2.71/2.710 Optics   **Spring '12**
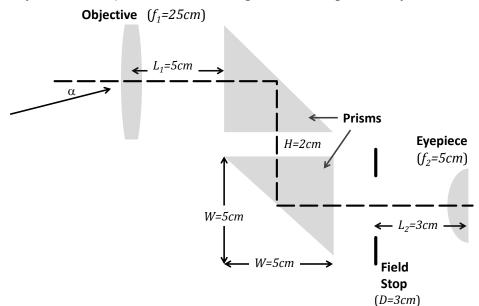Quiz #1 Solutions

1. **A Binocular.** The simplified optical diagram of an arm of a binocular can be considered as a telescope, which consists of two lenses of focal lengths f_1 = 25cm (objective) and f_2 = 5cm (eyepiece). The normal observer's eye is intended to be relaxed and the nominal focal length of the eye lens is taken to be f_{EL} = 40mm. **The first prism is placed 5cm away from the objective and the two prisms are separated by 2cm.**
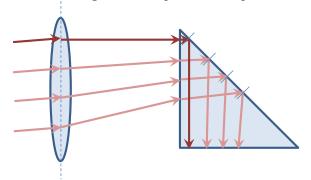

a) **(10%)** In order to make the binocular compact, a pair of 45° prisms (5cm wide) are used, each of them is designed for total internal reflection of incoming rays. Estimate the index of refraction needed to meet such a requirement under paraxial beam approximation.

b-e) Assume the index of refraction of both prisms is 1.5.

b) **(15%)** Please estimate the distance from the eyepiece to the back side of the second prism.

c) **(20%)** If two distant objects are separated by 10^{-3}rad to an observer with naked eye, how far apart (in units of length) will the images form on the observer's retina when the observer is using the binocular?

d) **(15%)** An aperture (D=3cm) is placed inside the binocular, at a distance of 3**cm** to the left of the eyepiece. Please locate the Entrance Window and Exit Window, and calculate the Field of View.

e) **(**extra credit **10%)** Where is the optimum location of the observer's eye pupil in the configuration described by **d)**?

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Solution:

a) The plane wave incident on the lens is converted into a converging spherical wave. Although each ray is initially parallel when it enters into the system at the objective lens, each exits the lens at a different angle with respect to the optical axis.

By applying Snell's law at the right interface of the prism, we can calculate the index of refraction that is required for total internal reflection. However, the angles incident on the interface depend upon the angle of the initial plane wave entering the system. Since no information is given about this angle, the simplest way to *estimate* the index of the prism is to consider a ray that strikes the interfaces with an incident angle of 45°.

$$n_{prism} sin\theta_{inc} = 1 \cdot sin\theta_{trans} \tag{1-1}$$

The index of the prism can be calculated from the criterion for total internal reflection, which occurs when the transmitted angle is 90°, *i.e.* it travels along the glass-air interface. For an incident critical angle, $\theta_{inc} = 45°$, the index can be found to be:

$$n_{prism} sin45° = 1 \cdot sin90°$$

$$n_{prism} \frac{1}{\sqrt{2}} = 1 \tag{1-2}$$

$$n_{prism} = \sqrt{2} = 1.41$$

Therefore, the index of refraction of the prism is estimated to be around **1.4**, which is consistent with the 1.5 value typical of glass.

b) By ray tracing we can gain some intuition about what is happening in the system. We see that the prisms act to essentially collapse the system into a more compact space. The angle of inclination of each ray entering into the top prism is identical to the angle of inclination of that same ray as it departs the lower prism. However, the rays leaving the lower prism are bundled closer together than they were when they entered into the upper one. Therefore, the focal point of the binoculars is located at a different plane than would have occurred had the prisms been replaced with mirrors instead. We can find the location where the rays are focused by invoking Fermat's principle, which says that all rays that emerge from the objective lens must travel the same distance to the focal plane. Here, the easiest ray to trace is the one that enters the upper prism parallel to the optical axis.

If the prisms were absent the focal plane would simply occur one focal length away from the objective lens. However, the presence of the prisms alters the trajectory of light rays by bending them towards the optical axis as they pass through the glass. This causes the rays to emerge from the exit end of the prism at a location further from the optical axis then they would have if the prisms were made of air (i.e. replaced with mirrors).

In order to find the distance between the second prism and the eyepiece, the location of the focal plane of the rays emerging from the second prism must first be found. There are two ways this can be accomplished:

Method 1: The focal plane can be most simply located by matching the optical path length that the binocular system takes with the focal length of the objective lens. However, care needs to be taken because the optical path length of light through the binocular system must be adjusted so that the path occurs *completely in air*. This means we must determine the optical path length of the rays through the prism and determine the equivalent distance along the optical axis light would take through air only.
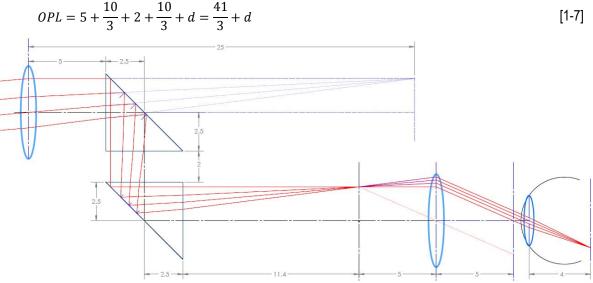
$$OPL_{air} = OPL_{prism}$$ [1-3]

$$d_{air} \cdot n_{air} = d_{prism} \cdot n_{prism}$$ [1-4]

$$5cm \cdot 1 = d_{prism} \cdot 1.5 \rightarrow d_{prism} = \frac{5}{1.5}cm = \frac{10}{3}cm$$ [1-5]

The optical path length from the objective lens to the focal plane through the binoculars, with an equivalent path through air:

$$OPL = L_1 + d_{prism} + H + d_{prism} + d$$ [1-6]

where L_1 is the distance from the objective lens to the first prism, d_{prism} is the equivalent distance light takes through the prism as if it were in air instead, and d is the distance from the lower prism to the focal plane. For a prism index of 1.5:

$$OPL = 5 + \frac{10}{3} + 2 + \frac{10}{3} + d = \frac{41}{3} + d$$ [1-7]

Now we can match the OPL found in equation [1-7] with the focal length of the objective lens:

$$OPL = f_1$$

$$\frac{41}{3}cm + d = 25cm \rightarrow d = \frac{34}{3}cm = 11.33cm$$ [1-8]

Method 2: Another way of viewing this system is to essentially unwrap it into a linear configuration and apply the matrix method:

$$r_i = \mathbf{T_{iP2}A_{P2}T_{P2-P1}A_{P1}T_{P1-L1}A_{L1}}r_o = \mathbf{A_{sys1}}r_o$$ [1-9]

Using Hecht's notation, the system matrix in equation [1-6] can be written as:

$$A_{sys1} = \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{w} & 0 \\ n & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ H & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{w} & 0 \\ n & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ L_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{f_1} \\ 0 & 1 \end{bmatrix}$$

$$A_{sys1} = \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & 0 \\ 1.5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & 0 \\ 1.5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{25} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{25} \\ \frac{41}{3} & \frac{34}{75} \end{bmatrix} \qquad \text{[1-10]}$$

$$A_{sys1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{25} \\ d + \frac{41}{3} & \frac{34}{75} - \frac{d}{25} \end{bmatrix}$$

Merging equations [1-6] and [1-7] two equations can be found:

$$\alpha_{out} = a_{11}\alpha_{in} + a_{12}y_{in} = \alpha_{in} - \frac{1}{25}y_{in} \qquad \text{[1-11]}$$

$$y_{out} = a_{21}\alpha_{in} + a_{22}y_{in} = \left(d + \frac{41}{3}\right)\alpha_{in} + \left(\frac{34}{75} - \frac{d}{25}\right)y_{in} \qquad \text{[1-12]}$$

The height of an image of an object located at infinity should not depend on the height of the initial rays. Therefore, a_{22} must be equal to zero:

$$a_{22} = \left(\frac{34}{75} - \frac{d}{25}\right) = 0 \rightarrow d = \frac{34}{3}cm = 11.33cm \qquad \text{[1-13]}$$

Notice, this is the same result as that produced by Method 1 (Equation [1-8]).

In order for the eyepiece to produce an image at infinity for comfortable viewing with the eye, it must be located one focal length away from the intermediate image produced by the binocular objective and prisms. Therefore, the distance from the right side of the lower prism to the eyepiece is:

$$d_{P2-Eyepiece} = d + f_2 = \frac{34}{3}cm + 5cm = \frac{49}{3}cm = 16.33cm \qquad \text{[1-14]}$$

c) Continuing with the matrix method, we can determine the locations of the two objects on the retina.

$$r_i = T_{Retina}A_{Eye}T_{Eye-EP}A_{EP}T_{EP-F1}A_{sys1}r_o = A_{sys}r_o \qquad \text{[1-15]}$$

As before, the system matrix can be rewritten as:

$$A_{sys} = \begin{bmatrix} 1 & 0 \\ f_e & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{f_e} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ L & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{f_2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ f_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{25} \\ d + \frac{41}{3} & \frac{34}{75} - \frac{d}{25} \end{bmatrix} \qquad \text{[1-16]}$$

It is noted that the distance from the eye-piece to the eye, L, is not yet known. Also from equation [1-8] it was found that $d = 34/3$ cm. Therefore:

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$$A_{sys} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{4} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ L & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{5} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{25} \\ 25 & 0 \end{bmatrix}$$

$$A_{sys} = \begin{bmatrix} 1 & -\frac{1}{4} \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ L & 1 \end{bmatrix} \begin{bmatrix} -5 & 0 \\ 30 & -1/5 \end{bmatrix} = \begin{bmatrix} -\frac{25}{2} + \frac{5L}{4} & \frac{1}{20} \\ -20 & 0 \end{bmatrix}$$ [1-17]

Merging equations [1-14] and [1-4], two equations can be found:

$$\alpha_{out} = \left(-\frac{25}{2} + \frac{5L}{4}\right)\alpha_{in} + \frac{1}{20}y_{in}$$ [1-18]

$$y_{out} = -20\alpha_{in}[cm]$$ [1-19]

From equation [1-19] the differential between the off-axis locations of objects with a difference in incident angles of 10⁻³ rad can be found:

$$\Delta y_{out} = 20\Delta\alpha_{in} = 20(0.001 rad) = 0.02 cm = 200\mu m$$ [1-20]

d) The entrance and exit pupils associated with the aperture can be found by imaging it through the preceding and following elements respectively with the imaging condition:

$$\frac{1}{s_0} + \frac{1}{s_i} = \frac{1}{f}$$ [1-21]

The entrance pupil can be found by using the focal length of the objective lens and an object distance of: $s_0 = f_1 + f_2 - L_2 = 25 + 5 - 3 = 27cm$

$$\frac{1}{27} + \frac{1}{s_{EnP}} = \frac{1}{25} \rightarrow s_{EnP} = \frac{675}{2}cm = 337.5cm$$ [1-22]

*Therefore, the entrance pupil that corresponds to the aperture is located **337.5cm** to the **left** of the objective lens.*

Likewise, the exit pupil associated with the aperture can be found by imaging it through the eyepiece lens. The object distance is now simply $s_o = 3cm$:

$$\frac{1}{3} + \frac{1}{s_{ExP}} = \frac{1}{5} \rightarrow s_{ExP} = -\frac{15}{2}cm = -7.5cm$$ [1-23]

*Therefore, the exit pupil corresponding to the aperture is located **7.5cm** to the **left** of the eyepiece lens.*

From the problem statement, the aperture discussed above is taken as the field stop. The field of view in this case is defined as twice the angle that the outer edge of the field stop makes with the center of the objective lens, α. Therefore the FOV is:

$$FOV = 2\alpha = 2\tan^{-1}\left(\frac{\frac{D}{2}}{f_o}\right) \approx 2\left(\frac{1.5cm}{25cm}\right) = \frac{3}{25} rad = 0.12\ rad$$ [1-24]

Note that the field stop is currently located at a poor position along the optical axis. To be effective in limiting the spatial extent of the image, it should ideally be located at a location where a real image is formed. In this configuration, it should be one focal length to the left of the eyepiece, i.e. 5cm instead of 3cm.

e) The optimal location of the eye is at the exit pupil of the binocular. In this situation, it is assumed that objective lens acts as the aperture stop. Therefore, the exit pupil is located when the ray passing through the center of the objective lens also passes through the center of the crystalline lens of the eye. The system matrix for this situation would be:

$$r_i = T_{Eye-EP}A_{EP}T_{EP-F1}A_{sys1}r_o = A_{sys}r_o$$ [1-25]

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As before, the system matrix can be rewritten as:

$$\mathbf{A_{sys}} = \begin{bmatrix} 1 & 0 \\ L & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{5} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{25} \\ 25 & 0 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 30 - 5L & -1/5 \end{bmatrix}$$ [1-26]
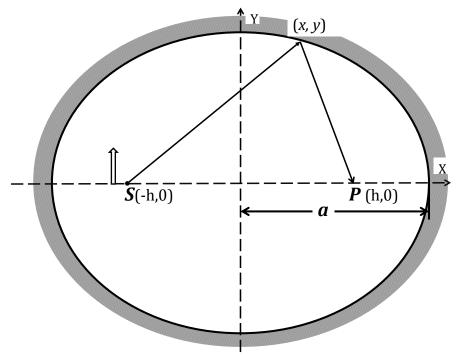
$$\alpha_{out} = -5\alpha_{in}$$ [1-27]

$$y_{out} = (30 - 5L)\alpha_{in} - \frac{y_{in}}{5}$$ [1-28]

Since we are interested in the ray that passes through both the center of the objective lens and the center of the eye, we take $y_{out} = y_{in} = 0$. Therefore, from equation [1-28], the distance from the eyepiece to the eye must be:

$$0 = (30 - 5L)\alpha_{in} \rightarrow L = 6cm$$ [1-28]

2. **Reflection from a concave cavity.** Figure 2 shows a reflective cavity made of concave mirrors, with light source **s**. The cavity is designed to reflect all rays leaving the source **s** to a point **p** *along* the long axis of the cavity.

a) **(10%)** Following Fermat's principles, the optical path length from **s** to **p** on any point (*x*, *y*) on the reflective cavity should be a constant. Please show such a constant is **2a,** the length of the long axis of the cavity.

b) **(15%)** Using Cartesian coordinates, please prove that any point (*x*, *y*) on the reflective cavity must satisfy the following relationship:

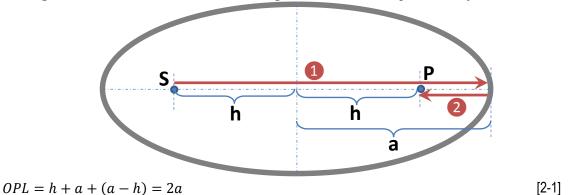$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Where $2h = |SP|$ is the distance from **s** to **p**, and $b = \sqrt{a^2 - h^2}$ is the length of the short axis of the cavity. Therefore, the cavity is an *ellipse*.

c) **(15%)** Assume the cavity is large enough so you can go in, and a small object is placed on
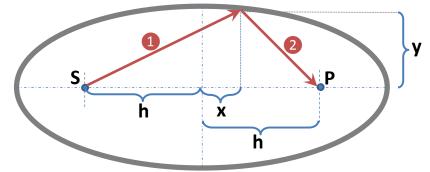
the left side of the source **s,** *as* shown by the arrow in Figure 1. Use ray tracing, please locate the first reflected image of the object. Is it real or virtual?

Solution:

a) Fermat's theorem tells us that all rays traveling from point S and converging at point P will travel the same distance. The simplest ray that can be considered is one that travels along the long-axis and makes a reflection at the right vertex of the elliptical cavity:

$$OPL = h + a + (a - h) = 2a \qquad\qquad [2\text{-}1]$$

b) Since all of the rays that leave point S converge to point P, the ray that travels from point S to P must both travel the same distance of 2a.

The total path travelled by the ray from point S to point P is the sum of arms 1 and 2. The optical path length is:

$$OPL = \sqrt{(h+x)^2 + y^2} + \sqrt{(h-x)^2 + y^2} = 2a \qquad\qquad [2\text{-}2]$$

Simplifying:

$$\sqrt{(h-x)^2 + y^2} = 2a - \sqrt{(h+x)^2 + y^2}$$

$$(h-x)^2 + y^2 = 4a^2 - 4a\sqrt{(h+x)^2 + y^2} + (h+x)^2 + y^2$$

$$-2hx = 4a^2 - 4a\sqrt{(h+x)^2 + y^2} + 2hx$$

$$\left(\frac{hx + a^2}{a}\right)^2 = (h+x)^2 + y^2 \qquad\qquad [2\text{-}3]$$

$$\frac{h^2 x^2}{a^2} + a^2 = h^2 + x^2 + y^2$$

$$y^2 = x^2 \left(\frac{h^2}{a^2} - 1\right) + (a^2 - h^2)$$
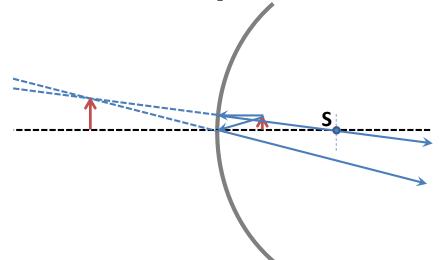
Given than the minor radius of the ellipse, b, is defined as $b^2 = a^2 - h^2$ equation [2-3] can be simplified to yield the classic equation of an ellipse.

$$y^2 = x^2 \left(\frac{h^2 - a^2}{a^2} \right) + b^2$$

$$y^2 = -x^2 \left(\frac{b^2}{a^2} \right) + b^2 = b^2 \left(1 - \frac{x^2}{a^2} \right)$$

[2-4]

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

[2-5]

c) Since the object is small, we can consider paraxial rays that leave from the object with slight angles of inclination and take advantage of the same special rays used to ray trace in a spherical mirror system. Two simple rays can be considered, a ray leaving horizontal to the long axis will pass through the focus of the cavity, and a ray striking the vertex of the cavity will have the same reflection and incident angles.

As seen in the figure above, the image formed from a single reflection off of the cavity wall is **virtual**. Notice that this is the same situation that happens for an object located between the focal point and the vertex point of a concave spherical mirror. Other images can be formed elsewhere by the cavity, only if paraxial rays are considered. Outside of the paraxial approximation, it becomes increasingly unlikely that all of the rays leaving a point on the object will converge to a focused image point.

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