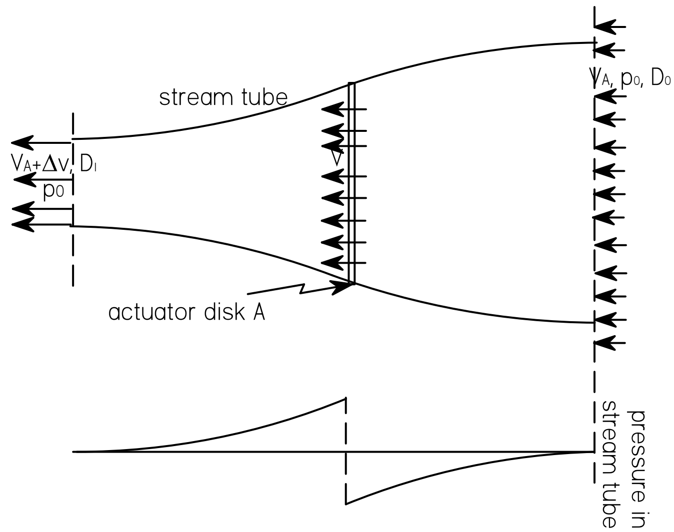
Actuator Disk

assume: propeller is a disk with diameter D and area A

frictionless

no rotation - upstream or downstream model propeller as thin "actuator disk" causing instantaneous increase in pressure



$A_1, D_1, V_A + \Delta v$ $\qquad$ D, A, V $\qquad$ D_0, A_0, V_A

$$\text{Thrust} = T = A \cdot \Delta p \qquad (10.1)$$

continuity ...

$$\rho \cdot V \cdot A = \text{constant}$$

$$\frac{\text{m\_dot}}{\rho} = V_A \cdot A_0 = V \cdot A = \left(V_A + \Delta v\right) \cdot A_1 \qquad V_A \cdot {D_0}^2 = V \cdot D^2 = \left(V_A + \Delta v\right) \cdot {D_1}^2 \qquad (10.2)$$

$${D_0}^2 = \frac{V}{V_A} \cdot D^2 \qquad {D_1}^2 = \frac{V}{V_A + \Delta v} \cdot D^2 \qquad (10.3)$$

$$D_0 := \sqrt{\frac{V}{V_A}} \cdot D \qquad D_1 := \sqrt{\frac{V}{V_A + \Delta v}} \cdot D \qquad (10.3a)$$

$$\text{\Delta\_in\_momentum} = \text{thrust\_on\_disk} = T = \text{m\_dot}_{out}\left(V_A + \Delta v\right) - \text{m\_dot}_{in} \cdot V_A$$

(force = mass flow * delta velocity)

$$T = \rho \cdot A_1 \cdot \left(V_A + \Delta v\right)^2 - \rho \cdot A_0 \cdot {V_A}^2$$

$$T := \rho \cdot \pi \cdot \frac{{D_1}^2}{4} \cdot \left(V_A + \Delta v\right)^2 - \rho \cdot \pi \cdot \frac{{D_0}^2}{4} \cdot {V_A}^2 \qquad (10.4)$$

$$T \text{ simplify } \rightarrow \frac{1}{4} \cdot \rho \cdot \pi \cdot V \cdot D^2 \cdot \Delta v$$

using (10.3a) above $\qquad$ (10.5)

now using Bernoulli equation

$$p + \frac{1}{2} \cdot \rho \cdot v^2 = \text{constant}$$

on both sides of the disk (a force is applied at the disk)

9/8/2006

ahead ... $p + \frac{1}{2}\cdot\rho\cdot V^2 = p_0 + \frac{1}{2}\cdot\rho\cdot V_A{}^2$ aft ... $p + \Delta p + \frac{1}{2}\cdot\rho\cdot V^2 = p_0 + \frac{1}{2}\cdot\rho\cdot\left(V_A + \Delta v\right)^2$

subtract ahead from aft ... $\Delta p = \frac{1}{2}\cdot\rho\cdot\left[\left(V_A + \Delta v\right)^2 - V_A{}^2\right] = \frac{1}{2}\rho\cdot\Delta v\cdot\left(2\cdot V_A + \Delta v\right)$ (10.6)

result ... $\frac{\left(V_A + \Delta v\right)^2 - V_A{}^2}{\Delta v}$ simplify $\rightarrow 2\cdot V_A + \Delta v$ $\Delta p := \frac{1}{2}\rho\cdot\Delta v\cdot\left(2\cdot V_A + \Delta v\right)$

now using (10.1) and equating to (10.5) $A := \frac{\pi}{4}\cdot D^2$

$$T := A\cdot\Delta p \rightarrow \frac{1}{8}\cdot\pi\cdot D^2\cdot\rho\cdot\Delta v\cdot\left(2\cdot V_A + \Delta v\right)$$

(10.5) $T := \frac{1}{4}\cdot\rho\cdot\pi\cdot V\cdot D^2\cdot\Delta v$ from which ... $V := V_A + \frac{\Delta v}{2}$

so .... $T \rightarrow \frac{1}{4}\cdot\pi\cdot D^2\cdot\rho\cdot\left(V_A + \frac{1}{2}\cdot\Delta v\right)\cdot\Delta v$ $T := \frac{\pi}{4}\cdot D^2\cdot\rho\cdot\left(V_A + \frac{\Delta v}{2}\right)\cdot\Delta v$ (10.9)

define a thrust loading coefficient ...

$C_T := \frac{T}{\frac{1}{2}\cdot\rho\cdot\frac{\pi}{4}\cdot D^2\cdot V_A{}^2}$ substitute (10.9) $C_T \rightarrow 2\cdot\left(V_A + \frac{1}{2}\cdot\Delta v\right)\cdot\frac{\Delta v}{V_A{}^2}$ a quadratic in Δv (10.10)

Given

$C_T = 2\cdot\left(V_A + \frac{1}{2}\cdot\Delta v\right)\cdot\frac{\Delta v}{V_A{}^2}$ $\frac{\text{Find}(\Delta v)}{V_A} \rightarrow \left[(-1) + \left(1 + C_T\right)^{\frac{1}{2}} \quad (-1) - \left(1 + C_T\right)^{\frac{1}{2}}\right]$

taking only positive root $\frac{\Delta v}{V_A} = (-1) + \left(1 + C_T\right)^{\frac{1}{2}}$

$$\eta_I = \text{ideal\_efficiency} = \frac{\text{useful\_work\_from\_disk}}{\text{work\_done\_on\_fluid\_by\_thrust\_per\_unit\_time}} = \frac{P_T}{P_{added}} = \frac{T\cdot V_A}{T\cdot V}$$

$\eta_I := \frac{T\cdot V_A}{T\cdot V} \rightarrow \frac{1}{V_A + \frac{1}{2}\cdot\Delta v}\cdot V_A$ uses relationship for V above (10.9) (10.11)

with ... $\Delta v := V_A\cdot\left[(-1) + \left(1 + C_T\right)^{\frac{1}{2}}\right]$ $\eta_I := \frac{1}{1 + \frac{1}{2}\cdot\frac{\Delta v}{V_A}}$ simplify $\rightarrow \frac{2}{1 + \left(1 + C_T\right)^{\frac{1}{2}}}$ (10.12)

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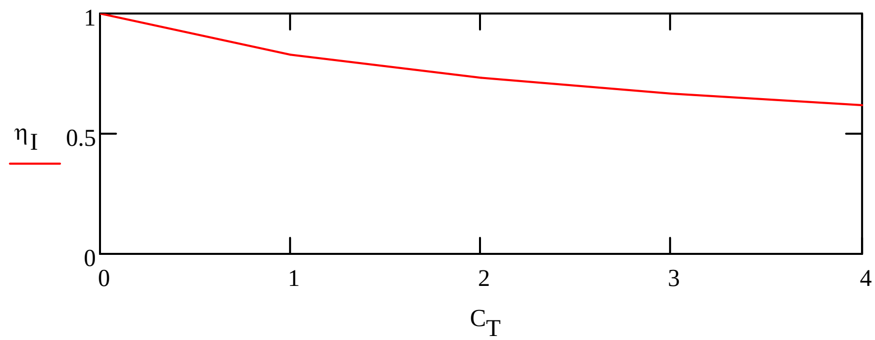
create plot with loading

$$C_T := \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \qquad i := 0..4 \qquad \eta_{I_i} := \frac{2}{1 + \sqrt{1 + C_{T_i}}} \qquad \eta_I = \begin{pmatrix} 1 \\ 0.828 \\ 0.732 \\ 0.667 \\ 0.618 \end{pmatrix}$$

as shown in PNA



Observations: 1). Propeller at high load coefficient C_T less efficient

2). $$\eta_I := \frac{1}{1 + \frac{1}{2} \cdot \frac{\Delta v}{V_A}}$$ => efficiency maximum when Δv small

3) for given thrust T, $$T \rightarrow \frac{1}{4} \cdot \pi \cdot D^2 \cdot \rho \cdot \left(V_A + \frac{1}{2} \cdot \Delta v\right) \cdot \Delta v$$ Δv small => D large => propeller diameter large

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