Lecture Notes on Geometrical Optics (02/10/14)

2.71/2.710 Introduction to Optics –Nick Fang

Outline:
- *Geometrical light rays*
- *Fermat's Principle of least time*
- *Fermat's Principle applied to Reflection and Refraction*
- *Snell's Law examples*
- *Imaging and the Lens Law*
- *The Lens Maker's Formula*
- *Thin lens and keys for ray tracing*
- *Optical Invariant*
- *Composite Lenses*

Introduction:

While Maxwell's equations can solve light propagation in a rigorous way, the exact solutions can be found in fairly limited cases, and most practical examples require approximations.

Based on the specific method of approximation, optics has been broadly divided into two categories, namely:

i. **Geometrical Optics** (ray optics), treated in the first half of the class

- Emphasizes on finding the light path
- Especially useful for studying the optical behavior of the system which has length scale much larger than the wavelength of light, such as:

    ▪ designing optical instruments,
    ▪ tracing the path of propagation in inhomogeneous media.

ii. **Wave Optics** (physical optics).

- Emphasizes on analyzing interference and diffraction
- Gives more accurate determination of light distributions, because both the amplitude and phase of the light are considered.

A. Geometrical Light Rays
- Geometrical optics is an intuitive and efficient approximation:

Radios and mobile phones make use of same Maxwell Equations to transfer information as carried by light waves, but our perception is quite different. **Why?**

We tend to think of light as bundles of rays in our daily life. This is because we observe the processes (emission, reflection, scattering) at a distance (> 10cms with bare eyes) that are much longer than the wavelength of light (10^{-7}m or 400-700nm), and our receivers (retina and CCD pixels) are also considerably large. In the other end, the wavelength of radio-frequency waves (10cm at 3GHz) is comparable or sometimes larger than the size and spacing between transmitting/receiving devices (say, the antennas in your cell phones).

$$\frac{\text{Distance of event}}{\text{wavelength}}$$

Geometric Optics

(Scalar) Fourier Optics

Nano-Optics

Vector Field, Polarization

Radio Engineering,

Antennas, Transmission lines, cavities, amplifiers

$$\frac{\text{Wavelength}}{\text{min feature size}}$$
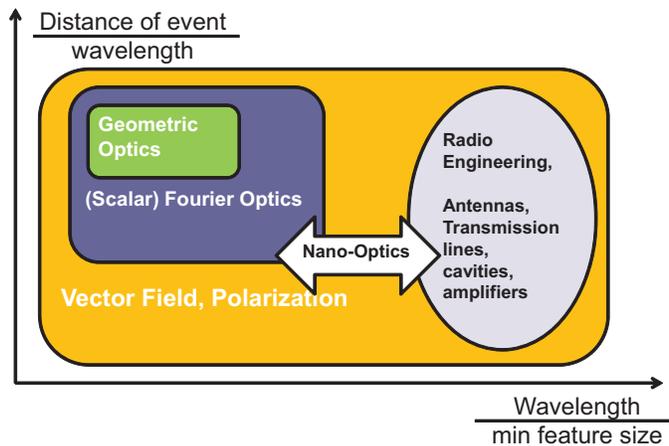
- With a common reference (the **optical axis** in your optical system) to make the problem as simple as possible, a light ray can be defined by two co-ordinates:

  o its position, x

  o its slope, θ

optical ray

θ

x

Optical axis
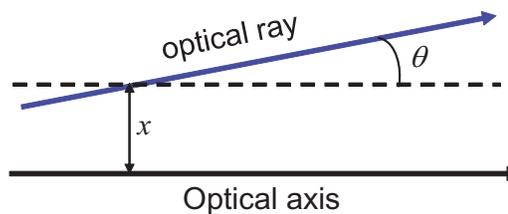
These parameters define a **ray vector**,   which will change with distance and as the ray propagates through optics.

Sign Conventions:
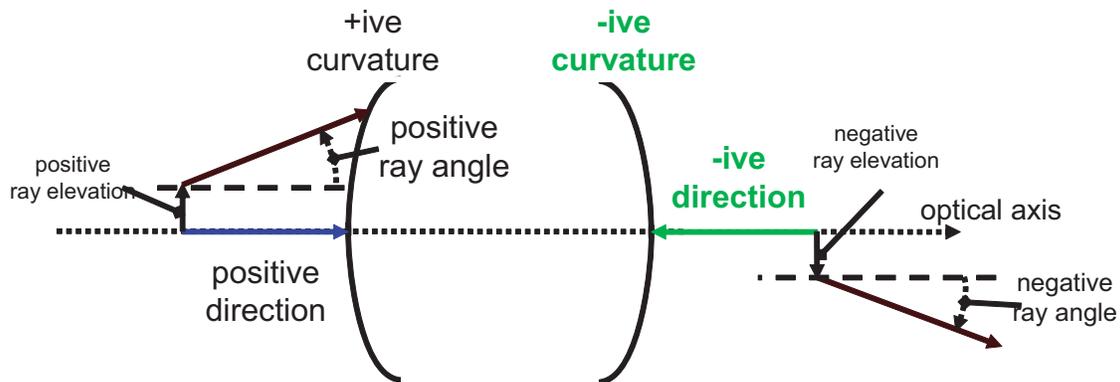
○ Light travels **from left to right**

○ A radius of curvature is **positive** if the surface is **convex towards the left**

○ Longitudinal distances are positive if pointing to the right

○ Lateral distances are positive if pointing up

○ Ray angles are positive if the ray direction is obtained by rotating the _+z_ axis counterclockwise through an acute angle

Properties of rays:

1) trajectories of "particles of light"
2) Normal to the wavefront surfaces
3) Continuous and piece-wise differentiable
4) Ray trajectories are such as to minimize the "optical path"

   ⇒ in free space, ray trajectories are straight lines

B. Fermat's Principle of least time:
At first glance, Fermat's principle is similar to **_the problem of classical mechanics_**: finding a possible trajectory of a moving body under a given potential field (We will introduce such _Lagrangian_ or _Hamiltonian_ approach in more detail in the coming lectures about _gradient index optics_).

The underlying argument is, light propagating between two given points **P** and **P'**, would take the **shortest path** (in time). In order to quantify the variation of light speed in different medium, we introduce of an index of refraction **n:**

$$n \equiv \frac{c}{v}$$

Where: **c~3x10⁸ m/s** is speed of light in vacuum;

and ***v*** is the speed of light in the medium.

Using the index of refraction, we can define an "Optical Path Length"(OPL):

$$OPL(\Gamma) = \int_{\Gamma} n(\vec{r})ds \qquad \qquad (1)$$

This is equivalent to finding the total time (ΔT= OPL/c) required for signals to travel from **P** to **P'**, and vice versa.

How is it consistent with wave picture? Modern theorists like Feynman take more rigorous approach to show that all other paths that do not require an **extreme time** ((shortest, longest or stationary) are cancelled out, leaving only the paths defined by Fermat's principle.

Under this formulation, we need to find a class of optical paths **Γ** (one or more) that are extreme value $(\frac{\partial}{\partial s} OPL(\Gamma) = 0)$.
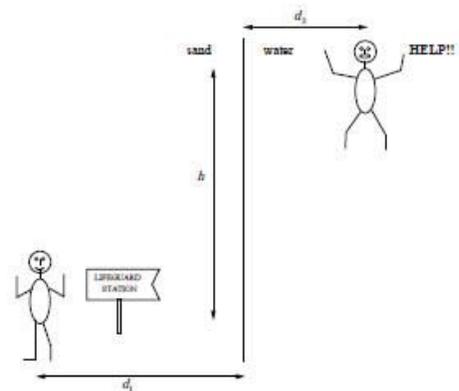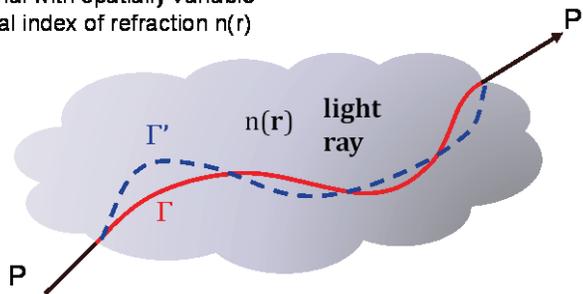
Analogy between Light path and a problem of lifeguard on the beach: Which path should the lifeguard follow to reach the drowning person in minimum time?

C. Fermat's Principle applied to Reflection and Refraction

a. Application on flat interfaces:

mirror
or dielectric interface
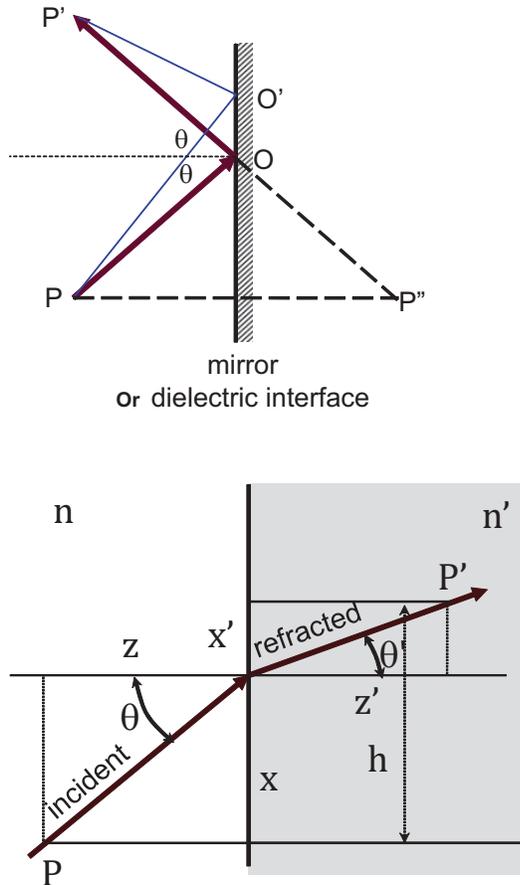
Figure: The law of reflection (Left) and refraction (Right) over a flat interface using Fermat's Principles. Results are the famous Snell's (Decartes') Law of refraction and reflection.

b. Application on curved surfaces:
   o **Parabolic reflector**

- *What should the shape function* s(x) *be in order for the incoming parallel ray bundle to come to a geometric spot?* (Hint: *for every path from the incoming plan wave reflected by the mirror,* OPL=2*f*)

focus at F
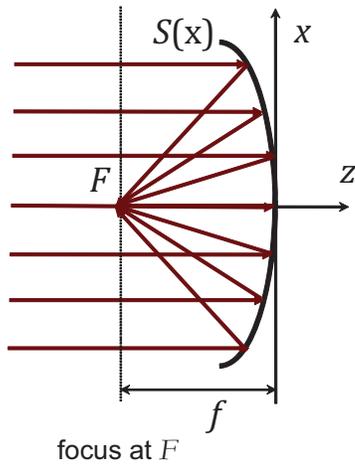
Examples of Reflectors: **solar concentrators, satellite dishes, radio telescope**

o **Elliptic refractor**
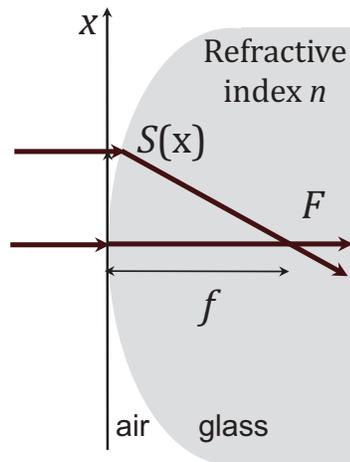
D. **A Mechanics View of Snell's Law and examples**

We can compare the reflection and refraction of light to that of a ball bouncing off a hard wall.

While the velocity of the ball changes immediately after collision, its projection parallel to the wall does not change. This is because there is no force acting on the ball during the collision process, therefore the momentum does not change along this direction. The same argument applies to the light ray: the momentum of light particles in the direction parallel to the "wall" (an ideally flat surface) does not change during reflection and transmission. Such process is often visualized with the following graph method with the help of Decartes spheres:
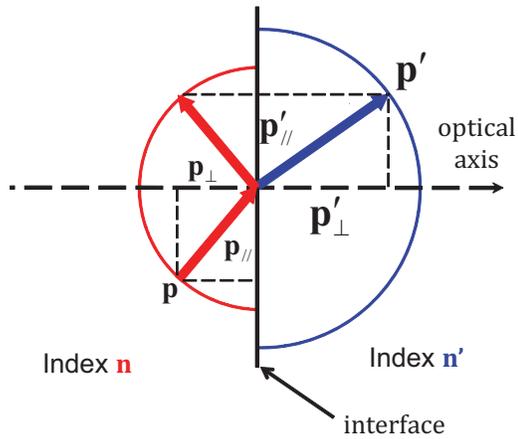
The half circles represent the amplitude of momentum in different medium ($|\mathbf{p}| = n\frac{\hbar\omega}{c}$), and the colored arrows indicates the direction of incident, reflected and transmitted light beams, and momentum is conserved in the parallel direction, indicating $\mathbf{p}_{//} = \mathbf{p}'_{//}$.

Snell's Law for many parallel layers

If the layers are parallel, then these angles are always equal.
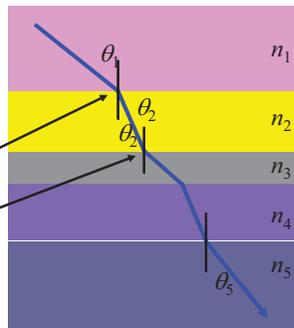
E. Refraction from a sphere: paraxial approximation
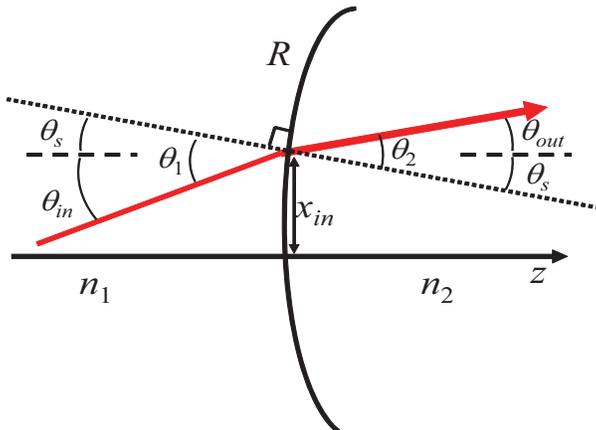
Snell's Law is approximated: $\sin\theta_s \approx x_{in}/R$

$$n_1(\theta_{in} + x_{in}/R) \approx n_2(\theta_{out} + x_{in}/R) \qquad \textbf{(2)}$$

Two special cases:

- $\theta_{in} = 0$: The incoming beams are collimated, and the refracted beams are converging to a spot at a distance.
$$(n_1 - n_2)\, x_{in}/R \approx n_2\theta_{out} \approx -n_2\, x_{in}/s_i$$

- $\theta_{out} = 0$: The beams emitted by a spot at a distance are converted to collimated outgoing beams after refraction.
$$n_1\theta_{in} \approx (n_2 - n_1)\, x_{in}/R \approx n_1\, x_{in}/s_o$$

F. Imaging and the Lens Law

In the general case

$$n_1\,(x_{in}/s_o + x_{in}/R) \approx n_2(-x_{in}/s_i + x_{in}/R)$$

We find the following relationship:

$$\frac{n_1}{s_o} + \frac{n_2}{s_i} \approx \frac{n_2 - n_1}{R} \qquad \textbf{(3)}$$

To form a **real** image with converging beams in both sides, we see that both s_o and s_i has to be larger than a minimum distance. If either of these conditions are broken then we will obtain a **virtual** image.

G. The Lens Maker's Formula

Typically a lens is formed with two spherical glass surfaces. Let's place a second interface to the right of our spherical glass (without loss of generality, let's assume the first surface is generating a virtual image to help our drawing).
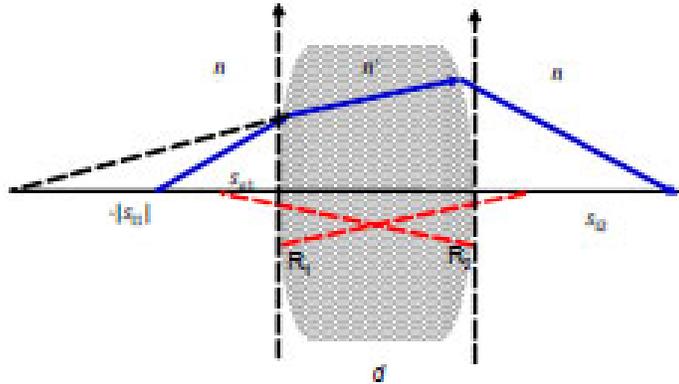
On the first interface we obtain:

$$\frac{n}{s_{o1}} + \frac{n\prime}{s_{i1}} \approx \frac{n\prime - n}{R_1} \qquad (4)$$

Note that S_{i1} is negative since the image is **virtual**. For the second interface, we again use the Eq(3)

$$\frac{n\prime}{s_{o2}} + \frac{n}{s_{i2}} \approx \frac{n - n\prime}{R_2} \qquad (5)$$

Also, $s_{o2} = d - s_{i1}$. Adding (4) and (5) we find:

$$\frac{n}{s_{o1}} + \frac{n}{s_{i2}} \approx (n' - n)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{n'd}{(d - s_{i1})s_{i1}}$$

When the lens is thin (d<<R₁, R₂), we can neglect the last term. Further, if the medium is air (n=1), we arrive at the famous Lens maker's equation:

$$\frac{1}{s_{o1}} + \frac{1}{s_{i2}} \approx (n' - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad (6)$$

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Practice Example 1:

      I found a glass plano-convex lens thin lens on my shelf and I am using it to image a light bulb 36cm in front of the lens. Assume the index of refraction is 1.5 and the radius of the lens is 6cm, where can I find the image behind the lens?

</div>

H. **Thin Lenses, Ray tracing**
   3 Simple Rules:

   1) Rays parallel to the optical axis (from the **left** side of lens) are deflected through the **right** focal point.

   2) Rays passing through the **left** focal point becomes parallel to the optical axis;

   3) Rays passing through the **center** of the lens remain in the same direction.

Figure (Figure 2-22 from Pedrotti)

Practice problem 2: A luminous object marked by the arrow and an observing screen are separated by a fixed distance L>4*f* (the focal length of the lens). Show that there are two positions separated by a distance *D*, where a thin convex lens can be placed to give a focused image on the observing screen. What is the expression of *f* as a function of *L* and *D*? This is known as *Bessel's method* to find the focal length.
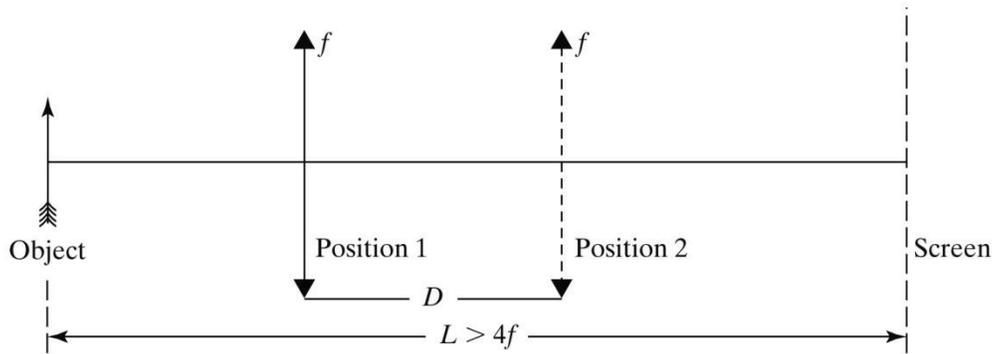
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Observation from ray tracing: Magnification $M \equiv \dfrac{h_i}{h_o} = -\dfrac{s_i}{s_o}$ **(since image is inverted)**

I. Optical Invariant

-What happens to an arbitrary "axial" ray that originates from the axial intercept of the object, after passing through a series of lenses?

If we make use of the relationship between launching angle and the imaging conditions, we have:

$$\theta_{in} = \frac{x_{in}}{s_o} \text{ and } \theta_{out} = \frac{x_{in}}{s_i}$$

$$\frac{\theta_{in}}{\theta_{out}} = \frac{s_i}{s_o} = -\frac{h_i}{h_o}$$

Rearranging, we obtain:

$$\theta_{in} h_o = \theta_{out} h_i$$

We see that the product of the image height and the angle with respect to the axis remains a **constant**. Indeed a more general result, $nh_o \sin\theta_{in} = n'h_i \theta_{out}$ is a constant (often referred as a *Lagrange invariant* in different textbooks) across any surface of the imaging system.

- The invariant may be used to deduce other quantities of the optical system, without the necessity of certain intermediate ray-tracing calculations.
- You may regard it as a precursor to *wave optics*: the angles are approximately proportional to lateral momentum of light, and the image height is equivalent to separation of two geometric points. For two points that are separated far apart, there is a limiting angle to transmit their information across the imaging system.

J. Composite Lenses:

To elaborate the effect of lens in combinations, let's consider first two lenses separated by a distance d. We may apply the *thin lens* equation and cascade the imaging process by taking the image formed by lens 1 as the object for lens 2.
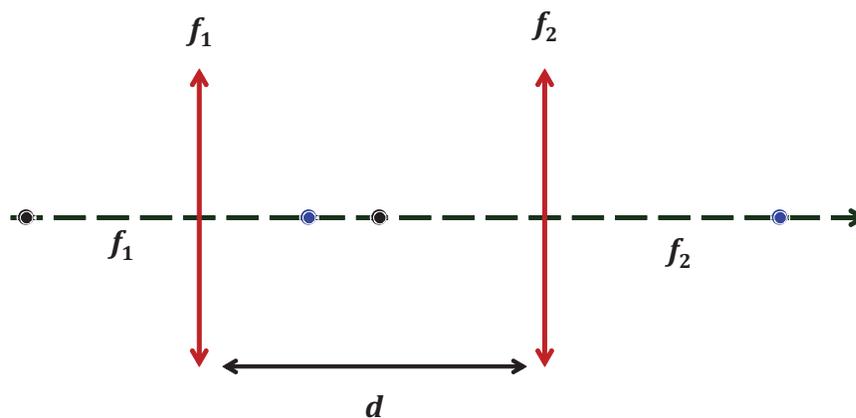
$$\frac{1}{s_{o1}} + \frac{1}{s_{i2}} = \left(\frac{1}{f_1} + \frac{1}{f_2}\right) - \frac{d}{(d - s_{i1})s_{i1}}$$

A few limiting cases:

a) Parallel beams from the left: s_{i2} is the back-focal length (BFL)

$$\frac{1}{\text{BFL}} = \left(\frac{1}{f_1} + \frac{1}{f_2}\right) - \frac{d}{(d - f_1)f_1}$$

b) collimated beams to the right: s_{o1} is the front-focal length (FFL)

$$\frac{1}{\text{L}} = \left(\frac{1}{f_1} + \frac{1}{f_2}\right) - \frac{d}{(d - f_2)f_2}$$

The composite lens does not have the same apparent focusing length in front and back end!

c) *d=f₁+f₂*: parallel beams illuminating the composite lens will remain parallel at the exit; the system is often called *afocal.* This is in fact the principle used in most **telescopes**, as the object is located at infinity and the function of the instrument is to send the image to the eye with a large angle of view. On the other hand, a point source located at the left focus of the first lens is imaged at the right focus of the second lens (the two are called *conjugate points*). This is often used as a **condenser** for illumination.

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