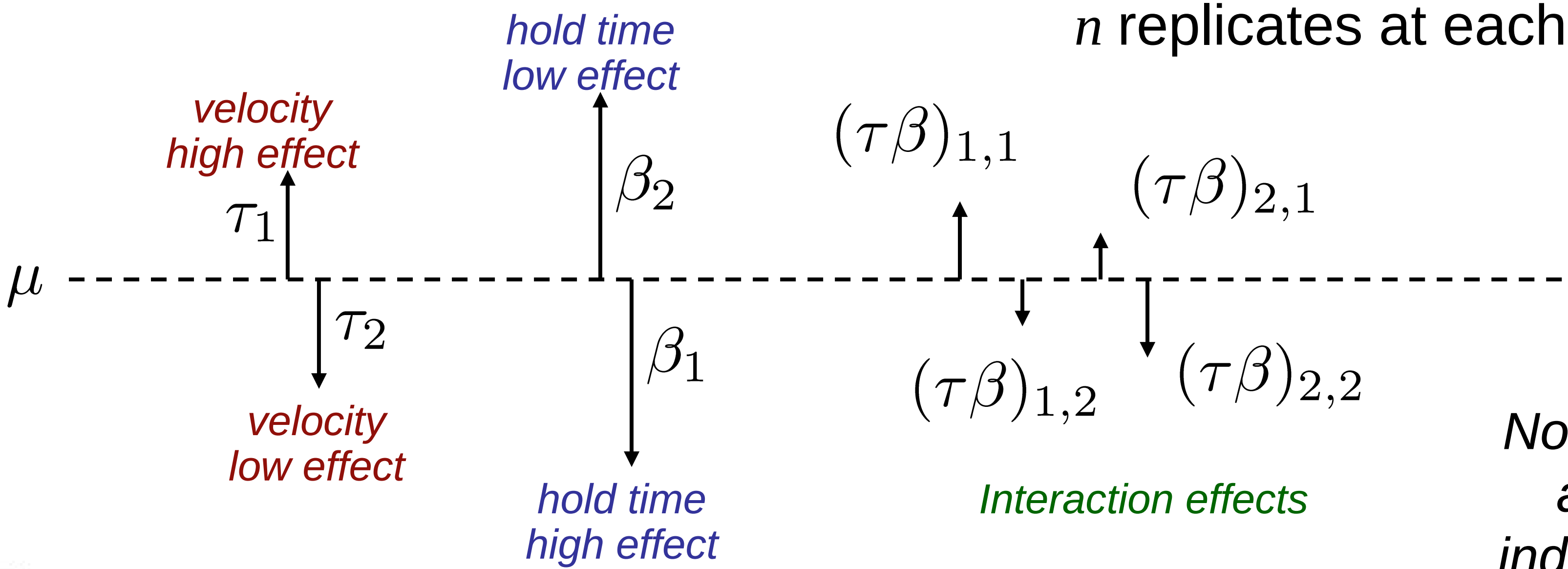


Full Effects Model

velocity effects → τ_i; hold time effects → β_j; Interaction effects → $(\tau\beta)_{ij}$

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk} \begin{cases} i = 1 \ldots a \\ j = 1 \ldots b \\ k = 1 \ldots n \end{cases}$$

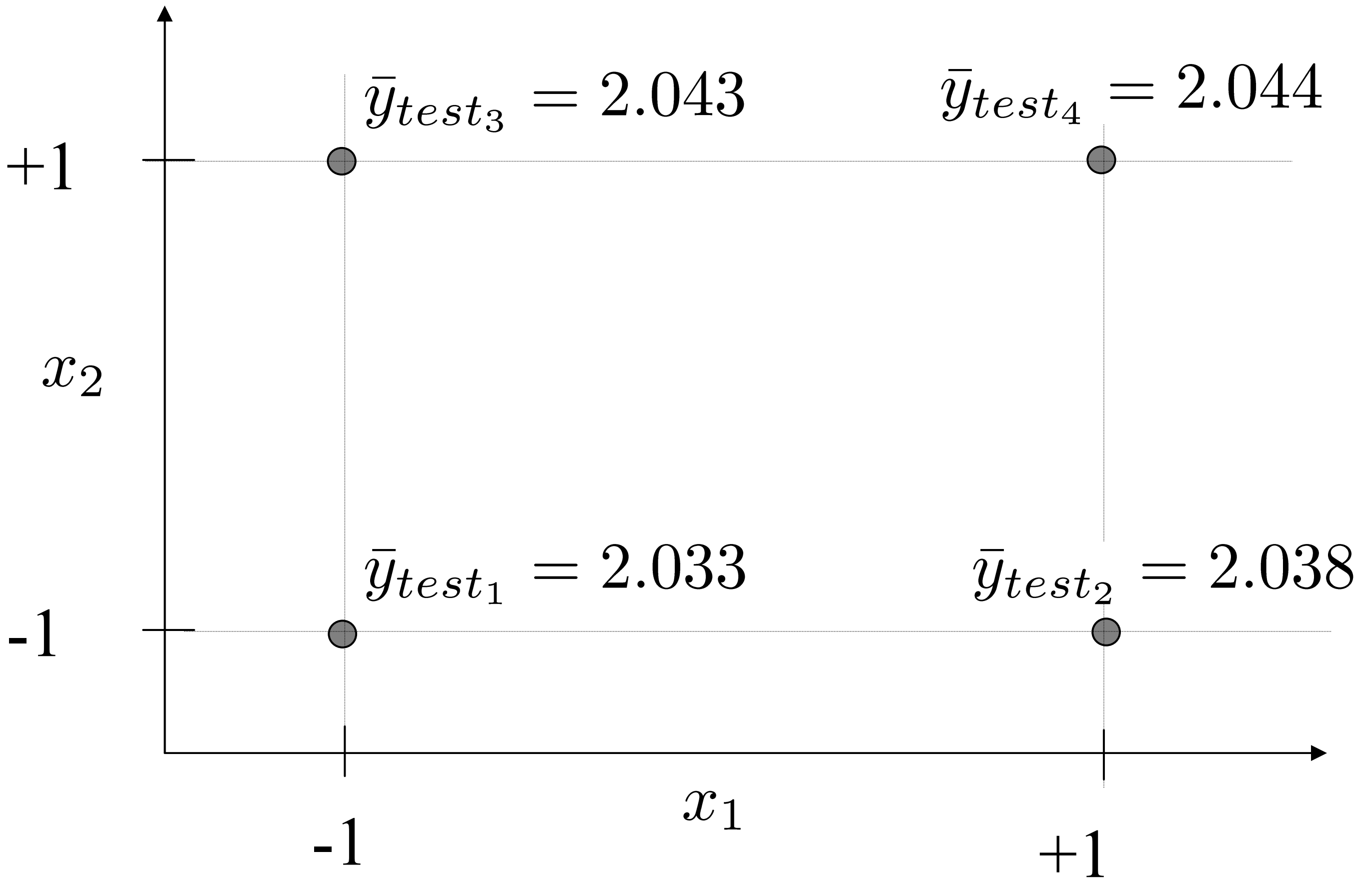
a levels for factor τ
b levels for factor β
n replicates at each treatment



*Note: coeffs are **not** independent*

Full Effects

test	x_1 (velocity)	x_2 (hold time)
1	–1	–1
2	+1	–1
3	–1	+1
4	+1	+1



$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}$$

Hypothesis?

Test on Each Term

$$H_0 : \tau_1 = \tau_2 \cdots \ \tau_a$$
$$H_1 : \tau_i \neq 0$$

$$H_0 : \beta_1 = \beta_2 \cdots \ \beta_a$$
$$H_1 : \beta_i \neq 0$$

$$H_0 : (\tau\beta)_{ij} = 0$$
$$H_1 : (\tau\beta)_{ij} \neq 0$$

Definitions

$$\bar{y}_i = \frac{1}{bn}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk}$$ responses from A at *a* levels averaged over *b* and *n*

$$\bar{y}_j = \frac{1}{an}\sum_{i=1}^{a}\sum_{k=1}^{n} y_{ijk}$$ responses from B at *b* levels averaged over *a* and *n*

$$\bar{y}_{ij} = \frac{1}{n}\sum_{k=1}^{n} y_{ijk}$$ responses from A & B at *ab* levels averaged over all *n*

$$\bar{\bar{y}} = \frac{1}{abn}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk}$$

ANOVA for Multiple Effects

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk} - \bar{\bar{y}})^2 =$$

$$bn\sum_{i=1}^{a}(\bar{y}_i - \bar{\bar{y}})^2 + an\sum_{j=1}^{a}(\bar{y}_j - \bar{\bar{y}})^2 +$$

$$n\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{y}_{ij} - \bar{y}_i - \bar{y}_j - \bar{\bar{y}})^2 + \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk} - \bar{y}_{ij})^2$$

Total sum of squared deviations

$$SS_T = SS_{Treatment\ A} + SS_{Treatment\ B} + SS_{InteractionAB} + SS_{Error}$$

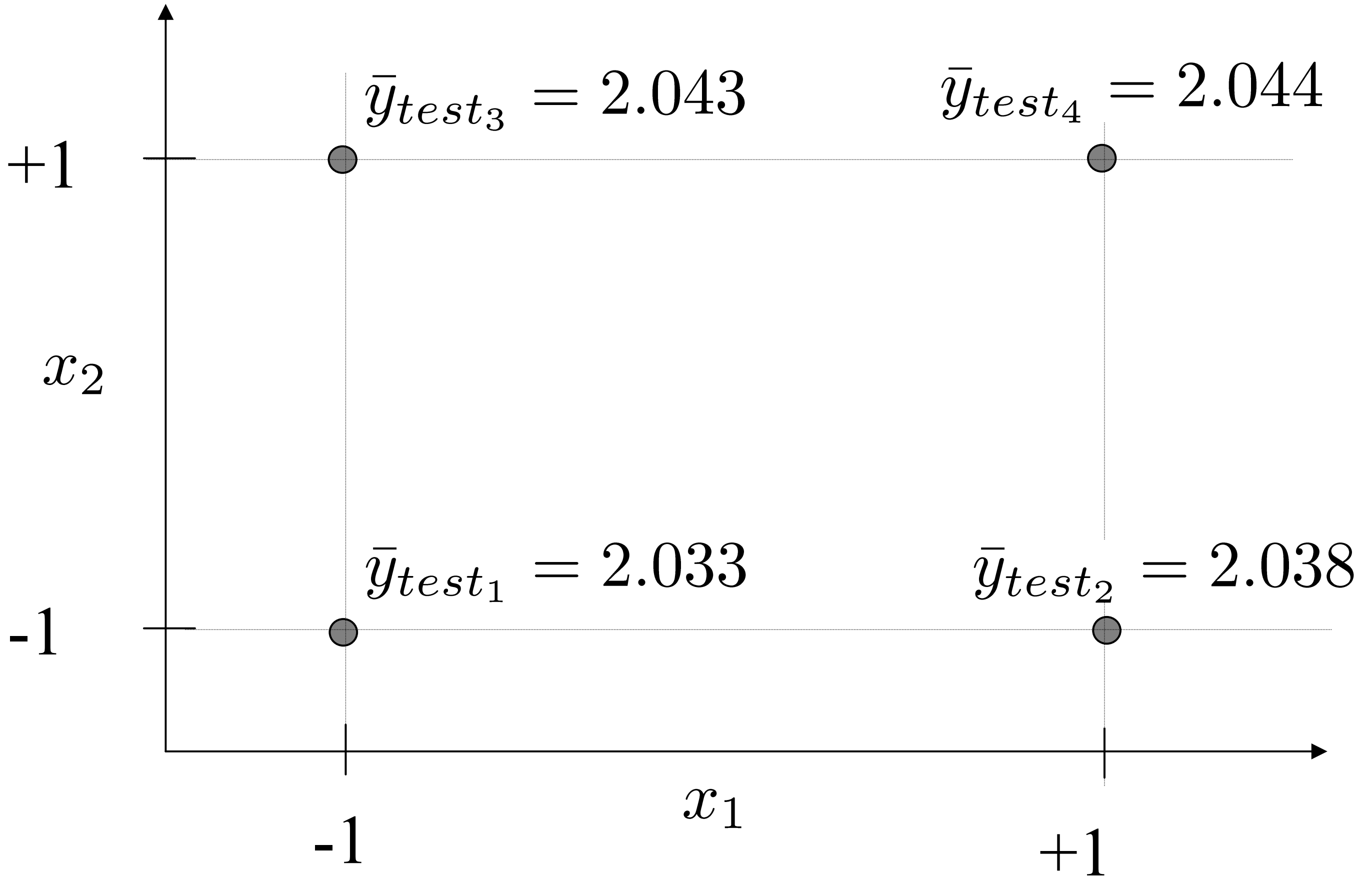
Degrees of Freedom?

$$abn\text{-}1 = a\text{-}1 + b\text{-}1 + (a\text{-}1)(b\text{-}1) + ab(n\text{-}1)$$

ANOVA Table for Multiple Factors (Treatments)

Source	SS	dof	MS	F	F_o
Factor A	SS_A	a-1	$\frac{SS_A}{a-1}$	$\frac{MS_A}{MS_E}$	$F_{(1-\alpha),a-1,ab(n-1)}$
Factor B	SS_B	b-1	$\frac{SS_B}{b-1}$	$\frac{MS_b}{MS_E}$	$F_{(1-\alpha),b-1,ab(n-1)}$
Interaction AB	SS_{AB}	(a-1)(b-1)	$\frac{SS_{AB}}{(a-1)(b-1)}$	$\frac{MS_{AB}}{MS_E}$	$F_{(1-\alpha),(a-1)(b-1),ab(n-1)}$
Within Tests (Pure Error)	SS_E	ab(n-1)	$\frac{SS_E}{ab(n-1)}$		
Total	SS_T	abn-1			

Now Consider a Linear Model



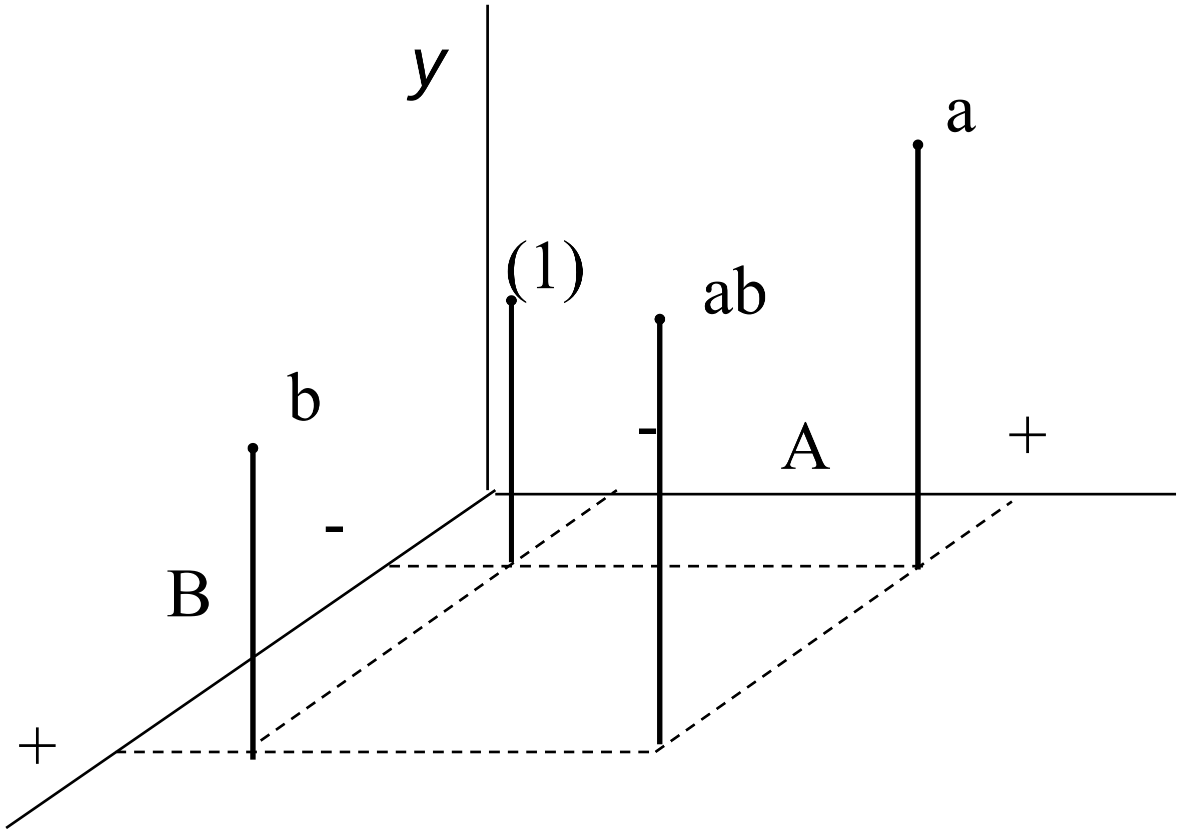
	test	x_1	x_2
(1)	1	−1	−1
a	2	+1	−1
b	3	−1	+1
ab	4	+1	+1

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2$$

Regression Model

Classical Design-of-Experiments (DOE) for 2^2

- Same graph, different labels



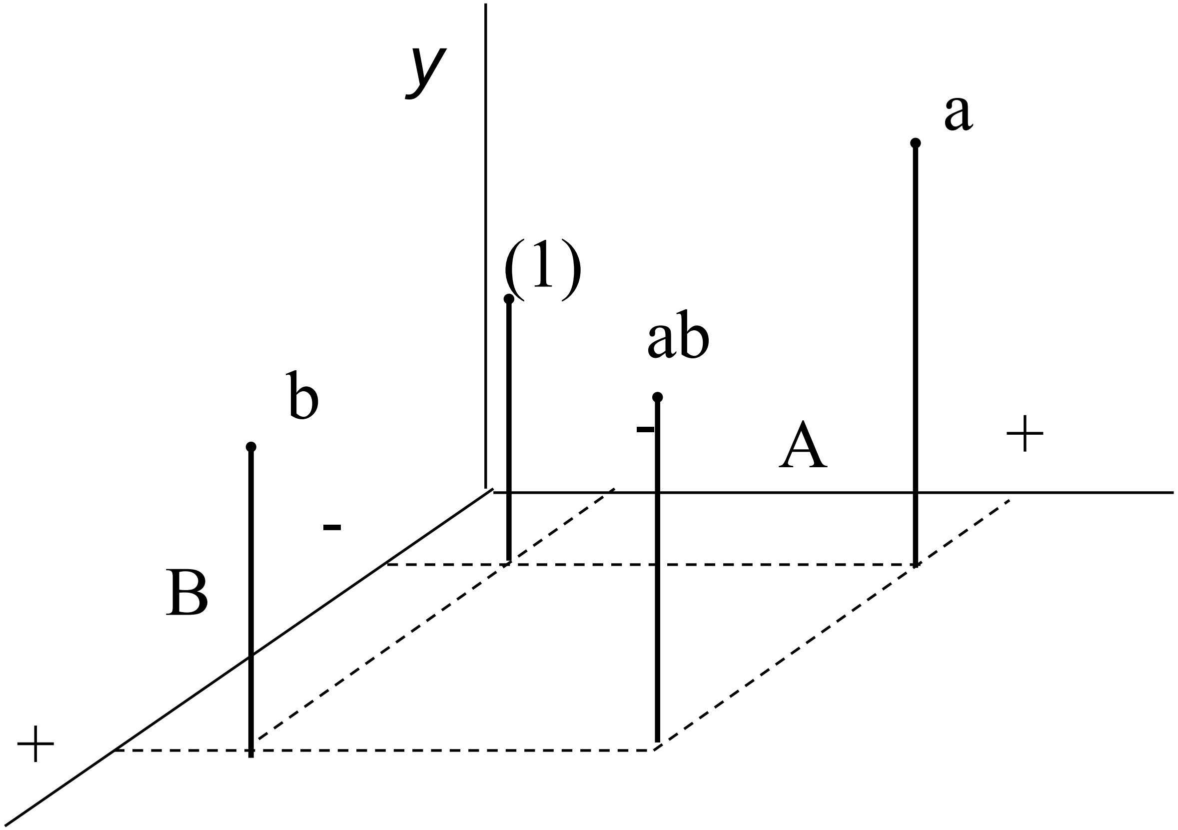
	A	B
(1)	-	-
a	+	-
b	-	+
ab	+	+

- treatment condition
- average of *n* responses at that condition

Effects

- The Effect of an input term on the Output
  - A and B are "Main Effects"
  - AB is the Interaction Effect
- Main Effects
  - Change caused by a single factor <u>averaged over all other changes</u>

Main Effects

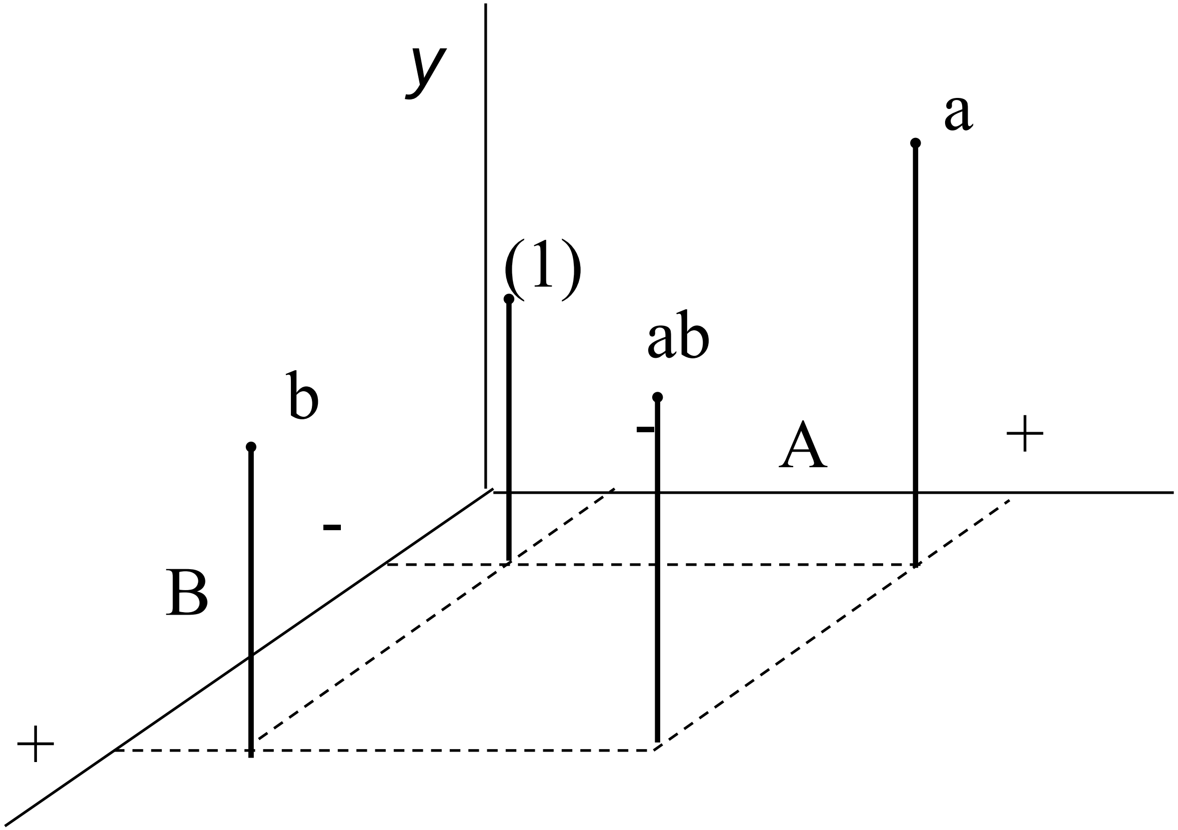


$$A = \bar{y}_A^{\,+} - \bar{y}_A^{\,-}$$
$$= \frac{a + ab}{2n} - \frac{b + (1)}{2n}$$

$$B = \bar{y}_B^{\,+} - \bar{y}_B^{\,-}$$
$$= \frac{b + ab}{2n} - \frac{a + (1)}{2n}$$

Interaction Effect

- Diagonal Averages



$$AB = \bar{y}_{AB}^{\,+} - \bar{y}_{AB}^{\,-}$$

$$= \frac{ab + (1)}{2n} - \frac{a + b}{2n}$$

Definition: Contrasts

$$A = \frac{1}{2n}\underbrace{[a + ab - b - (1)]}$$

$$B = \frac{1}{2n}\underbrace{[b + ab - a - (1)]}$$

$$AB = \frac{1}{2n}\underbrace{[ab + (1) - a - b]}$$

[…..] = "Contrast"

$$\hat{y} = \bar{\bar{y}} + \frac{A}{2}x_1 + \frac{B}{2}x_2 + \frac{AB}{2}x_1x_2$$

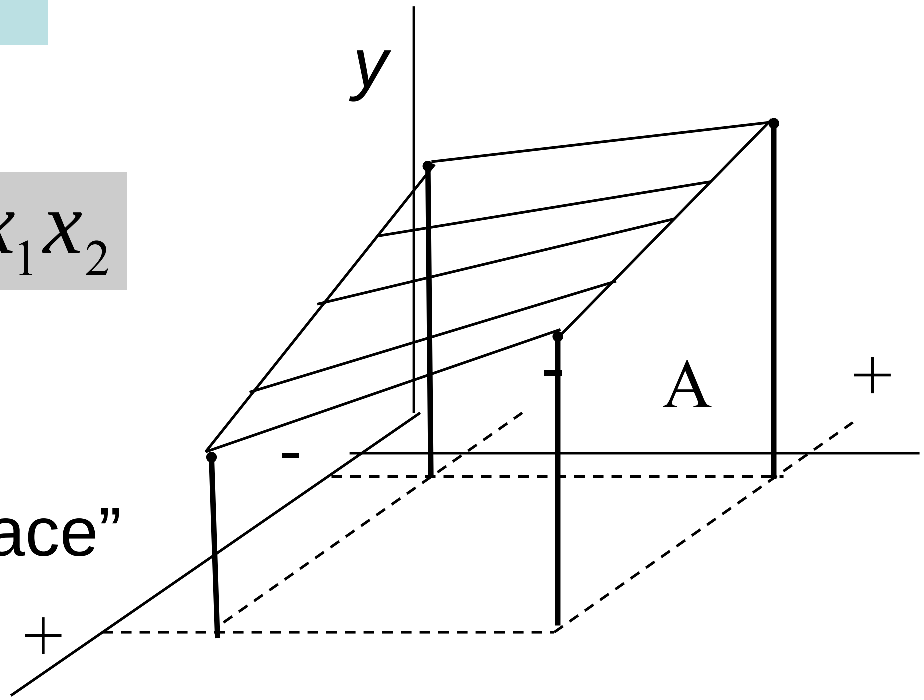
Model Based on Contrasts

$$\hat{y} = \bar{\bar{y}} + \frac{A}{2}x_1 + \frac{B}{2}x_2 + \frac{AB}{2}x_1x_2$$

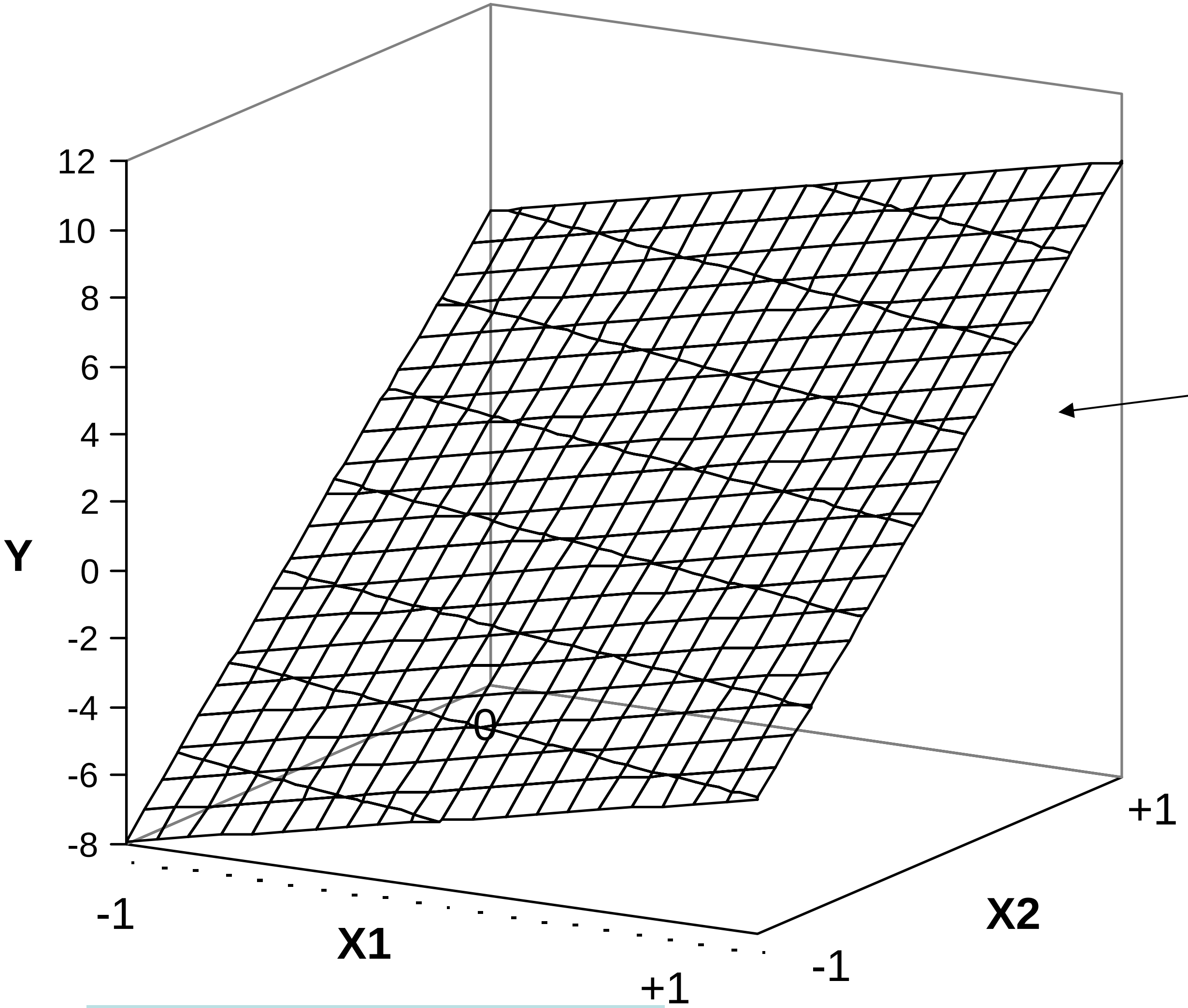
$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2$$

(Regression model)

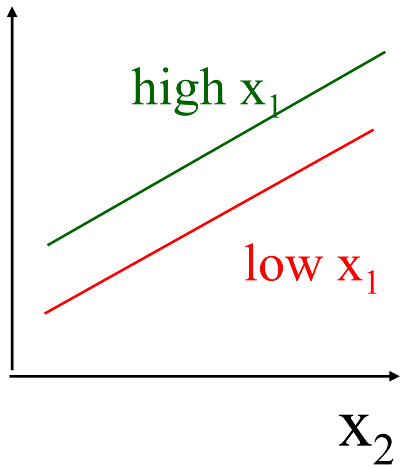
This defines a 3-D "ruled surface"



Response Surface without Interaction

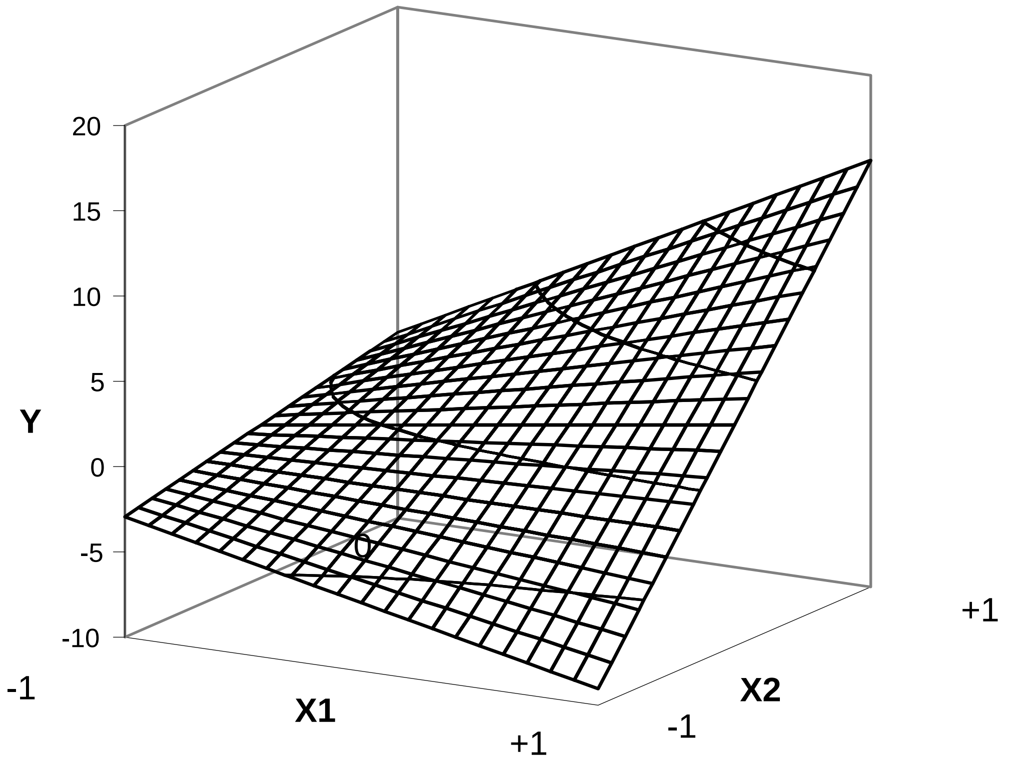


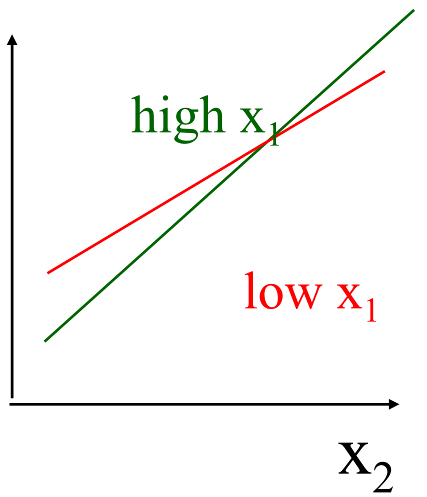
Ruled Surface



$$y = 1 + 7x_1 + 2x_2$$

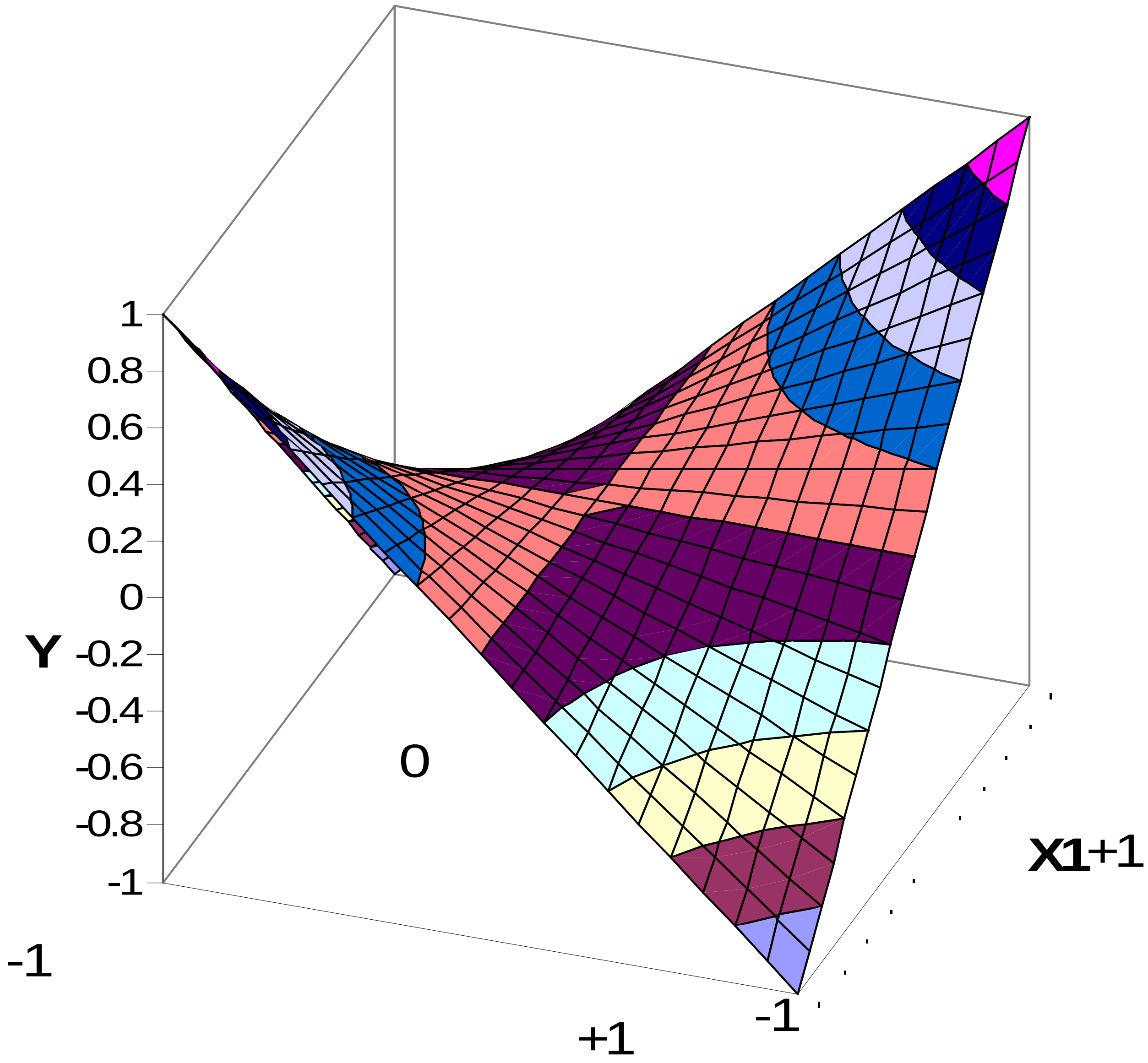
Response Surface: Positive Interaction

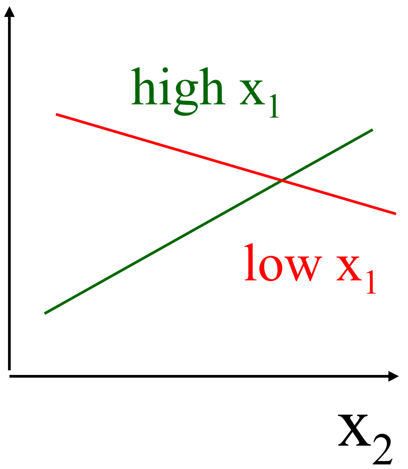




$$y = 1 + 7x_1 + 2x_2 + 5x_1x_2$$

Response Surface: Negative Interaction





$$y = 1 + 7x_1 + 2x_2 - 5x_1x_2$$

General Form for Contrasts

Trial	A	B	AB
(1)	-	-	+
a	+	-	-
b	-	+	-
ab	+	+	+

$$A:[a+ab-b-(1)]$$

$$B:[b+ab-a-(1)]$$

$$AB:[ab+(1)-a-b]$$

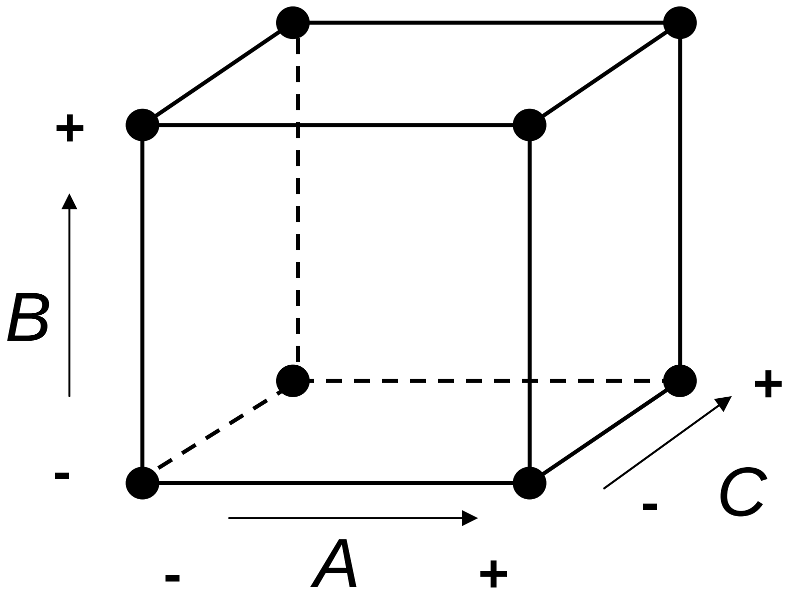
$$\text{Contrast}_{\text{A}} = \text{Trial Column} \cdot \text{A}$$
$$\text{Contrast}_{\text{B}} = \text{Trial Column} \cdot \text{B}$$
$$\text{Contrast}_{\text{AB}} = \text{Trial Column} \cdot \text{AB}$$

Extension to 2^k

Consider 2^3:

Run Number	Treatment Combination		Factor Levels x_1 A	x_2 B	x_3 C
1	(1)	y_1	-1	-1	-1
2	a	y_2	1	-1	-1
3	b	y_3	-1	1	-1
4	ab	y_4	1	1	-1
5	c	y_5	-1	-1	1
6	ac	y_6	1	-1	1
7	bc	y_7	-1	1	1
8	abc	y_8	1	1	1

Generalization

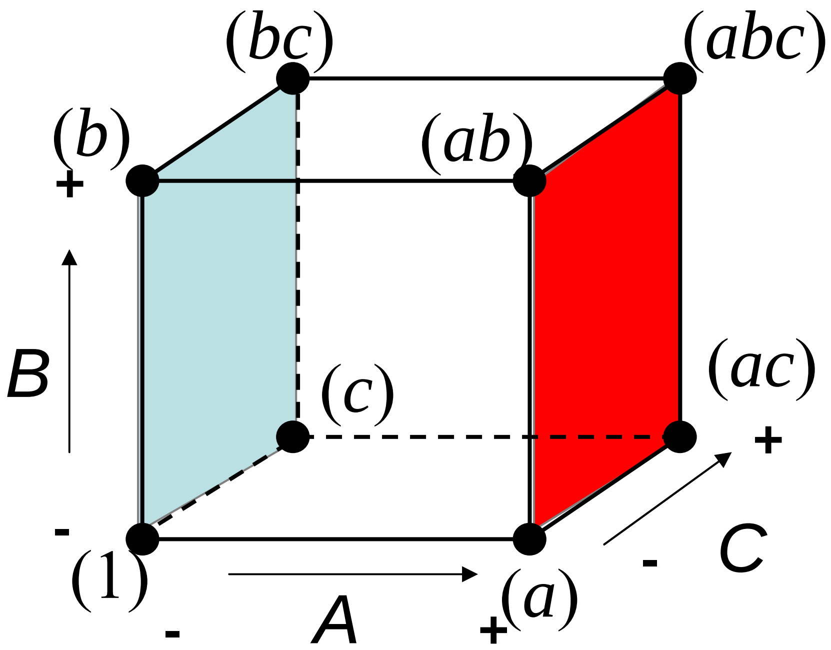


$$2^k$$

number of factors

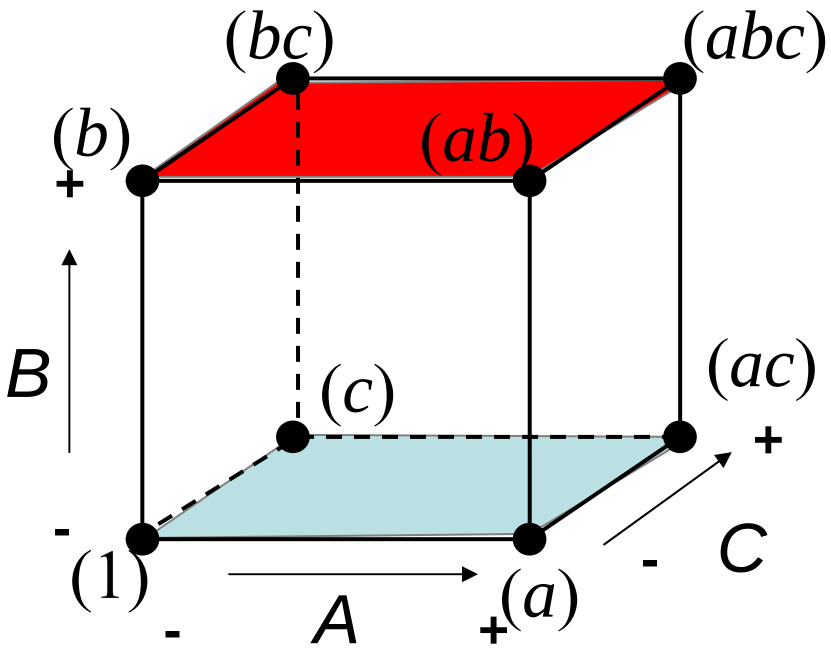
number of levels

"Surface" Averages



$$A = \frac{1}{4}\left[(abc)+(ab)+(ac)+(a)\right] - \frac{1}{4}\left[(b)+(c)+(bc)+(1)\right]$$

Surface Averages



$$B = \frac{1}{4}\left[(abc)+(ab)+(bc)+(b)\right] - \frac{1}{4}\left[(a)+(c)+(ac)+(1)\right]$$

Factorial Combinations

Treament Combination	Factorial Combination I	A	B	AB	C	AC	BC	ABC
(1)	1	-1	-1	1	-1	1	1	-1
a	1	1	-1	-1	-1	-1	1	1
b	1	-1	1	-1	-1	1	-1	1
ab	1	1	1	1	-1	-1	-1	-1
c	1	-1	-1	1	1	-1	-1	1
ac	1	1	-1	-1	1	1	-1	-1
bc	1	-1	1	-1	1	-1	1	-1
abc	1	1	1	1	1	1	1	1

Note: this is the scaled $\boldsymbol{X}$ matrix in the regression model

Contrasts for 2^3

Treament Combination	Factorial Combination							
	I	A	B	AB	C	AC	BC	ABC
(1)	1	-1	-1	1	-1	1	1	-1
a	1	1	-1	-1	-1	-1	1	1
b	1	-1	1	-1	-1	1	-1	1
ab	1	1	1	1	-1	-1	-1	-1
c	1	-1	-1	1	1	-1	-1	1
ac	1	1	-1	-1	1	1	-1	-1
bc	1	-1	1	-1	1	-1	1	-1
abc	1	1	1	1	1	1	1	1

$$\text{Contrast } A : [a + ab + ac + abc - b - c - bc - (1)]$$

$$\text{Contrast } ABC : [a + b + c + abc - ab - ac - bc - (1)]$$

$$\text{Effect} = \frac{\text{Contrast}}{n2^{k-1}}$$

$$\Rightarrow \quad A = \frac{1}{4n}[a + ab + ac + abc - b - c - bc - (1)]$$

Relationship to Regression Model

$$\underline{\beta} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \underline{y}$$

$\Downarrow$

$\underline{y}$ is data from experimental design $\mathbf{X}$

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2$$ regression model

- A is the Effect of input 1 averaged over all other input changes (-1 to +1 or a total range of 2)
- B is the Effect of input 2 averaged over all other input changes, …….

$$\beta_0 = \bar{y} \quad \beta_1 = \frac{A}{2}; \quad \beta_2 = \frac{B}{2}; \quad \beta_{12} = \frac{AB}{2}$$

or

$$\hat{y} = \bar{\bar{y}} + \frac{A}{2} x_1 + \frac{B}{2} x_2 + \frac{AB}{2} x_1 x_2$$

ANOVA for 2^k

- Now have more than one "effect"
- We can derive:

$$SS_{Effect} = (Contrast)^2 / n2^k$$

- And it can be shown that:

$$SS_{Total} = SS_A + SS_B + SS_{AB} + SS_{Error}$$

ANOVA Table

Source	SS	d.o.f.	MS	F_0	F_{crit}
A	$\frac{Contrast_A^2}{2^2 n}$	1	SS_A	$\frac{MS_A}{MS_E}$	$F_{1,2n-4,\alpha}$
B	$\frac{Contrast_B^2}{2^2 n}$	1	SS_B	$\frac{MS_B}{MS_E}$	
AB	$\frac{Contrast_{AB}^2}{2^2 n}$	1	SS_C	$\frac{MS_{AB}}{MS_E}$	
Error	SS_E	$2^2 *n-3$	$\frac{SS_E}{2^2 *n-3}$		
Total	$\Sigma\Sigma(y_{ij}-\bar{y})^2$	$2^2 *n-1$			

Alternative Form

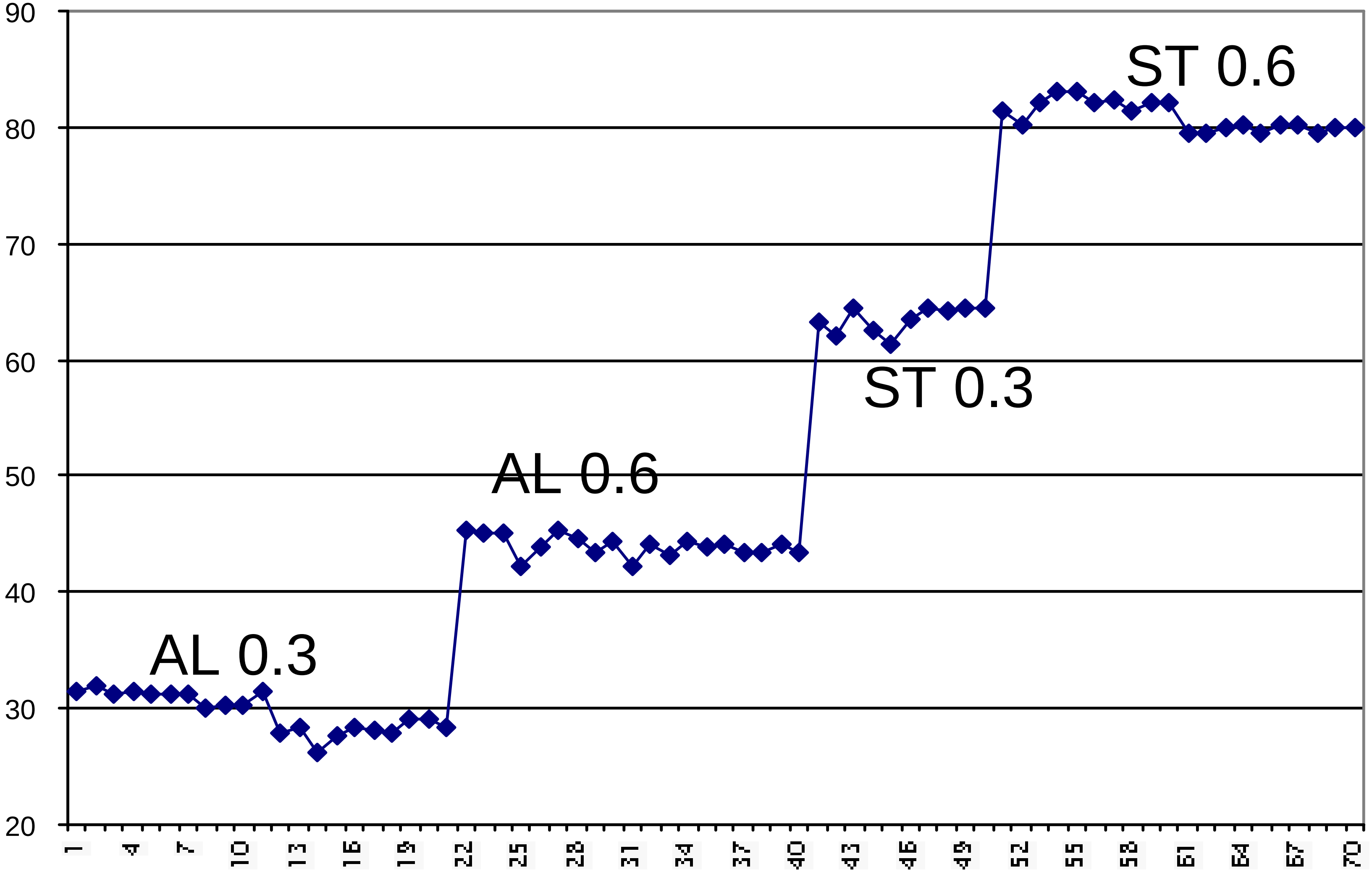
Source	SS	d.o.f.	MS	F
mean	$nm\,\beta_0^{\,2}$	1	$\frac{SS(\beta_0)}{1}$	$\frac{MS(\beta_0)}{MS(\varepsilon)}$
x_1	$nm\,\beta_1^{\,2}$	1	$\frac{SS(\beta_1)}{1}$	$\frac{MS(\beta_1)}{MS(\varepsilon)}$
x_2	$nm\,\beta_2^{\,2}$	1	$\frac{SS(\beta_2)}{1}$	$\frac{MS(\beta_2)}{MS(\varepsilon)}$
x_{12}	$nm\,\beta_{12}^{\,2}$	1	$\frac{SS(\beta_{12})}{1}$	$\frac{MS(\beta_{12})}{MS(\varepsilon)}$
ε	$\sum_{i=1}^{m}\sum_{j=1}^{n}\varepsilon_{ij}$	$mn-4$	$\frac{SS(\varepsilon)}{(mn-4)}$	
total	$\sum_{i=1}^{m}\sum_{j=1}^{n}y_{ij}$	mn		

n = replicates
m = 2^k

SS_{Total} includes the grand mean in this formulation

For all terms $F_{crit} = F_{1,\,mn-4,(1-\alpha)}$

Recall the Brakeforming Data (MIT 2002)



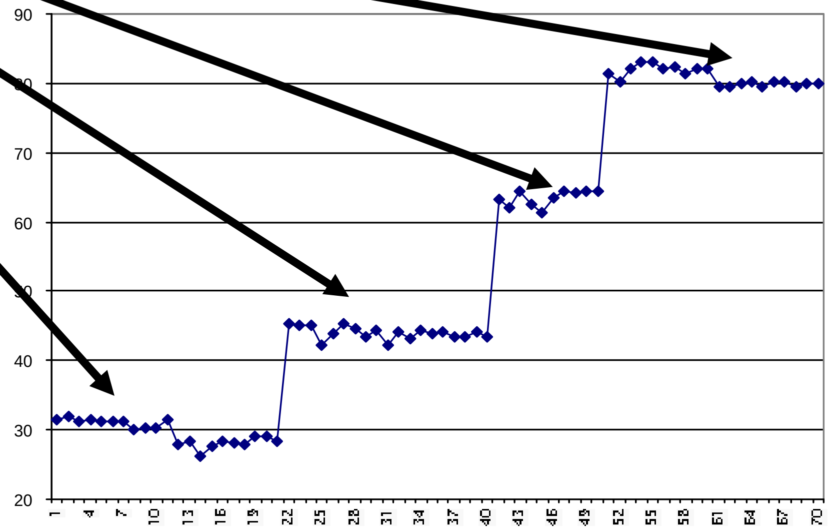
Inputs and Levels

- Inputs
  - – Punch Depth (x_1)
    - 0.3 In (-1)
    - 0.6 in (+1)
  - – Material Type/Thickness (x_2) (e.g.. bending stiffness)
    - Aluminum (-`1)
    - Steel (+1)
- 2 Inputs 2 levels each - 2^2 Model
- Output: Angle (y)

Data Table for 2^2 Model

Test	x1	x2	yi1	yi2	yi3	yi4	yi5	yi6	yi7	yi8	yi9	yi10
1	-1	-1	31.45	32.00	31.15	31.45	31.15	31.15	31.15	30.15	30.20	30.30
2	-1	1	45.30	45.10	45.00	42.15	44.00	45.35	44.55	43.30	44.30	42.15
3	1	-1	63.15	62.00	64.50	62.55	61.30	63.45	64.40	64.10	64.45	64.35
4	1	1	81.45	80.15	82.20	83.00	83.05	82.20	82.25	81.45	82.15	82.00

- x_1 : Material
- x_2 : Depth
- 4 Tests
- 10 Replicates



Looking only at Mean Response

Test	x1	x2	yibar
1	-1	-1	31.02
2	-1	1	44.12
3	1	-1	63.43
4	1	1	81.99

$\underline{y} =$

31
44.1
63.4
82

$\mathbf{X} =$

1	x1	x2	x1x2
1	-1	-1	1
1	-1	1	-1
1	1	-1	-1
1	1	1	1

Model and Interpretation

- Solving $\underline{\beta} = X^{-1}\underline{y}$

$\underline{\beta}=$

55.1
17.6
7.92
1.36

$$y = 55.1 + 17.6x_1 + 7.9x_2 + 1.4x_1x_2 + \varepsilon$$

Residual Analysis

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + h.o.t. + \varepsilon$$

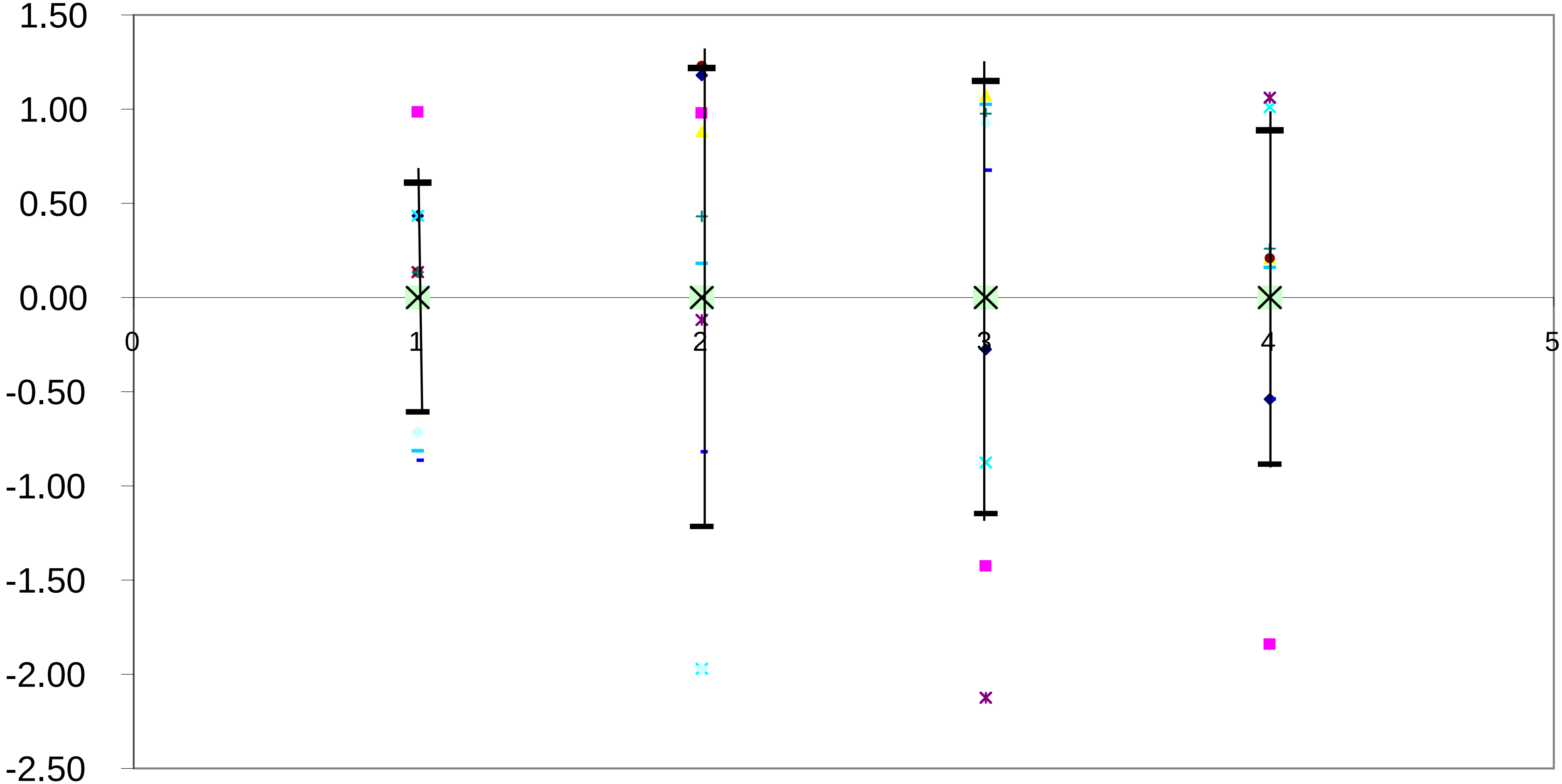
$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2$$

$$y - \hat{y} = h.o.t. + \varepsilon = \text{residual}$$

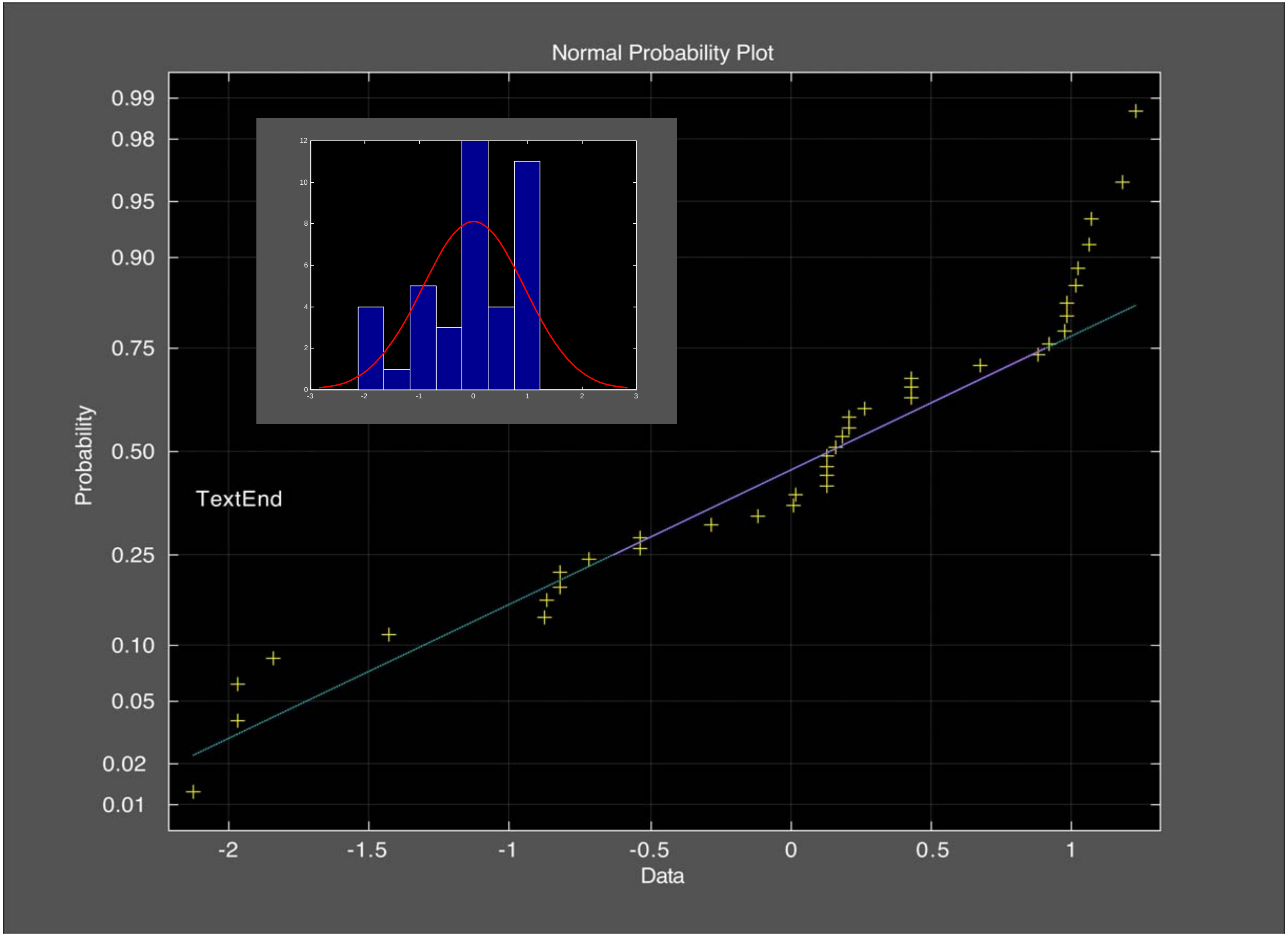
Properties of residual?

- if model is "correct"
- if model of error is $\sim N(0,\sigma^2)$

Residuals (ε) with Test



Residual Distribution



Aside: *Use of All Data*

X η

1	x1	x2	x1x2	y
1	-1	-1	1	31.45
1	-1	1	-1	45.30
1	1	-1	-1	63.15
1	1	1	1	81.45
1	-1	-1	1	32.00
1	-1	1	-1	45.10
1	1	-1	-1	62.00
1	1	1	1	80.15
1	-1	-1	1	31.15
1	-1	1	-1	45.00
1	1	-1	-1	64.50
1	1	1	1	82.20
1	-1	-1	1	31.45
1	-1	1	-1	42.15
1	1	-1	-1	62.55
1	1	1	1	83.00
1	-1	-1	1	31.15
1	-1	1	-1	44.00
1	1	-1	-1	61.30
1	1	1	1	83.05
1	-1	-1	1	31.15
1	-1	1	-1	45.35
1	1	-1	-1	63.45
1	1	1	1	82.20
1	-1	-1	1	31.15
1	-1	1	-1	44.55
1	1	-1	-1	64.40
1	1	1	1	82.25
1	-1	-1	1	30.15
1	-1	1	-1	43.30
1	1	-1	-1	64.10
1	1	1	1	81.45
1	-1	-1	1	30.20
1	-1	1	-1	44.30
1	1	-1	-1	64.45
1	1	1	1	82.15
1	-1	-1	1	30.30
1	-1	1	-1	42.15
1	1	-1	-1	64.35
1	1	1	1	82.00

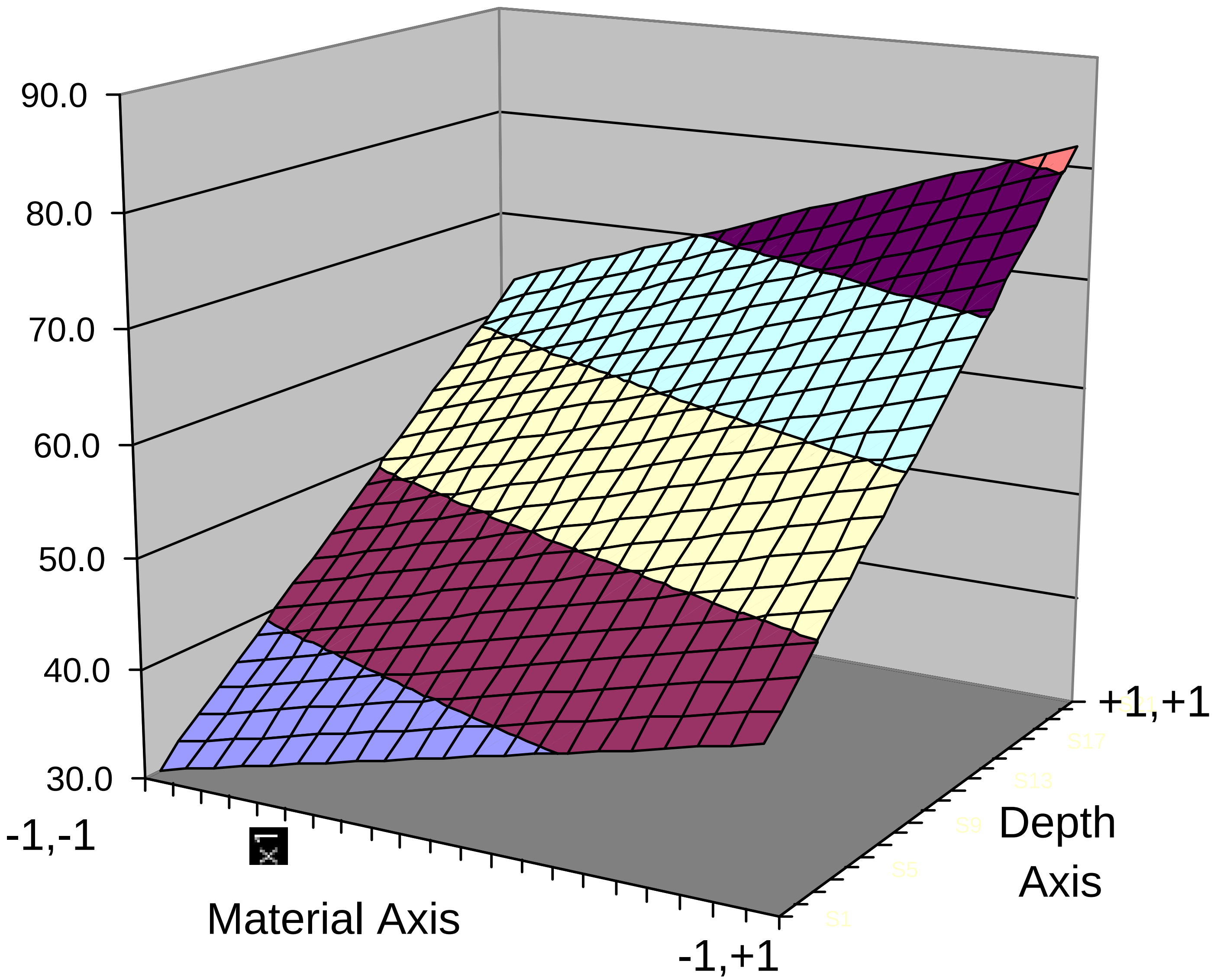
$$\underline{\beta} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\underline{\eta}$$

$$\underline{\beta} = \begin{bmatrix} 55.1 \\ 17.6 \\ 7.92 \\ 1.36 \end{bmatrix}$$

Same as before!

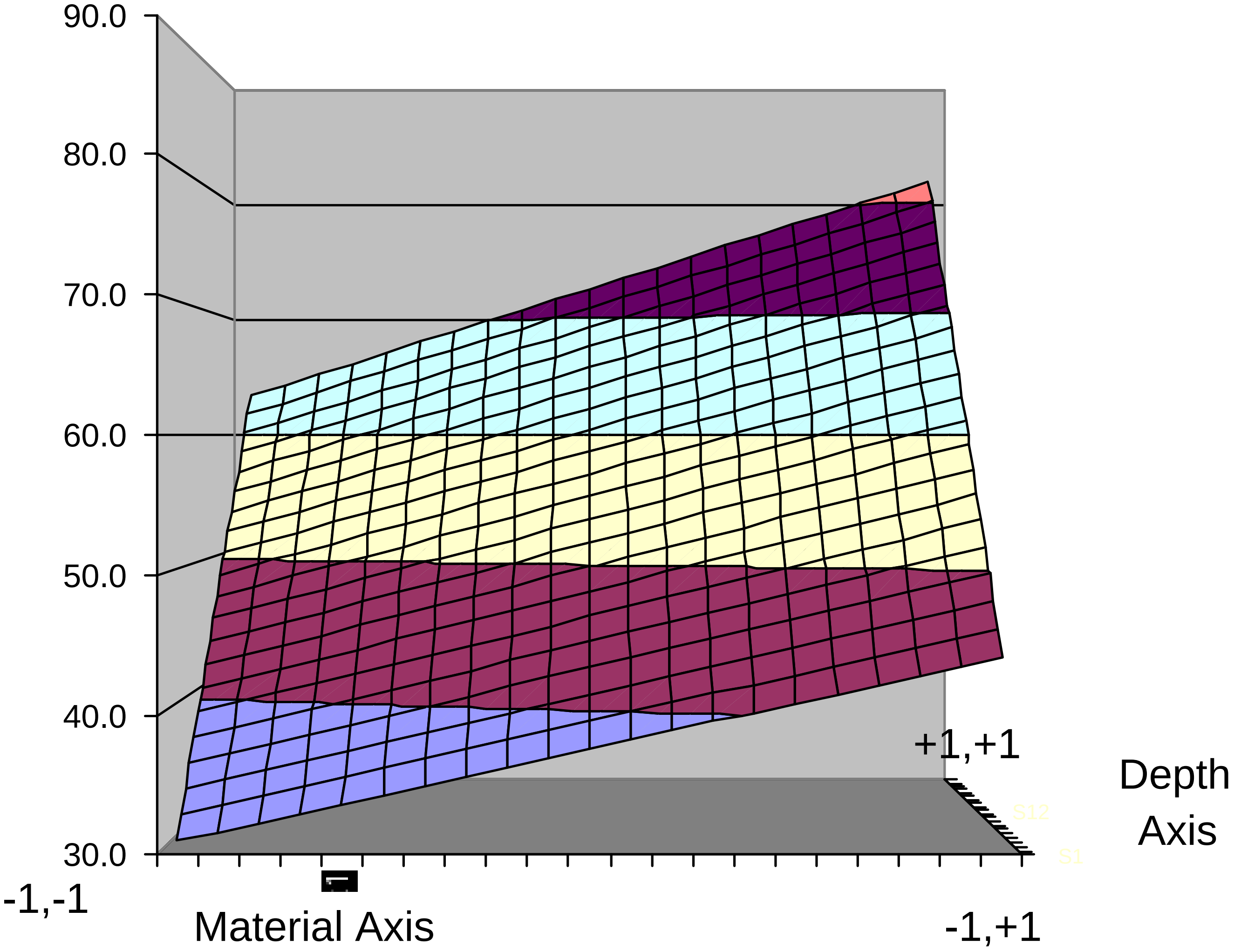
Response Surface

-1 0.3 in
+1 0.6 in
-1 Al
+1 St



Side View of Surface

-1 0.3 in
+1 0.6 in
-1 Al
+1 St



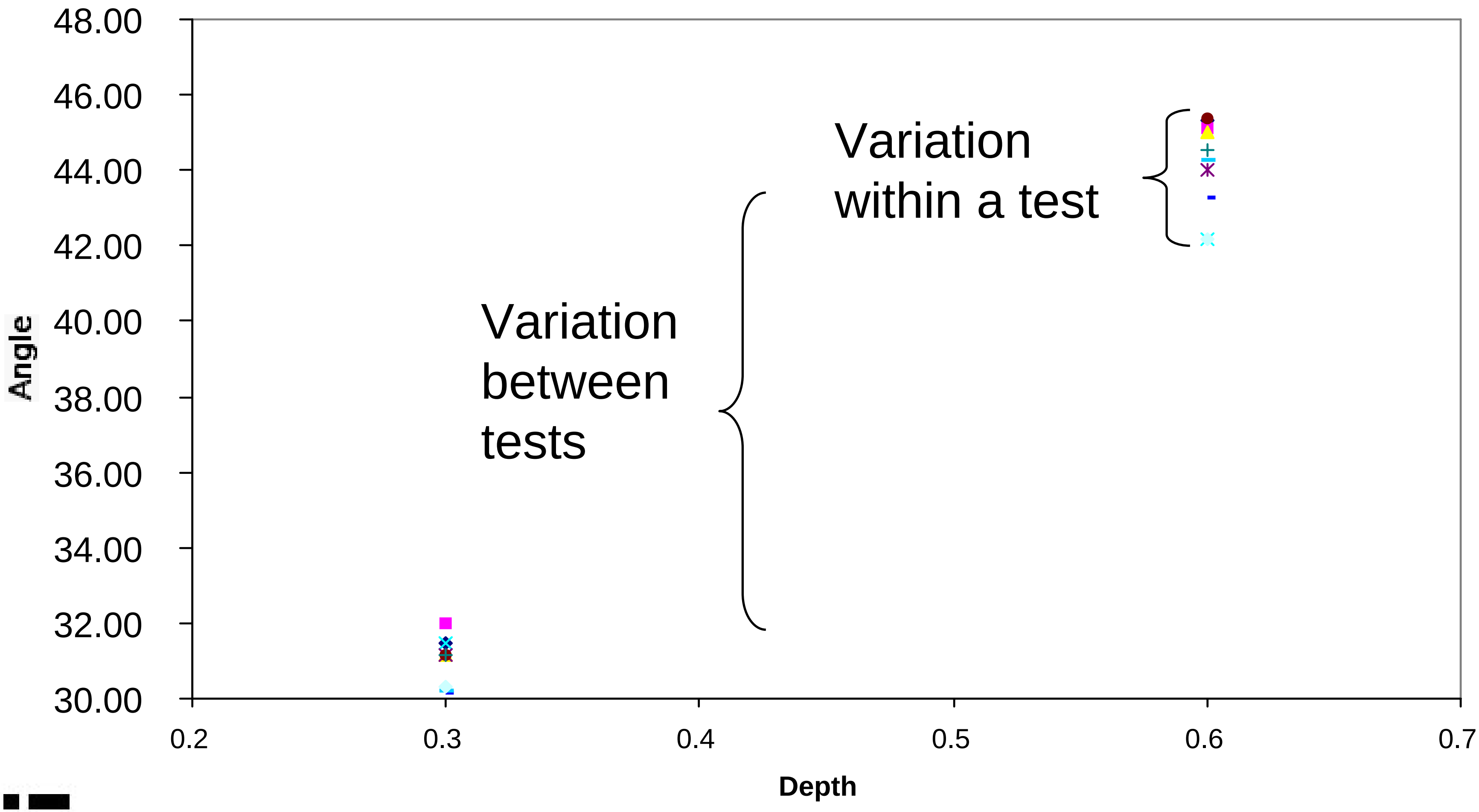
- Degree of interaction?

Are the Model Terms Significant?

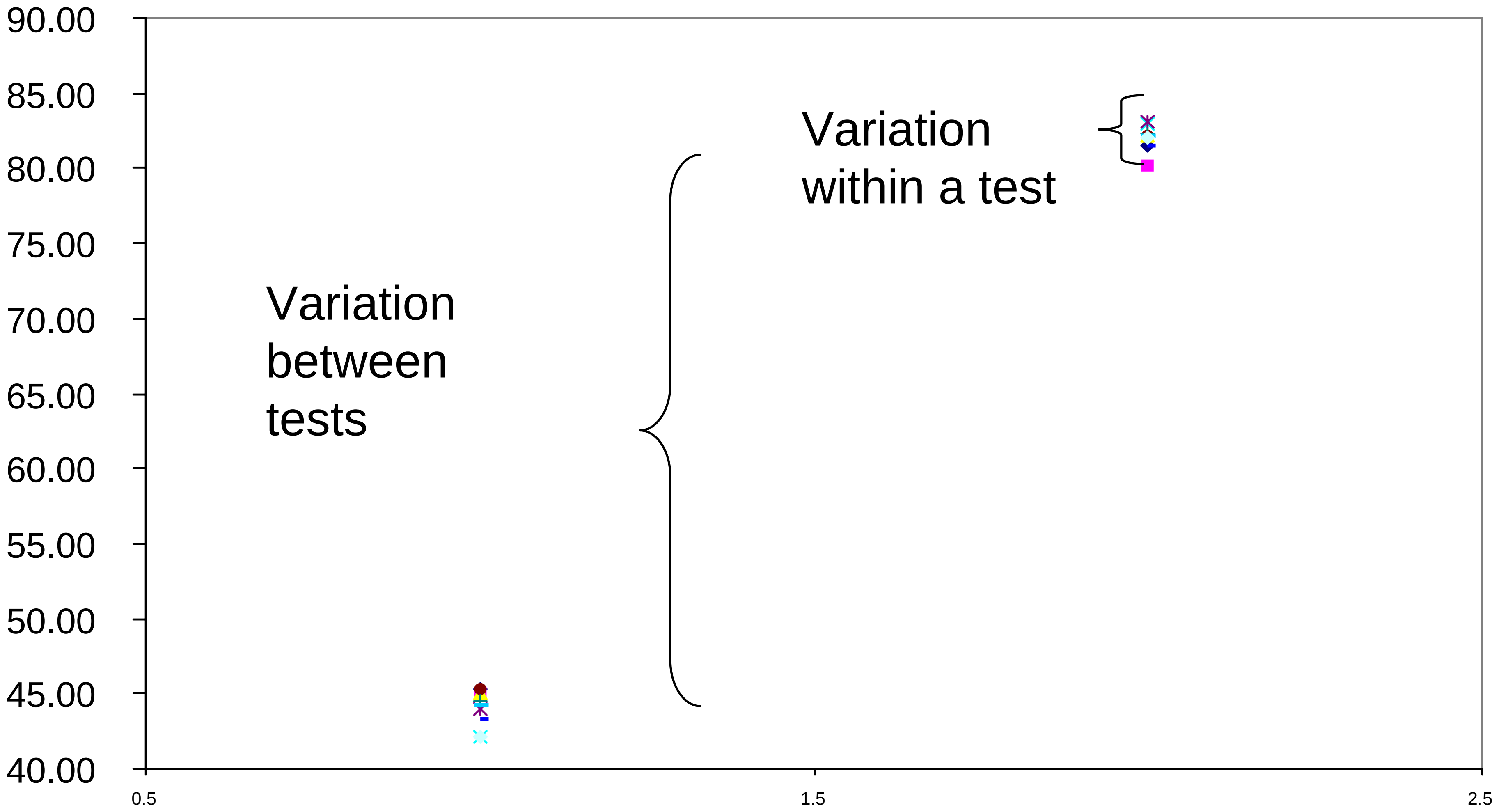
- The Mean Effect β_0
- The Effect of Depth β_1
- The Effect of Material β_2
  - Contaminated by simultaneous change of modulus, thickness and yield
- The Interaction of Depth and Material β_{12}

Look at Single Variable Plots

- Effect of Depth with Aluminum Only

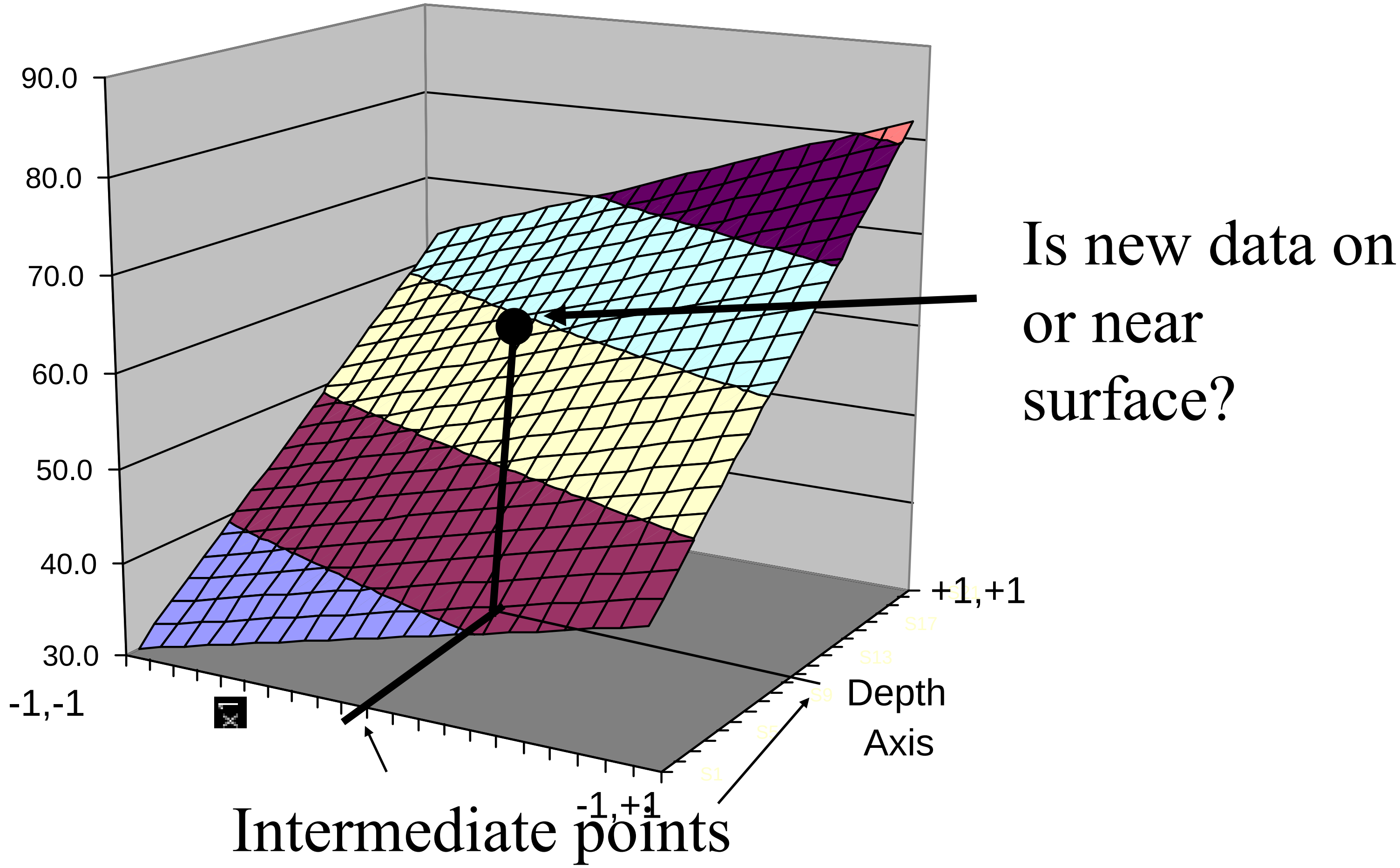


Single Variable Plot: Material Effect



Is Model Form Adequate?

- How to Test?



Next Time

- Checking adequacy of model form
  - Tests for higher order fits
- Experimental Design
  - Blocks and Confounding
- Single Replicate Designs
- Fractional Factorial Designs