Outline:

A. *Superposition of waves, Interference*
B. *Interferometry*
   - Amplitude-splitting *(e.g.* Michelson *interferometry)*
   - Wavefront-splitting (*e.g. Young's Double Slits*)
C. *More Interferometry: Fabry Perot, etc*

A. Superposition of Waves, Interference

The nature of linear wave equation guarantees that waves can be superimposed: we may combine an array of waves by algebra, as far as each of them are proper solution of the wave equation. To facilitate this process, we make use of the following complex number to represent the wave field.

- *Waves in complex numbers*

For example, the electric field of a monochromatic light field can be expressed as:

$$E(z,t) = Acos(kz - \omega t + \varphi) \tag{1}$$

$$\text{Since } \exp(ix) = \cos(x) + i\sin(x) \tag{2}$$

$$E(z,t) = Re\{Aexp[i(kz - \omega t + \varphi)]\} \tag{3}$$

Or

$$E(z,t) = \frac{1}{2}\{Aexp[i(kz - \omega t + \varphi)]\} + c.c.(complex\ conjugate) \tag{4}$$

- Complex numbers simplify optics! Interference with two beams

e.g. 2 plane waves propagating in +z direction

$$E_{1x} = E_{1x}(0)exp[i(k_1 z - \omega_1 t + \varphi_1)] \tag{5}$$
$$E_{2x} = E_{2x}(0)exp[i(k_2 z - \omega_2 t + \varphi_2)] \tag{6}$$

We can define phase of each waves:

$$\delta_1(z,t) = k_1 z - \omega_1 t + \varphi_1 \tag{7}$$
$$\delta_2(z,t) = k_2 z - \omega_2 t + \varphi_2 \tag{8}$$

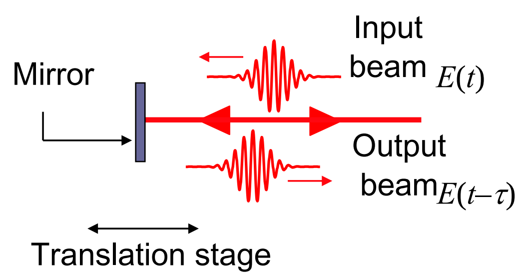
For a point P located at z=z_0 the combined field is:

$$E_x = E_{1x} + E_{2x} = E_{1x}(0)\exp[i\delta_1(z_0,t)] + E_{2x}(0)\exp[i\delta_2(z_0,t)] \tag{9}$$

B. Interferometry

We often use optical interferometers to facilitate the study of interference. A few common setups are discussed here. Based on the operation principle to split the beams, we may find the so-called amplitude-splitting or wavefront-splitting devices.

- Michelson Interferometry

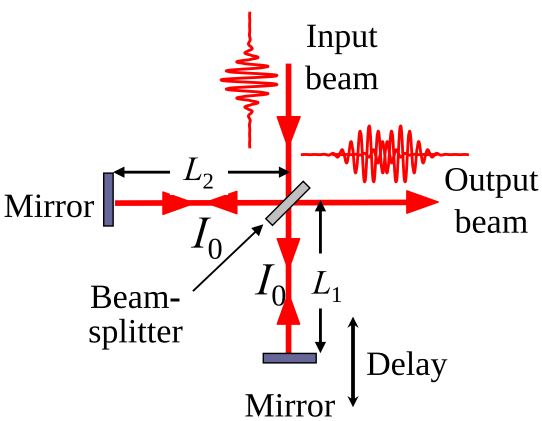


The Michelson Interferometer is named after Albert Michelson, who used it with Edward Morley in 1887, in an attempt to measure the existence of the "ether".

In Michelson- Morley's famous experiment, the delay time $\Delta\tau$ is achieved simply by moving a mirror along the optical axis. Moving a mirror backward by a distance L yields a delay of:

$$\Delta\tau = \frac{2L}{c} \quad (10)$$

(e.g. 300 μm of mirror displacement yields a delay of 2×10^{-12}s=2ps).



A Michelson Interferometer as shown in left schematic, split a beam of incident light into two arms using a thin glass window. Both beams travel to mirrors that are precisely aligned to reflect them. Before recombining them at the beam splitter, the two beams traveled with different optical path length L_1 and L_2.

$$I = 2I_0 + 2I_0 \left\langle \cos\left(2\omega \frac{L_1 - L_2}{c}\right)\right\rangle \quad (11)$$

The variation of intensity as a function of the path length L_1 gives us a measure of wavelength of light! Recent effort is to apply such technology in measurement of gravity waves.

Observation:

- Michelson Interferometer measures (auto)-correlation in time.

To see this effect, we suppose the input light beam is not monochromatic. Thus

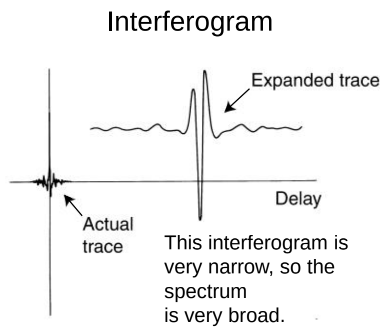
$$I = \frac{c}{2}\varepsilon\langle E_x \cdot E_x^*\rangle = \frac{c}{2}\varepsilon\langle (E_{1x} + E_{1x}(t-\tau)) \cdot (E_{1x}^* + E_{1x}^*(t-\tau))\rangle \quad (12)$$

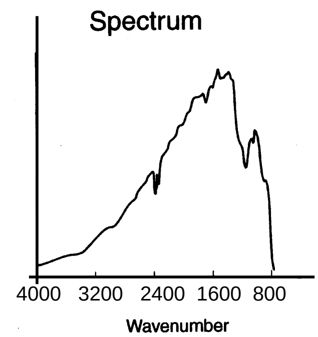
$$I = I_1(t) + I_1(t-\tau) + 2\langle E_{1x} \cdot E_{1x}^*(t-\tau)\rangle \tag{13}$$

Such auto-correlation function tells the similarity of the field over given period of time. The Fourier transform of the auto-correlated signal yields the *Power Spectrum*:

$$\int_0^\infty E_{1x} \cdot E_{1x}^*(t-\tau)dt \overset{FT}{\Rightarrow} E_{1x}(\omega) \cdot E_{1x}^*(\omega) = |E_{1x}(\omega)|^2 \tag{14}$$

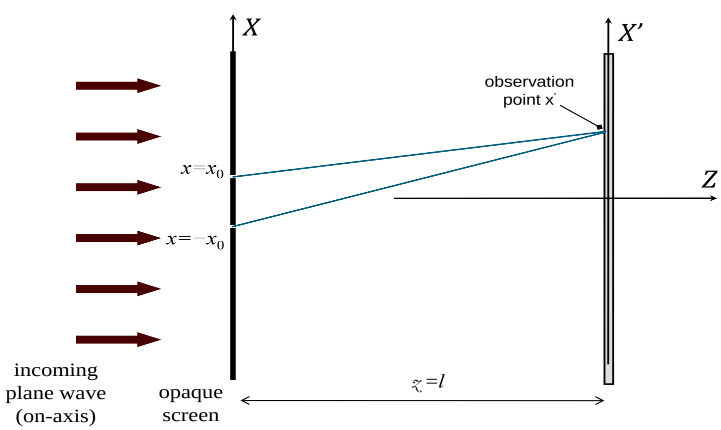
This is how Fourier Transform Spectroscopy (often abbreviated as FTIR) are constructed nowadays. (See **Pedrotti 21-2** for more discussion)





- In the above analysis we assumed the input beams are ideal plane waves so we only focused on intensity variation of a single spot as the arm length changes. In reality, a set of nested rings are often observed on the receiving screen as the pattern on right. This is due to the variation of phase as a function of momentum difference $(k_1 - k_2)z$.

- Young's Double Slit Interferometry



To analyze Young's experiment, we assume the screen X with two narrow slit is illuminated with a monochromatic plane wave. After the slits, two cylindrical waves are excited. This creates fringes at the observation plane X' , after travelling a distance $z = l$.

For a position x' on the screen, the phase difference of the two waves becomes:

$$\delta_1 - \delta_2 = k(r_1 - r_2) + (\varphi_1 - \varphi_2) \tag{15}$$

where $r_1 = \sqrt{(x' - x_0)^2 + l^2}, r_2 = \sqrt{(x' + x_0)^2 + l^2}$. When the slits are arranged symmetrically and the incoming plane wave is at normal incidence, then $\varphi_1 - \varphi_2 = 0$

In a typical experiment the screen is placed far away from the slits ($l \gg x_0$) so we can further take approximation:

$$r_1 = l\sqrt{1 + \frac{(x'-x_0)^2}{l^2}} \approx l\left(1 + \frac{(x'-x_0)^2}{2l^2}\right) \tag{16}$$

$$r_2 = l\sqrt{1 + \frac{(x'+x_0)^2}{l^2}} \approx l\left(1 + \frac{(x'+x_0)^2}{2l^2}\right) \tag{17}$$

$$(r_1 - r_2) = -\frac{2x'x_0}{l} \tag{18}$$

In order for the constructive interference to occur, $\cos(\delta_1 - \delta_2) = 1$, then

$$k(r_1 - r_2) = -k\frac{2x_0x'}{l} = 2m\pi \tag{19}$$

$m = 0, \pm1, \pm2, \ldots$. are called order number.

This gives rise to the famous condition:

$$2x_0x' = ml\lambda \tag{20}$$

Therefore Young's experiments directly measured the wavelength of light (in 18 century!)

Note: Generally fringes will form by two or more beams crossing at an angle. To quantify that we can modify our phase term for the crossing beams:

$$\delta_1(x, z, t) = k_{1x}x + k_{1z}z - \omega_1 t + \varphi_1 \tag{21}$$

or in terms of incident angle θ:

$$\delta_1(x, z, t) = k_1 x sin\theta_1 + k_1 z cos\theta_1 - \omega_1 t + \varphi_1 \tag{22}$$

$$\delta_1 - \delta_2 = (k_1 sin\theta_1 - k_2 sin\theta_2)x + (k_1 cos\theta_1 - k_2 cos\theta_2)z + (\varphi_1 - \varphi_2) \tag{23}$$

Fringes can vary both on x and z directions!

- ***Note: Young's double slit experiment measures the correlation in space.*** Assuming the input light beam is not a plane wave (i.e. inhomogeneous), at x'=0 we measure the interference:

$$I = \frac{c}{2}\varepsilon\langle E_x \cdot E_x^*\rangle = \frac{c}{2}\varepsilon\langle (E_{1x}(x - x_0) + E_{1x}(x+x_0)) \cdot (E_{1x}^*(x - x_0) + E_{1x}^*(x + x_0))\rangle \tag{24}$$

$$I = I_1(x - x_0) + I_1(x + x_0) + 2\langle E_{1x}(x - x_0) \cdot E_{1x}^*(x + x_0)\rangle \qquad (25)$$

Such correlation function tells the similarity of the field over a given **spatial period.** This effect is often used to measure the coherence of a remote star under the telescope, although the radiation is thought to be randomly distributed. A daily life example is the spatial coherence of ripples in the pool ("Spatial coherence from Ducks" by Emil Wolf et al, Physics today 2010).

- Comparison between Michelson and Young's double slits:
  Both can be regarded as interference of two spherical waves, but observed in different directions.

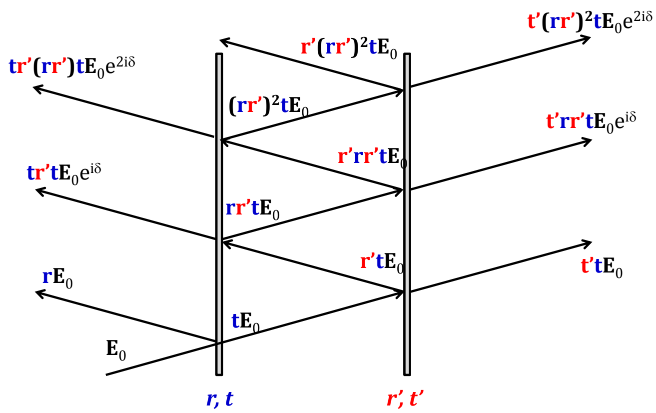
Image of wavefront splitting inferometry removed due to copyright restrictions.

- *Other wavefront splitting interferometry similar to Young's Double slits:*
  - Lloyd's Mirror
  - Fresnel's biprism
  - Fresnel's mirror
  - Billet's split lens

C. More examples of Interferometry

- Thin Film interference

- Fabry-Perot Interferometry



Consider two parallel reflective surfaces separated by distance *d*, the first one has a amplitude transmission and reflection coefficient ***t*** and ***r***, and the second has amplitude transmission and reflection coefficient ***t'*** and ***r'***. Multiple reflections between the two surfaces results in two series of reflected and transmitted terms. Due to the round trip travel path, there is a phase difference between successive transmitted terms:

$$\delta_0 = (2k_z d) \tag{26}$$

The transmitted series is

$$E_t = \exp(i\delta_0/2)(t'tE_0 + t'(rr')tE_0\exp(i\delta_0) + t'(rr')^2tE_0\exp(2i\delta_0) + \cdots) \tag{27}$$

$$E_t = t'tE_0\exp(i\delta_0/2)[1 + (rr')\exp(i\delta_0) + (rr')^2\exp(2i\delta_0) + \cdots] \tag{28}$$

$$E_t = t'tE_0\exp(i\delta_0/2)\sum_{n=0}^{\infty}(rr')^n\exp(in\delta_0) \tag{29}$$

$$E_t = \frac{t'\,t\exp(i\delta_0/2)E_0}{1 - rr'\exp(i\delta_0)} \tag{30}$$

The transmitted irradiance is given by:

$$I_t = \frac{T'TI_0}{|1 - rr'\exp(i\delta_0)|^2} \tag{31}$$

The denominator of the last result can be expressed as:

$$\begin{aligned}|1 - rr'\exp(i\delta_0)|^2 &= (1 - rr'\exp(i\delta_0))(1 - r^*r'^*\exp(-i\delta_0))\\ &= 1 - (r\,r'\exp(i\delta_0) + r^*r'^*\exp(-i\delta_0)) + RR'\\ &= 1 - 2\sqrt{RR'}\cos(\delta) + RR'\\ &= \left(1 - \sqrt{RR'}\right)^2 + 2\sqrt{RR'}(1 - \cos(\delta))\\ &= \left(1 - \sqrt{RR'}\right)^2 + 4\sqrt{RR'}\sin^2(\tfrac{\delta}{2})\end{aligned} \tag{32}$$

$$\text{Where } \delta = (2k_z d) + \phi_r + \phi_{r'} \tag{33}$$

We define the *coefficient of finesse F*:

$$\mathcal{F} = \frac{4\sqrt{RR'}}{\left(1-\sqrt{RR'}\right)^2} \tag{34}$$

To express the general form of I_t:

$$I_t = \frac{T'TI_0}{\left(1-\sqrt{RR'}\right)^2}\left[\frac{1}{1+\mathcal{F}\sin^2(\frac{\delta}{2})}\right] \tag{35}$$

Applications: Fabry-Perot cavities are often designed to distinguish closely spaced spectral lines of a gas medium. Higher values of Finesse *F* give a sharper transmission pass band and greater spectral resolution. To find the half-width of the pass band, we solve:

$$\frac{1}{1+\mathcal{F}\sin^2(\frac{\delta}{2})} = \frac{1}{2} \tag{36}$$

giving

$$sin\left(\frac{\delta_{1/2}}{2}\right) = \frac{1}{\sqrt{\mathcal{F}}} \tag{37}$$

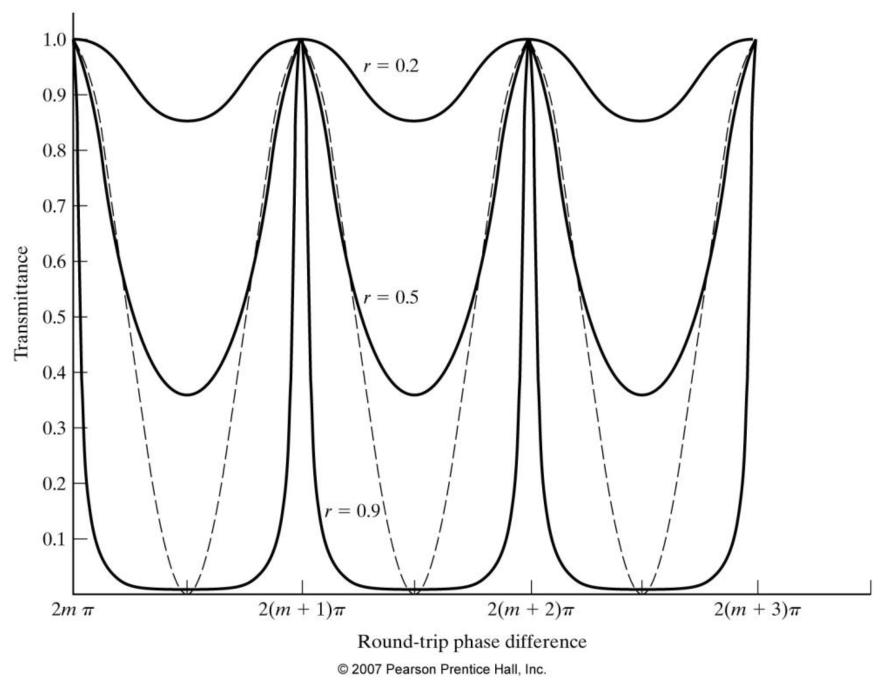


Figure 8.9 from Pedrotti: Transmittance of Fabry-Perot Cavity.