15P2003

13.024 - Numerical Methods in Incompressible Fluid Mechanics

Lecture Notes – Version 3

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INCOMPRESSIBLE FLUID MECHANICS BACKGROUND

$$\vec{V} = \vec{i}u + \vec{j}v + \vec{k}w$$

Conservation of Mass, Continuity Equation

$$\operatorname{div} \vec{V} = \nabla \cdot \vec{V} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Newtonian Dynamics, Navier-Stokes Equations

$$\frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \vec{V}$$

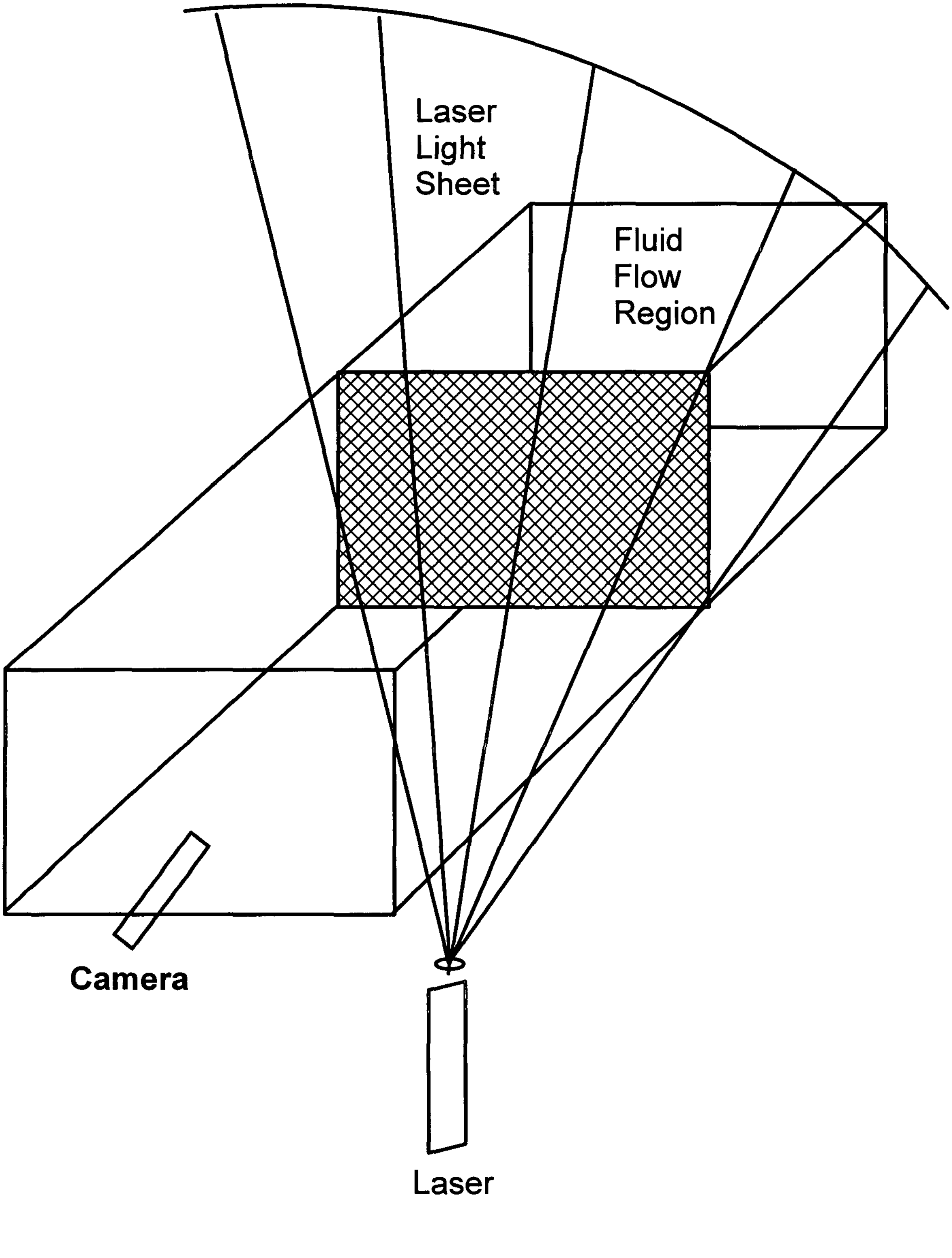
P is the dynamic pressure. The total pressure, P_T, is the sum of the dynamic pressure and the hydrostatic pressure, $-\rho\, g\, z$, where z is positive upwards. $P_T = P - \rho\, g\, z$.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

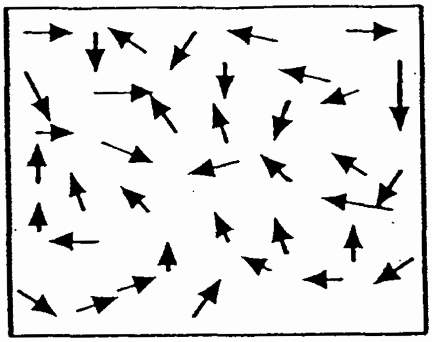
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

PARTICLE IMAGE VELOCIMETRY



PIV Example



u and v can be measured, so $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ are known.

$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}$$

If PIV is done on multiple planes inside a fluid domain, then $\frac{\partial w}{\partial z}$ is known over the whole domain. At a rigid boundary, $w = 0$ and, in principle, w can be found anywhere by:

$$w = \int_{boundary}^{z} \frac{\partial w}{\partial z} dz$$

Example

In a domain bounded by $0 \leq z \leq 4$, $u = (3e^z - ze^z - 3)\sin x$ and $v = 0$ over a range of x and in $0 \leq z \leq 2$. In this sub-domain,

$$\frac{\partial u}{\partial x} = (3e^z - ze^z - 3)\cos(x)$$

$$\frac{\partial w}{\partial z} = -(3e^z - ze^z - 3)\cos(x)$$

$$w = \int_0^z \frac{\partial w}{\partial z} dz = -\left[3e^z - 3 - ze^z + e^z - 1 - 3z\right]\cos(x)$$

$$w = \int_0^z \frac{\partial w}{\partial z} dz = -\left[4e^z - 4 - ze^z - 3z\right]\cos(x)$$

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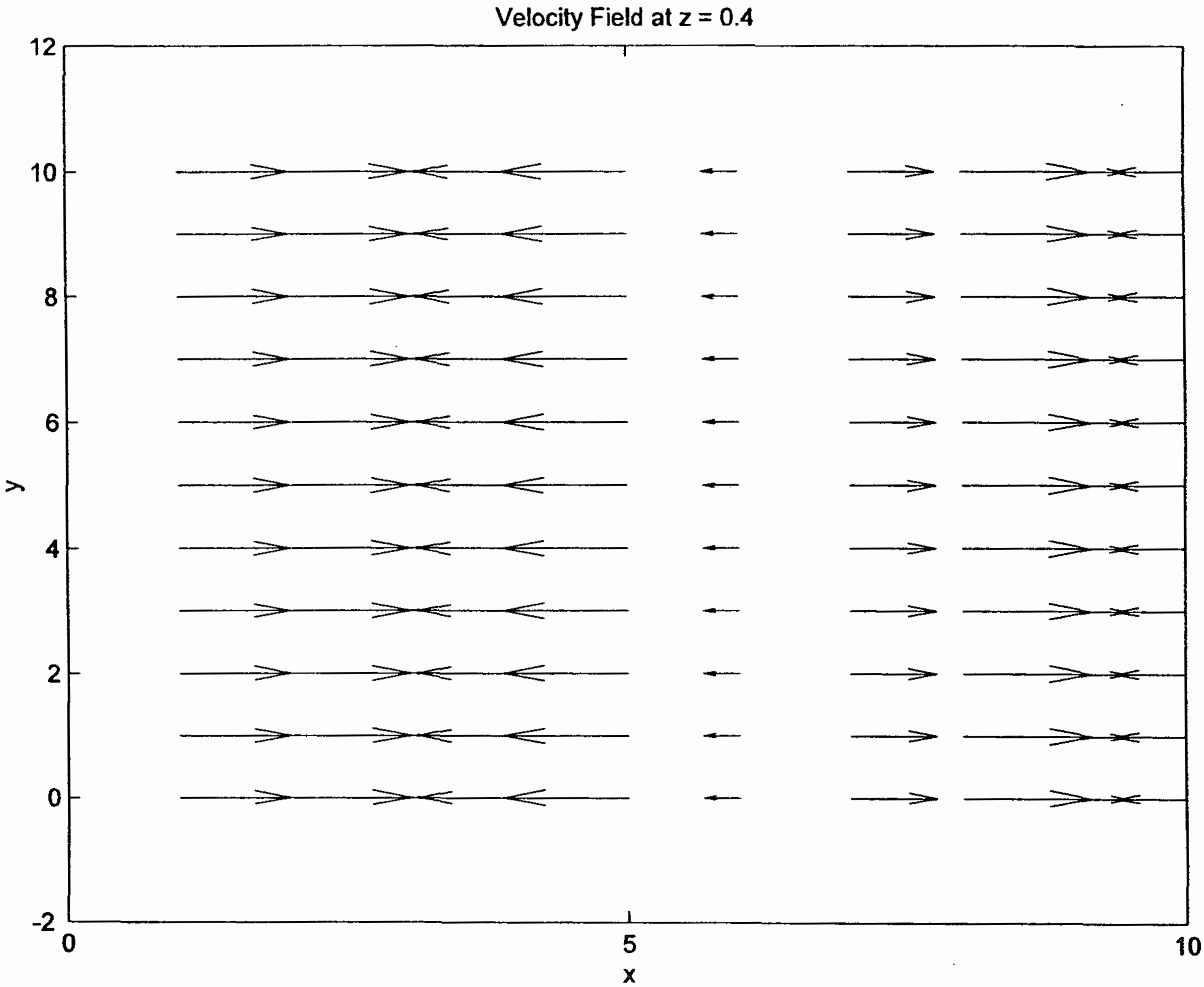

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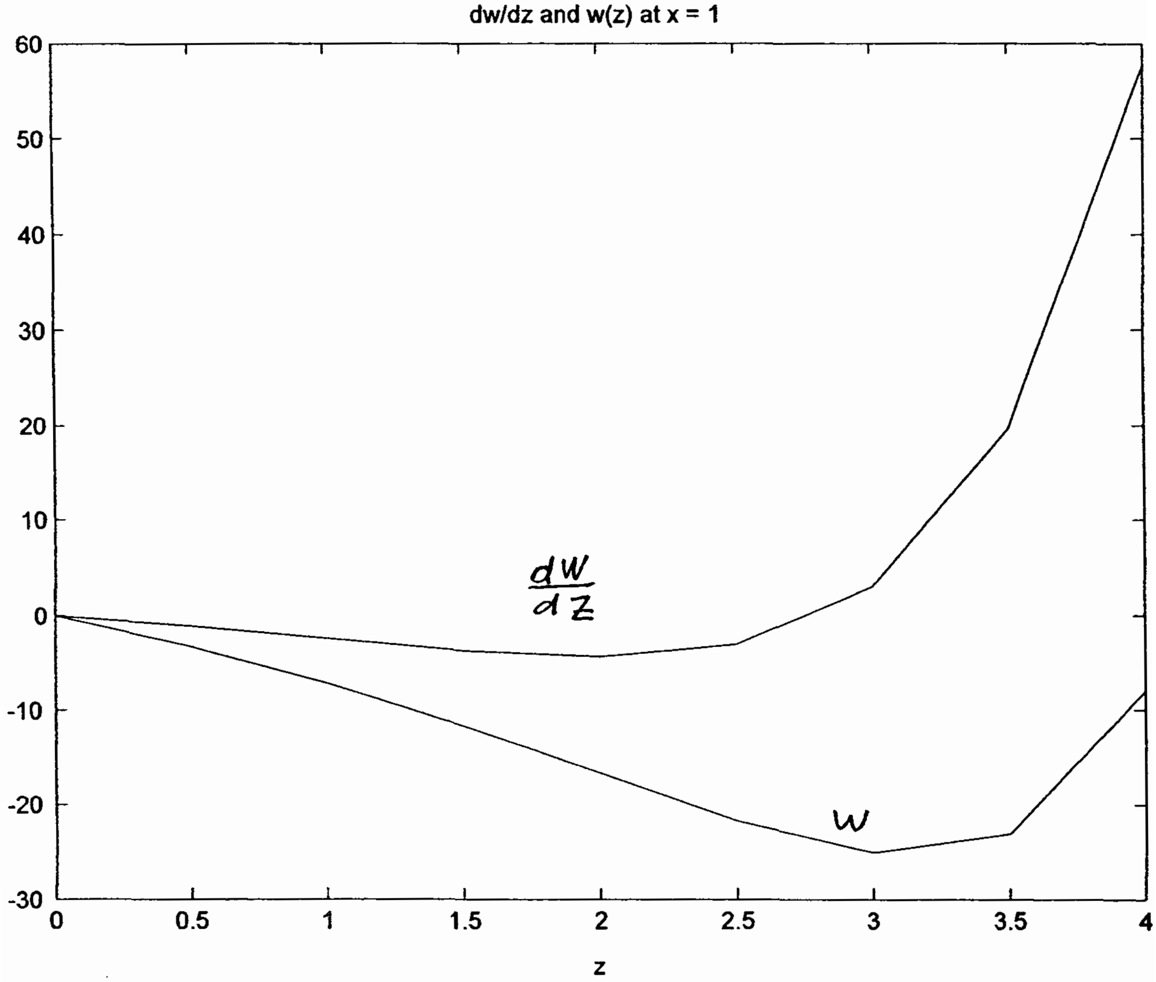
dw/dz and w(z) at x = 1
60
50
40
30
20
10
0
-10
-20
-30
0
0.5
1
1.5
2
2.5
3
3.5
4
z
dW/dZ
W

A More Interesting PIV Example

Consider the following flow for $z > 0$ in a range of x and y. Of course, in an experiment you would not know the mathematical formulation. Rather you would just measure u and v over a set of (x,y,z) points.

$$u(x, y, z) = \left(3e^{0.1z} - 3\cos\, z\right)\sin y \cos x$$

$$v(x, y, z) = \left(3e^{0.1z} - 3\cos\, z\right)\cos y$$

The x and y derivatives of the velocities can be computed numerically from the measurements. If the experiment were done well, they would have values according to the following formulae:

$$\frac{\partial u}{\partial x} = -\left(3e^{0.1z} - 3\cos\, z\right)\sin y \sin x$$

$$\frac{\partial v}{\partial y} = -\left(3e^{0.1z} - 3\cos\, z\right)\sin y$$

Then, the continuity equation is: $\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}$

The experimentally determined values would have the values given by:

$$\frac{\partial w}{\partial z} = \left(3e^{0.1z} - 3\cos\, z\right)\, \sin\, y(\sin\, x + 1)$$

Integrating $\frac{\partial w}{\partial z}$ from 0 to z at a prescribed value of (x,y) would give w(z) there. The values obtained would obey:

$$w = \left(30\, e^{0.1z} - 30 - 3\, \sin\, z\right)\, \sin\, y\, (\sin\, x + 1)$$

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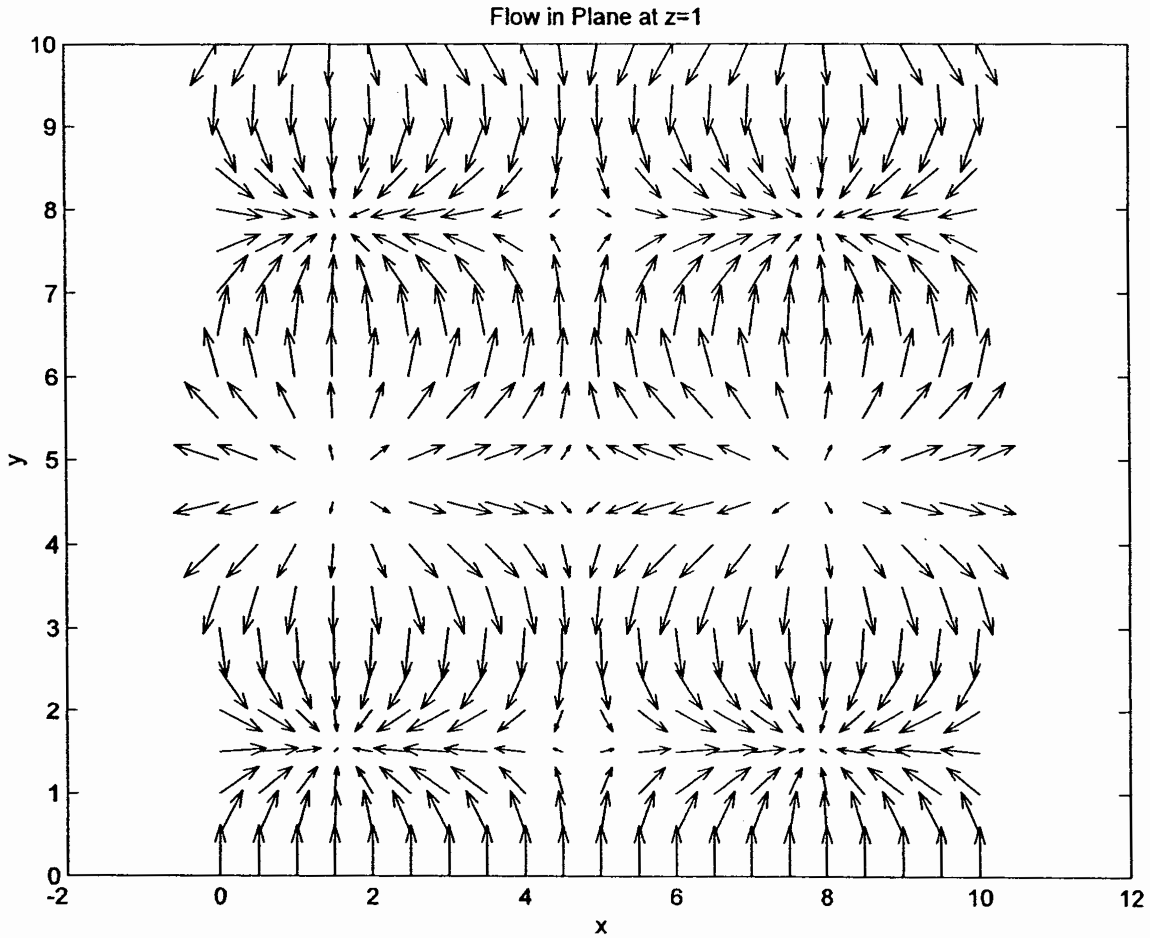

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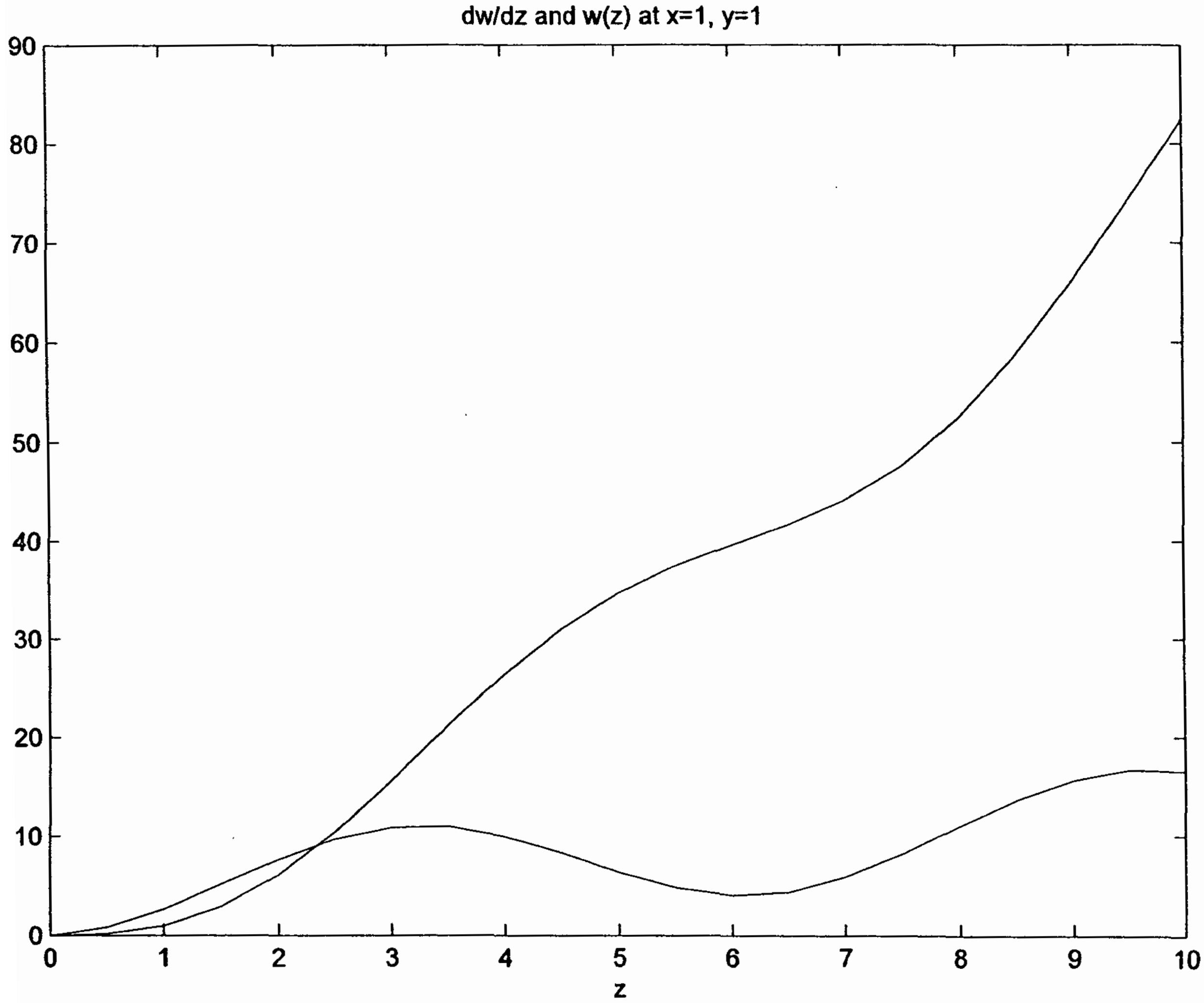

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AVERAGED NAVIER-STOKES EQUATIONS

$$\vec{V} = \overline{\vec{V}} + \vec{v'}$$

$$\frac{\partial\left(\overline{\vec{V}} + \vec{v'}\right)}{\partial t} + \left[\left(\overline{\vec{V}} + \vec{v'}\right) \cdot \nabla\right]\left(\overline{\vec{V}} + \vec{v'}\right) = -\frac{1}{\rho}\nabla\left(\overline{P} + p'\right) + \nu\nabla^2\left(\overline{\vec{V}} + \vec{v'}\right)$$

Take Average of above equation:

$$\frac{\partial\overline{\vec{V}}}{\partial t} + \overline{\left[\left(\overline{\vec{V}} + \vec{v'}\right) \cdot \nabla\right]\left(\overline{\vec{V}} + \vec{v'}\right)} = -\frac{1}{\rho}\nabla\overline{P} + \nu\nabla^2\overline{\vec{V}}$$

$$\left[\left(\overline{\vec{V}} + \vec{v'}\right) \cdot \nabla\right]\left(\overline{\vec{V}} + \vec{v'}\right) = (\overline{\vec{V}} \cdot \nabla)\overline{\vec{V}} + (\overline{\vec{V}} \cdot \nabla)\vec{v'} + (\vec{v'} \cdot \nabla)\overline{\vec{V}} + (\vec{v'} \cdot \nabla)\vec{v'}$$

$$\overline{\left[\left(\overline{\vec{V}} + \vec{v'}\right) \cdot \nabla\right]\left(\overline{\vec{V}} + \vec{v'}\right)} = (\overline{\vec{V}} \cdot \nabla)\overline{\vec{V}} + \overline{(\vec{v'} \cdot \nabla)\vec{v'}}$$

Thus, the Reynolds-Averaged Equation is:

$$\frac{\partial\overline{\vec{V}}}{\partial t} + (\overline{\vec{V}} \cdot \nabla)\overline{\vec{V}} + \overline{(\vec{v'} \cdot \nabla)\vec{v'}} = -\frac{1}{\rho}\nabla\overline{P} + \nu\nabla^2\overline{\vec{V}}$$

The "Reynolds Stress Term" is:

$$\begin{aligned}\overline{(\vec{v'} \cdot \nabla)\vec{v'}} \;=\; & \vec{i}\,\overline{\left(u'\frac{\partial u'}{\partial x} + v'\frac{\partial u'}{\partial y} + w'\frac{\partial u'}{\partial z}\right)} + \\ & \vec{j}\,\overline{\left(u'\frac{\partial v'}{\partial x} + v'\frac{\partial v'}{\partial y} + w'\frac{\partial v'}{\partial z}\right)} + \\ & \vec{k}\,\overline{\left(u'\frac{\partial w'}{\partial x} + v'\frac{\partial w'}{\partial y} + w'\frac{\partial w'}{\partial z}\right)}\end{aligned}$$

THE PRESSURE EQUATION FOR AN INCOMPRESSIBLE FLUID

Start with the Navier-Stokes Equation

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \vec{V}$$

Take its divergence.

Because $\nabla \cdot \vec{V} = 0$, the only non-zero terms are:

$$\mathrm{div}\,(\vec{V} \cdot \nabla)\vec{V} = -\frac{1}{\rho}\nabla^2 P$$

Working out the details of the LHS and interchanging the LHS and the RHS results in:

$$-\frac{1}{\rho}\nabla^2 P = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + 2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + 2\frac{\partial w}{\partial x}\frac{\partial u}{\partial z} + 2\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}$$

$$\nabla^2 P = -\rho\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + 2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + 2\frac{\partial w}{\partial x}\frac{\partial u}{\partial z} + 2\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}\right\}$$

The pressure, P, satisfies Poisson's Equation driven by products of the spatial derivatives of the velocity. This is different than the common Bernoulli Equations because here the flow can be unsteady and rotational.

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The Vorticity equation

$$\text{vorticity} = \varpi \equiv \text{curl}\,\vec{V} \equiv \nabla \times \vec{V}$$

Start with the Navier Stokes equation:

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \vec{V}$$

Take the curl of this equation, term by term:

$$\frac{\partial \varpi}{\partial t} + (\vec{V} \cdot \nabla)\varpi + (\varpi \cdot \nabla)\vec{V} = \nu \nabla^2 \varpi$$

$$\frac{D\varpi}{\partial t} = -(\varpi \cdot \nabla)\vec{V} + \nu \nabla^2 \varpi$$

The first term on the right hand side is the rotation and stretching of the vorticity by the non-uniform velocity field.

Inviscid Fluid Mechanics, Euler's Equation

Set the viscosity, μ and the kinematic viscosity, ν to zero. Apply these "settings" to the Navier Stokes Equation.

$$\frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} = -\frac{1}{\rho}\nabla P$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial y}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z}$$

Bernoulli Theorems for Inviscid Flow

Theorem 1 - Irrotational Flow

Vector Identity

$$(\vec{V}\cdot\nabla)\vec{V} = \frac{1}{2}\nabla(|\vec{V}|^2) - \vec{V}\times(\nabla\times\vec{V}) = \frac{1}{2}\nabla(|\vec{V}|^2) - \vec{V}\times\varpi$$

$$\text{Let}\quad H = \frac{1}{2}(|\vec{V}|)^2 + \frac{P}{\rho}$$

For Irrotational Flow, $\vec{V} = \nabla\phi$ and $\nabla\times\vec{V} = 0$

$$\frac{\partial}{\partial t}\nabla\phi + \nabla H \equiv \nabla(\frac{\partial\phi}{\partial t} + H) = 0$$

$$\frac{\partial\phi}{\partial t} + H = f(t)$$

The function $f(t)$ can be absorbed into ϕ by letting $\phi = \phi' + \int_{t_o}^{t} f(t)dt$ and $\vec{V} = \nabla\phi'$.

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi'}{\partial t} + f(t) \qquad \text{so,} \qquad \frac{\partial\phi'}{\partial t} + H = 0$$

Finally rename ϕ' as ϕ.

$$\frac{\partial\phi}{\partial t} + H = 0$$

Remember that the total pressure, $P_T = P - \rho g z$.

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\vec{V})^2 + \frac{P_T}{\rho} + gz = 0$$

Theorem 2 - Steady Flow

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} = -\frac{1}{\rho}\nabla P$$

$$\frac{\partial}{\partial t}\nabla\phi + \frac{1}{2}\nabla\left(|\vec{V}|^2\right) - \vec{V} \times \varpi + \frac{1}{\rho}\nabla P = 0$$

$$\frac{\partial}{\partial t}\nabla\phi + \nabla H - \vec{V} \times \varpi = 0$$

Thus, for steady flow:

$$\vec{V} \times \varpi = \nabla H$$

Streamlines and vortex lines are perpendicular to ∇H.
Along either a streamline or a vortex line, H is a constant. So on any one of these lines,

$$H = \frac{1}{2}(\vec{V})^2 + \frac{P}{\rho} = \frac{1}{2}(\vec{V})^2 + \frac{P_T}{\rho} + gz = \text{constant}$$

If the flow is both steady and irrotational, H is the same everywhere because $\nabla H = 0$.

Vorticity Dynamics and Kelvin's Circulation Theorem

$$\text{Circulation } = \Gamma = \oint \vec{V} \cdot d\mathbf{r} = \int_{\mathbf{S}} \varpi \cdot \mathbf{ds}$$

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\vec{V}$$

Identity for an incompressible fluid: $\nabla^2\vec{V} = -\nabla \times \varpi$

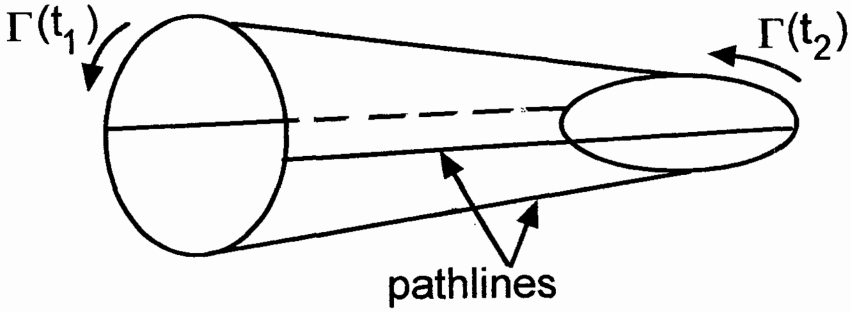
$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\nabla P - \nu\nabla \times \varpi$$

Following a closed material curve,

$$\begin{aligned} \frac{d\Gamma}{dt} &= \oint \frac{D\vec{V}}{Dt} \cdot d\mathbf{r} = -\oint \frac{\mathbf{1}}{\rho}\nabla\mathbf{P} \cdot \mathbf{dr} - \nu \oint (\nabla \times \varpi) \cdot \mathbf{dr} \\ &= -\nu \oint (\nabla \times \varpi) \cdot d\mathbf{r} \end{aligned}$$

Kelvin's Circulation Theorem

In an inviscid fluid, $\dfrac{d\Gamma}{dt} = 0$



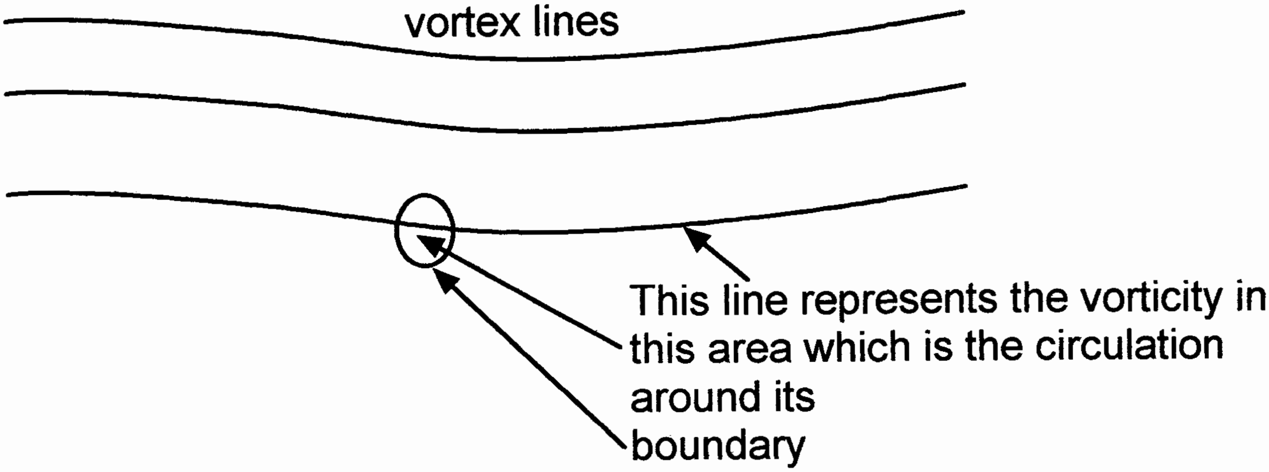
Corollary: In an inviscid fluid with no circulation (such as starting from rest) the circulation remains zero.

Practical Implication

In a high Reynolds number streaming flow, fluid which has not passed close to a boundary or a free surface has negligible vorticity.

Therefore, in high Reynolds number streaming flows, vorticity is limited to boundary layers, separated zones, and wakes.

Concept of Vortex Lines

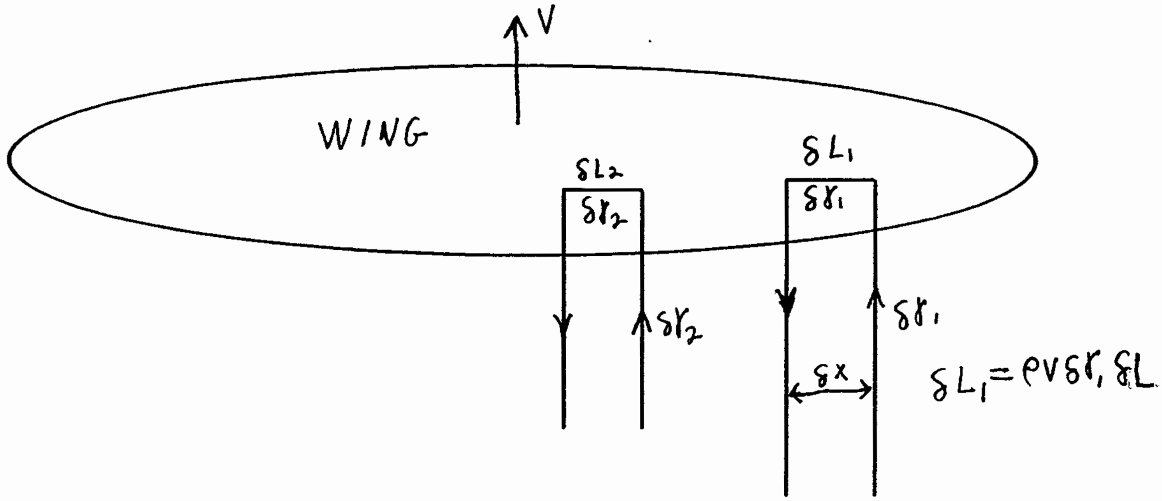


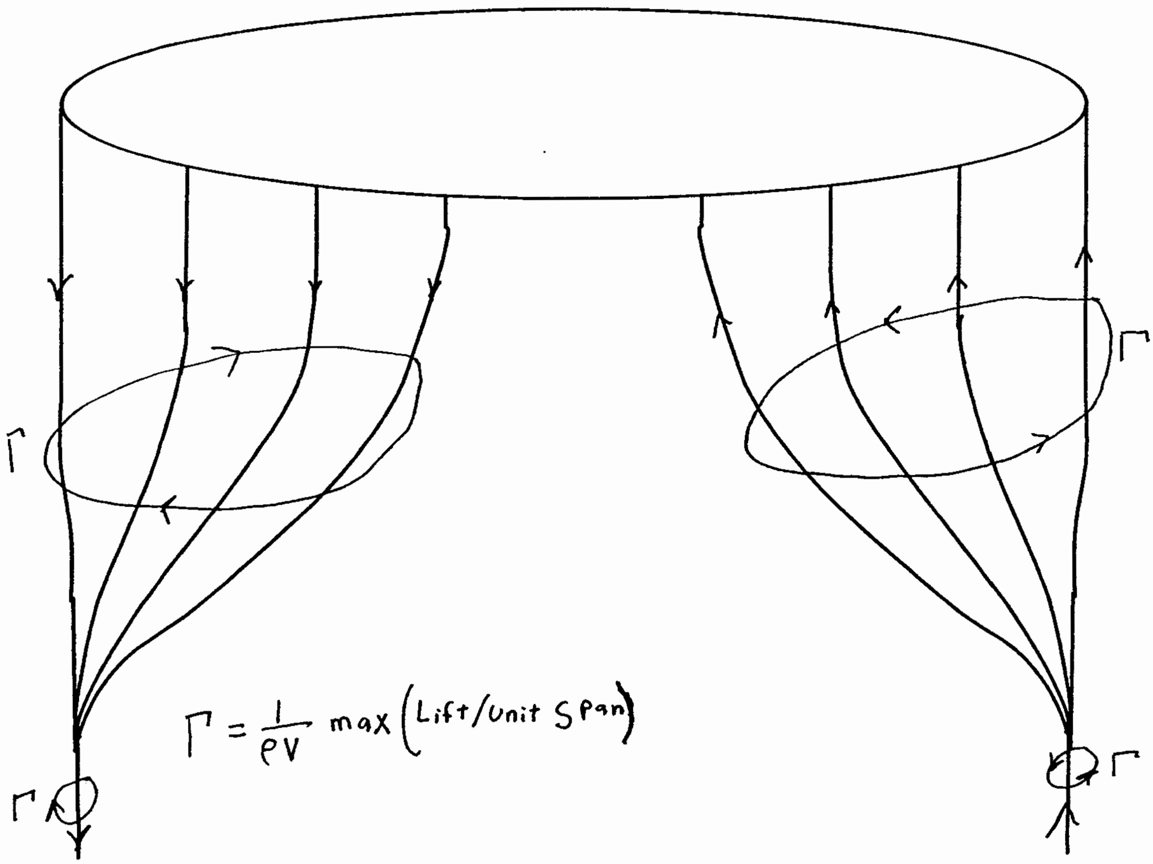
Each line represents circulation around an area which is the same as the vorticity inside the area.

$$\varpi = \mathrm{curl}\,\vec{V} = \nabla \times \vec{V}$$

$$\text{Therefore: } \mathrm{div}\,\varpi = \nabla \cdot \varpi = 0$$

The vortex field is solenoidal. Vortex lines are continuous. They can have curves and turns,but they cannot have ends in the fluid.


V
WING
δL₂
δγ₂
δL₁
δγ₁
δγ₂
δγ₁
δx
δL₁ = ρVδγ₁ δL


Γ
Γ
Γ = 1/ρV max (Lift/unit span)
Γ
Γ

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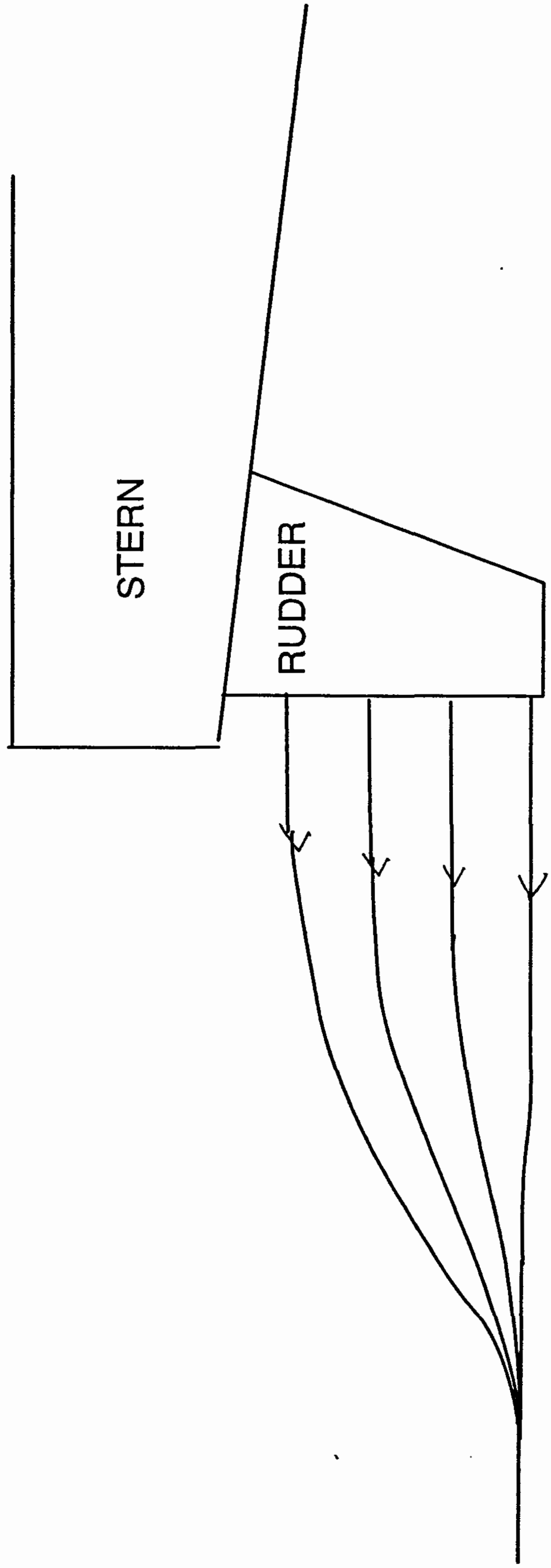

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Potential Flows and Mostly Potential Flows

For an irrotational fluid $\nabla \times \vec{V} = 0$

This means that there exists a *velocity potential*, ϕ, such that,

$$\vec{V} = \nabla \phi$$

For an incompressible fluid $\nabla \cdot \vec{V} = 0$

Thus, $\nabla \cdot (\nabla \phi) = 0$

$$\nabla^2 \phi = 0$$

For a completely potential flow, the velocity potential satisfies Laplace's equation.

For an incompressible flow that is "nearly" irrotational except in boundary layers and wakes, the flow outside these boundary layers and wakes is approximately described by a velocity potential that satisfies Laplace's equation.

Green Functions, Green's Theorem and Boundary Integral Equations

The following development is for three-dimensional flows. The development is similar for two-dimensional flows except that two dimensional source functions are involved and the dimensionality of some integrals and associated constants are different.

Green's Theorem

If ϕ and ψ both satisfy Laplace's equation ($\nabla^2\phi = 0, \;\; \nabla^2\psi = 0$), then:

$$\int\int_S \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right] dS = \int\int\int_V \left[\phi\nabla^2\psi - \psi\nabla^2\phi\right] dV = 0$$

Green Functions A three-dimensional (ξ, η, ζ) space is considered with a "sink" at location (x, y, z). The "sink" has the velocity potential ψ_s,

$$\psi_s = \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} = \frac{1}{r}$$

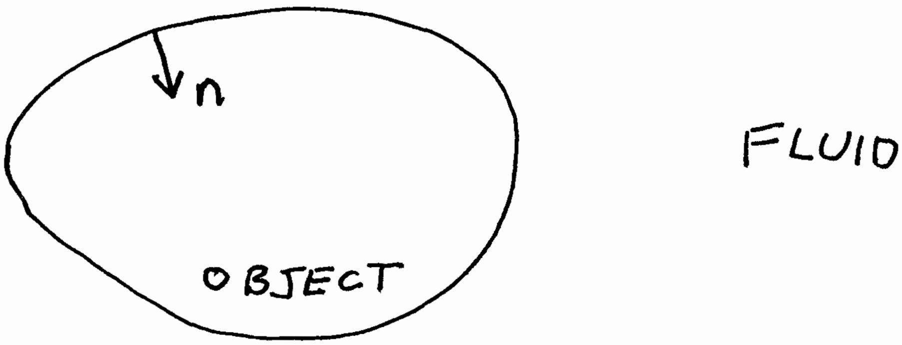
$$\text{where: } r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$$

Importantly, $\nabla^2\psi_s = 0$ both for the differentiations done in (ξ, η, ζ) space as well as for the differentiations done in (x, y, z) space. The following development can be formulated either way and we will choose to differentiate over (ξ, η, ζ).

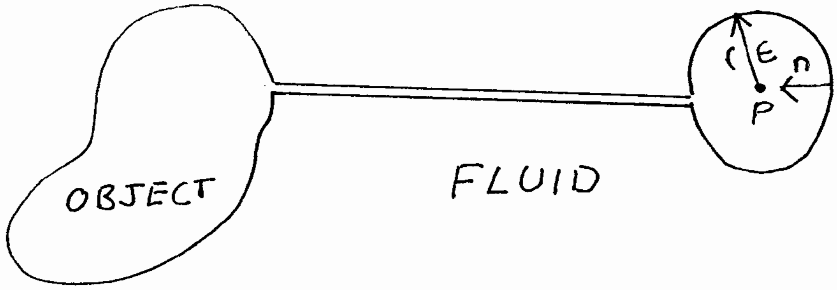
A Green Function, G is:

$$G = \psi_s(x, y, z, \xi, \eta, \zeta) + \psi_r(x, y, z, \xi, \eta, \zeta)$$

where, $\nabla^2\phi_r = 0$ in the fluid domain, and (x, y, z) is called the point P.



If P is outside the fluid domain, the bracketed terms on the right hand side in Green's Theorem are zero. However, if P is inside the fluid domain, $\nabla^2\psi \neq 0$ at P. Then, if P is enclosed by a small sphere of radius ϵ, which is excluded from the fluid domain, Green's Theorem applies in the modified fluid domain. However, now the integrals include an integral about the small sphere.

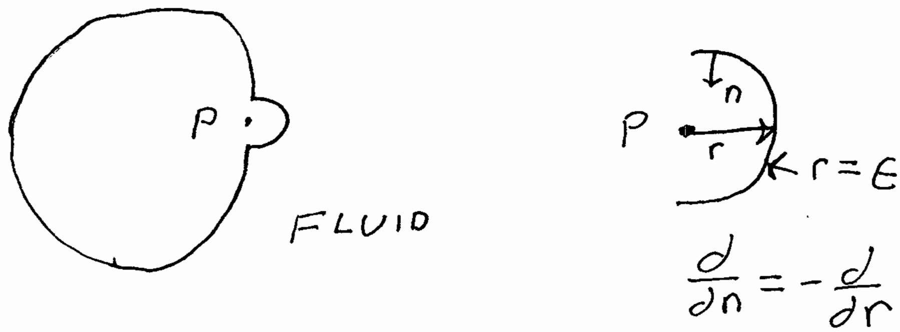


$$\int\int_{S+sphere}\left[\phi\frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n}\right] ds_{\xi,\eta,\zeta} = 0$$

$$\int\int_{sphere}\phi\frac{\partial G}{\partial n}ds = -\phi[P(x,y,z)]\frac{-1}{\epsilon^2}4\pi\epsilon = 4\pi\phi[P(x,y,x)]$$

$$\int\int_{S}\left[\phi\frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n}\right] ds_{\xi,\eta,\zeta} = -4\pi\phi[P(x,y,z)]$$

If $P(x,y,z)$ is on the boundary, the integral is not defined. However, if we replace the real boundary by one which has an infinitesmal hemisphere syrrounding P, the Green Function integral is zero because the functions have no singularities in the revised fluid domain.



$$\int\int_{S+hemisphere}\left[\phi\frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n}\right] ds_{\xi,\eta,\zeta} = 0$$

$$\int\int_{hemisphere}\phi\frac{\partial G}{\partial n}ds = -\phi[P(x,y,z)]\frac{-1}{\epsilon^2}2\pi\epsilon = 2\pi\phi[P(x,y,x)]$$

$$\int\int_{S}\left[\phi\frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n}\right] ds_{\xi,\eta,\zeta} = -4\pi\phi[P(x,y,z)]$$

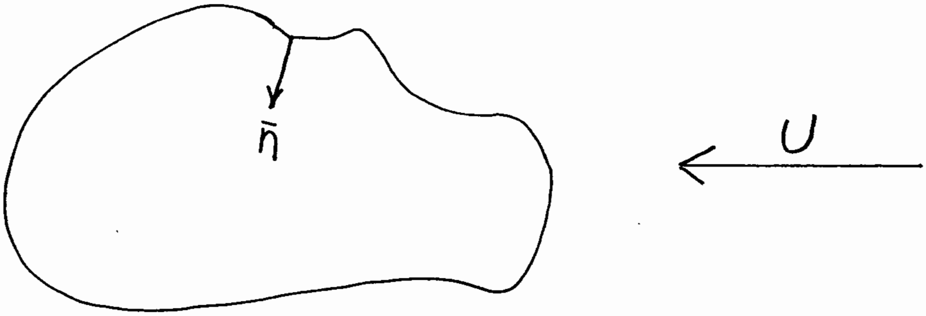
Putting the preceding parts together, if a closed fluid domain of surface S is considered with $\vec{n}$ being the outward normal vector (out of the fluid) and ψ is taken as G with proper exclusion of the singular point of G when (x, y, z) is inside the domain or on its boundary,

$$\int\int_S \left[\phi \frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n}\right] dS = \begin{cases} 0 & (x,y,z) \text{ outside } S \\ -2\pi\phi(x,y,x) & (x,y,z) \text{ on } S \\ -4\pi\phi(x,y,x) & (x,y,z) \text{ inside } S \end{cases}$$

The integral is over the closed area in (ξ, η, ζ). When the singular point is on the surface, an infinitesimally small circle surrounding the singular point is excluded from the integral.

Example of method of solution

Generate integral equation on surface of an object in a uniform flow.



Suppose uniform flow onto an object is known

$\frac{\partial \phi}{\partial n}$ is known.

$\Phi = -Ux + \phi$

Boundary condition: $\frac{\partial \Phi}{\partial n} = 0$,

$-U\,\hat{i} \cdot \vec{n} + \frac{\partial \phi}{\partial n} = 0, \quad \frac{\partial \phi}{\partial n} = U\,\hat{i} \cdot \vec{n}$

$$\int\int_S \left[\phi \frac{\partial G}{\partial n} - G\,U\,\hat{i} \cdot \vec{n}\right] dS = -2\pi\phi$$

$$\int\int_S \phi \frac{\partial G}{\partial n} dS + 2\pi\phi = \int\int_S G\,U\,\hat{i} \cdot \vec{n} dS$$

Right hand side is known in integral equation for ϕ on boundary.

Solve for values of ϕ on boundary (panel methods).

Then ϕ and $\frac{\partial \phi}{\partial n}$ are known on boundary.

Green's Theorem then gives ϕ in all space.

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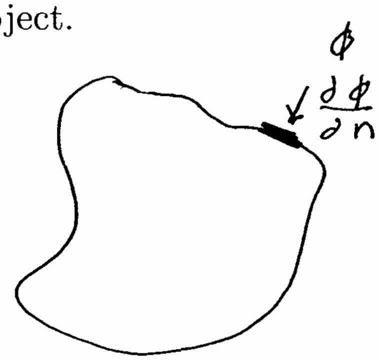
Interpretation of Boundary Integral Equation in terms of source and Dipole Layers

$$\int\int_S \left[\frac{1}{4\pi}\frac{\partial\phi}{\partial n}G - \frac{1}{4\pi}\phi\,\frac{\partial G}{\partial n}\right] dS = \begin{cases} 0 & (x,y,z) \text{ outside } S \\ \phi(x,y,x) & (x,y,z) \text{ inside } S \end{cases}$$

"Inside S" means inside the fluid and outside "S'' means outside the fluid. G is the potential of a "unit sink" and -$\partial g/\partial n$ is the potential of a unit dipole.

The Green's Theorem integrals are integrals of sink distributions per unit area of $(1/4\pi)\partial\phi/\partial n$ over the object and of dipole distributions of strength per unit area of $\phi/4\pi$ over the object.

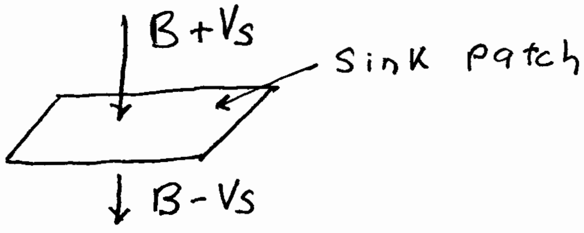
The sinks



Consider the effect of a unit sink.

$$\phi = \frac{1}{r}, \quad V_n = \frac{1}{r^2}, \quad \text{influx} = \frac{1}{r^2}4\pi r^2 = 4\pi$$

Now, look at the small patch of area A on the surface:



B is the effect of the integrals on the remainder of the object. Call the sink strength per unit area σ. Total sink Strength on the patch is σA.

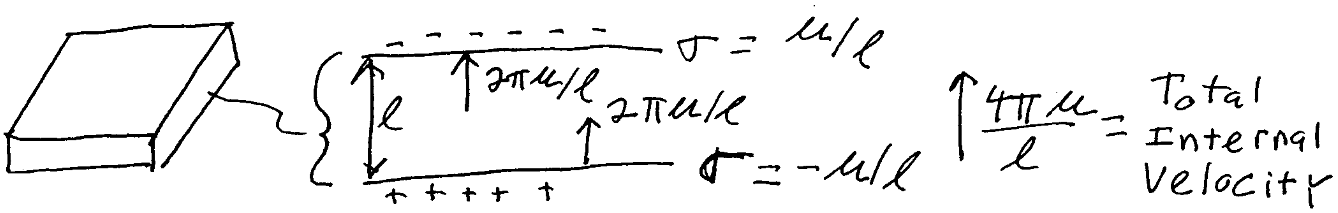
Net influx, based on the velocities is $2AV_s$.

$$2AV_s = 4\pi\sigma A \qquad\qquad 2V_s = 4\pi\sigma$$

$2V_s$ is the jump in normal velocity. This must equal the normal velocity, $\partial\phi/\partial n$ in the fluid at the boundary since $V = 0$ inside the object.

$$\frac{\partial\phi}{\partial n} = 4\pi\sigma \qquad\qquad \sigma = \frac{1}{4\pi}\frac{\partial\phi}{\partial n}$$

Now consider an infinitesmal dipole patch of strength μ



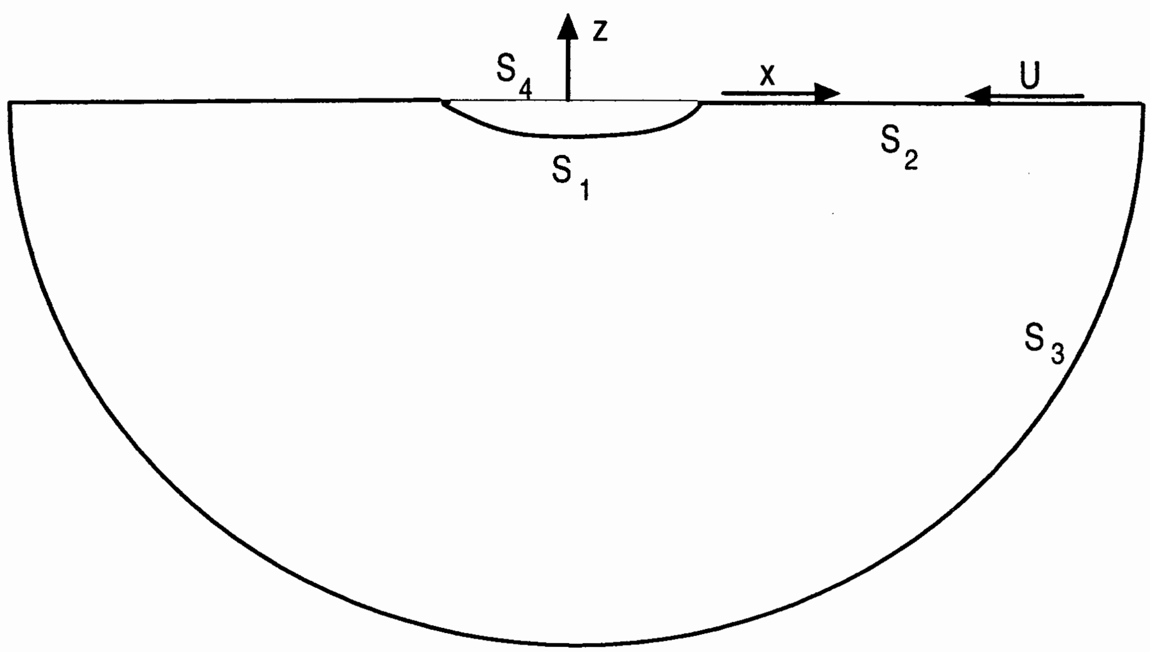
Inside the infinistemally thin dipole layer of thickness ℓ,

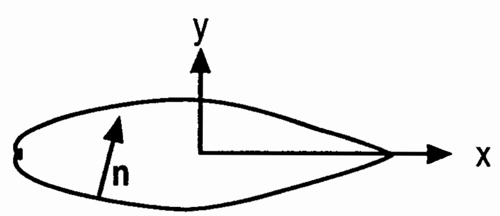
$$v = \frac{4\pi\mu}{\ell} \qquad\qquad \phi_{\text{in fluid}} - \phi_{\text{inside object}} = 4\pi\mu$$

since: $\phi_{\text{inside object}} = 0$,

$$\phi_{\text{in fluid}} = 4\pi\mu \qquad\qquad \mu = \frac{1}{4\pi}\phi_{\text{in fluid}}$$

Kelvin-Neumann Problem





The Kelvin–Neumann Problem

$$\phi = \frac{1}{4\pi} \int\int_{(S_1+S_2+S_3)} \left[G\frac{\partial \phi}{\partial n} - \phi\frac{\partial G}{\partial n} \right] dS$$

ϕ is the perturbation potential (does not include $-Ux$).

The integral over S_1, which is the part of the ship hull below the waterline is of the same form as a Green's theorem or "panel method" integral for any finite size body, except here the top is open.

ϕ and G decay with distance from the ship fast enough for the integral over S_3 to vanish.

The integral over S_2, which is the free surface external to the ship is special. We consider it here and call it ϕ_2.

$$\phi_2 = \frac{1}{4\pi} \int\int_{S_2} \left[G\frac{\partial \phi}{\partial n} - \phi\frac{\partial G}{\partial n} \right] dx\, dy$$

Since $\mathbf{n}$ is a unit vector in the z direction on the mean free surface,

$$\phi_2 = \frac{1}{4\pi} \int\int_{S_2} \left[G\frac{\partial \phi}{\partial z} - \phi\frac{\partial G}{\partial z} \right] dx\, dy$$

On the mean free surface, $\frac{\partial \phi}{\partial z} = -\frac{U^2}{g}\frac{\partial^2 \phi}{\partial x^2}$ and we choose G (the Kelvin-Neumann Green function) such that it satisfies the same boundary condition, $\frac{\partial G}{\partial z} = -\frac{U^2}{g}\frac{\partial^2 G}{\partial x^2}$

Then, applying these boundary conditions, ϕ_2 becomes,

$$\phi_2 = -\frac{1}{4\pi}\frac{U^2}{g} \int\int_{S_2} \left[G\frac{\partial^2 \phi}{\partial x^2} - \phi\frac{\partial^2 G}{\partial x^2} \right] dx\, dy$$

$$\phi_2 = -\frac{1}{4\pi}\frac{U^2}{g} \int\int_{S_2} \frac{\partial}{\partial x} \left[G\frac{\partial \phi}{\partial x} - \phi\frac{\partial G}{\partial x} \right] dx\, dy$$

Now, the integral over x can be done. The contributions at $x = \pm\infty$ vanish so the result is:

$$\phi_2 = \frac{1}{4\pi}\frac{U^2}{g}\int_{fore}\left[G\frac{\partial\phi}{\partial x} - \phi\frac{\partial G}{\partial x}\right]dy - \frac{1}{4\pi}\frac{U^2}{g}\int_{aft}\left[G\frac{\partial\phi}{\partial x} - \phi\frac{\partial G}{\partial x}\right]dy$$

The curve of the waterline is called C, with the part forward of the maximum beam called C_f and the part aft of this is called C_a. Consider the integrals taken along C in the counterclockwise direction. Then, dy is positive on the forebody (C_f) and negative on the afterbody (C_a).

$$\phi_2 = \frac{1}{4\pi}\frac{U^2}{g}\int_C\left[G\frac{\partial\phi}{\partial x} - \phi\frac{\partial G}{\partial x}\right]dy$$

If ν is the vector in the horizontal plane that is perpendicular to the waterline and pointed out of the fluid into the ship, and $d\ell$ is the differential of arc length along C, along the waterline $dy = -\nu_x d\ell$, so,

$$\phi = \frac{1}{4\pi}\int\int_{S_1}\left[G\frac{\partial\phi}{\partial n} - \phi\frac{\partial G}{\partial n}\right]dS - \frac{1}{4\pi}\frac{U^2}{g}\int_C\left[G\frac{\partial\phi}{\partial x} - \phi\frac{\partial G}{\partial x}\right]\nu_x d\ell$$

Now, consider a potential function, ϕ' defined in the region bounded by S_1 and S_4. In other words, ϕ' is some function of space such that $\nabla^2\phi' = 0$ in the region where it is defined outside the actual fluid. The boundary condition we impose on ϕ' on S_4 is the same as the one we impose on ϕ on S_2. On S_1 we impose $\phi' = \phi$. $\frac{\partial\phi'}{\partial z} = -\frac{U^2}{g}\frac{\partial^2\phi'}{\partial x^2}$ on S_4

In the fluid region,

$$0 = \frac{1}{4\pi}\int\int_{(S_1+S_4)}\left[G\frac{\partial\phi'}{\partial n'} - \phi'\frac{\partial G}{\partial n'}\right]dS$$

$\mathbf{n}' = -\mathbf{n}$ pointed into the fluid on S_1 and n' is a unit vector in the z direction on S_4.

Call the contribution to ϕ' from the integral on S_4 by ϕ'_4.

$$\phi'_4 = \frac{1}{4\pi}\int\int_{S_4}\left[G\frac{\partial\phi'}{\partial z} - \phi'\frac{\partial G}{\partial z}\right]dx\,dy = -\frac{1}{4\pi}\frac{U^2}{g}\int\int_{S_4}\left[G\frac{\partial^2\phi'}{\partial x^2} - \phi'\frac{\partial^2 G}{\partial x^2}\right]dx\,dy$$

$$\phi'_4 = -\frac{1}{4\pi}\frac{U^2}{g}\int\int_{S_4}\frac{\partial}{\partial x}\left[G\frac{\partial\phi'}{\partial x} - \phi'\frac{\partial G}{\partial x}\right]dx\,dy$$

Carrying out the integral over x gives,

$$\phi_4' = -\frac{1}{4\pi}\frac{U^2}{g}\int_{fore}\left[G\frac{\partial \phi'}{\partial x} - \phi'\frac{\partial G}{\partial x}\right]dy + \frac{1}{4\pi}\frac{U^2}{g}\int_{aft}\left[G\frac{\partial \phi'}{\partial x} - \phi'\frac{\partial G}{\partial x}\right]dy$$

$$\phi_4' = -\frac{1}{4\pi}\frac{U^2}{g}\int_C\left[G\frac{\partial \phi'}{\partial x} - \phi'\frac{\partial G}{\partial x}\right]dy = \frac{1}{4\pi}\frac{U^2}{g}\int_C\left[G\frac{\partial \phi'}{\partial x} - \phi'\frac{\partial G}{\partial x}\right]\nu_x d\ell$$

$$0 = \frac{1}{4\pi}\int\int_{S_1}\left[G\frac{\partial \phi'}{\partial n'} - \phi'\frac{\partial G}{\partial n'}\right]dS + \frac{1}{4\pi}\frac{U^2}{g}\int_C\left[G\frac{\partial \phi'}{\partial x} - \phi'\frac{\partial G}{\partial x}\right]\nu_x d\ell$$

$$0 = \frac{1}{4\pi}\int\int_{S_1}\left[G\frac{\partial \phi'}{\partial n'} + \phi\frac{\partial G}{\partial n}\right]dS + \frac{1}{4\pi}\frac{U^2}{g}\int_C\left[G\frac{\partial \phi'}{\partial x} - \phi\frac{\partial G}{\partial x}\right]\nu_x d\ell$$

To this, we add the equation for ϕ derived before:

$$\phi = \frac{1}{4\pi}\int\int_{S_1}\left[G\frac{\partial \phi}{\partial n} - \phi\frac{\partial G}{\partial n}\right]dS - \frac{1}{4\pi}\frac{U^2}{g}\int_C\left[G\frac{\partial \phi}{\partial x} - \phi\frac{\partial G}{\partial x}\right]\nu_x d\ell$$

The sum is:

$$\phi = \frac{1}{4\pi}\int\int_{S_1} G\left[\frac{\partial \phi}{\partial n} + \frac{\partial \phi'}{\partial n'}\right]dS - \frac{1}{4\pi}\frac{U^2}{g}\int_C G\left[\frac{\partial \phi}{\partial x} - \frac{\partial \phi'}{\partial x}\right]\nu_x d\ell$$

$\frac{1}{4\pi}\left[\frac{\partial \phi}{\partial n} + \frac{\partial \phi'}{\partial n'}\right]$ is the source (actually it is a sink) strength σ.

The normal derivative of $\frac{1}{4\pi}\phi$ jumps at the interface by the source strength, σ. The tangential derivative of ϕ is continuous across the interface because ϕ is continuous. $\frac{\partial \phi}{\partial x}$ jumps across the interface by the jump in the normal derivative times n_x. Therefore,

$$\phi = \int\int_{S_1} G\,\sigma\,dS - \frac{U^2}{g}\int_C G\,\sigma\,n_x\,\nu_x,d\ell$$

The Kelvin–Neumann Green Function

The Kelvin Neumann Green Function, $G^k(x, y, z)$ is the velocity potential for a source located at (a, b, c) and moving at speed U and which satisfies the linearized free surface boundary condition:

$$G^k_{xx}(x, y, 0) + \upsilon G^k_z = 0, \qquad \upsilon = \frac{g}{U^2}$$

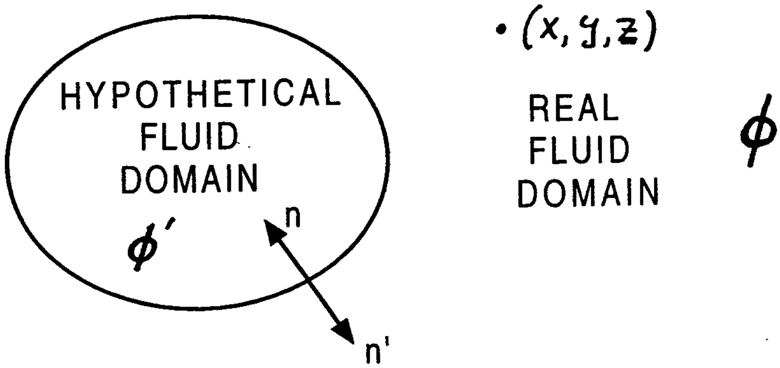
This function is:

$$\begin{aligned} G^k(x, y, z) &= -\frac{1}{r} + \frac{1}{r_1} + \\ &\frac{4\upsilon}{\pi}\int_0^{\pi/2} d\theta \oint_0^{\infty} \frac{e^{k(z+c)} \cos[k(x-a)\cos\theta] \cos[k(y-b)\sin\theta]}{k\cos^2\theta - \upsilon} dk + \\ &4\upsilon \int_0^{\pi/2} e^{\upsilon(z+c)\sec^2\theta} \sin[\upsilon(x-a)\sec\theta] \cos[\upsilon(y-b)\sin\theta\sec^2\theta] \sec^2\theta d\theta \end{aligned}$$

where:

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2, \qquad r_1^2 = (x-a)^2 + (y-b)^2 + (z+c)^2$$

Source Only and Dipole Only Distributions



$$\mathbf{n} = -\mathbf{n}' \qquad \frac{\partial}{\partial \mathbf{n}} = -\frac{\partial}{\partial \mathbf{n}'}$$

ϕ is the velocity potential in the fluid.

ϕ' is a function that satisfies $\nabla^2\phi' = 0$ in the region inside the object.

For a field point in the fluid domain, the following equations apply:

$$\phi = \frac{1}{4\pi}\int\int G\frac{\partial \phi}{\partial n}dS - \frac{1}{4\pi}\int\int \phi\frac{\partial G}{\partial n}dS$$

$$0 = \frac{1}{4\pi}\int\int G\frac{\partial \phi'}{\partial n'}dS - \frac{1}{4\pi}\int\int \phi'\frac{\partial G}{\partial n'}dS$$

$$0 = \frac{1}{4\pi}\int\int G\frac{\partial \phi'}{\partial n'}dS + \frac{1}{4\pi}\int\int \phi'\frac{\partial G}{\partial n}dS$$

$$\phi = \frac{1}{4\pi}\int\int G\left(\frac{\partial \phi}{\partial n} + \frac{\partial \phi'}{\partial n'}\right)dS - \frac{1}{4\pi}\int\int (\phi - \phi')\frac{\partial G}{\partial n}ds$$

Suppose ϕ' is chosen as the harmonic function whose values on S are the same as ϕ. The hypothetical interior flow would have the same tangential velocity on the object as the real outer flow. Then:

$$\phi = \frac{1}{4\pi} \int\int G \left(\frac{\partial \phi}{\partial n} + \frac{\partial \phi'}{\partial n'} \right) dS$$

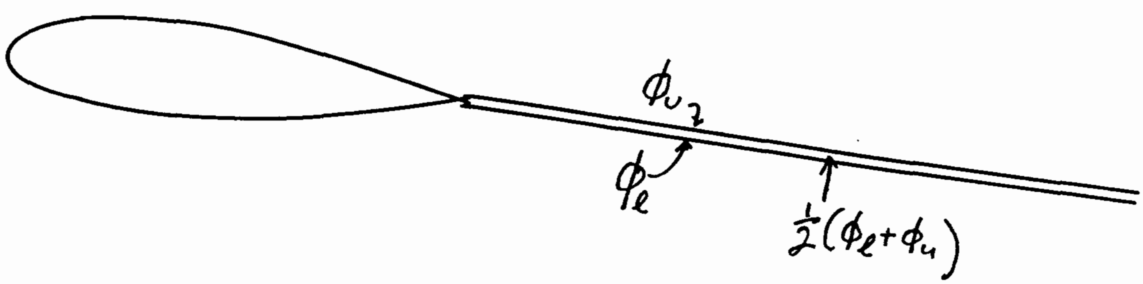
This is a representation for ϕ in terms of surface sources only.

This representation does not apply to lifting flows since they have wakes across which the potential jumps.

Now consider the case for which ϕ' is chosen such that on S, $\frac{\partial \phi}{\partial n} = -\frac{\partial \phi'}{\partial n'}$. The normal velocity is continuous across the surface for this case. Then:

$$\phi = -\frac{1}{4\pi} \int\int (\phi - \phi') \frac{\partial G}{\partial n} ds$$

This is a distribution of dipoles on the object surface S.



FMB8C (1)

Green's Theorem in Two Dimensions

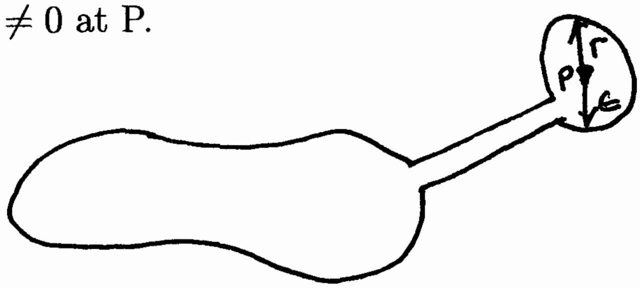
For two dimensional flow, the source potential is $\ln r$ and the Green function becomes:

$$G(x,y,\xi,\eta) = \ln r + \psi_r(x,y,\xi,\eta) \qquad \text{where: } \nabla^2\psi_r = 0$$

The analysis proceeds exactly the same as in the 3D case. When a point $P = (x,y)$, which is the local origin for $\ln r$, is outside the fluid domain,

$$\int_s \left[\phi\frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n}\right] ds = 0$$

When the point P is inside the fluid domain, Green's Theorem is valid in a domain in which the point P is excluded by a small circle, *circle* ϵ surrounding it since $\nabla^2 \neq 0$ at P.



$$\text{Then:} \quad -\int_{circle\,\epsilon} \left[\phi\frac{\partial G}{\partial r} - G\frac{\partial \phi}{\partial r}\right] ds + \int_S \left[\phi\frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n}\right] ds = 0$$

$$\text{Here:} \quad -\int_{circle\,\epsilon} \left[\phi\frac{\partial G}{\partial r} - G\frac{\partial \phi}{\partial r}\right] ds = -2\pi\phi(P)$$

$$\text{Therefore:} \quad \int_S \left[\phi\frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n}\right] ds = \begin{cases} 0 & (x,y) \text{ outside } S \\ \pi\phi(x,y) & (x,y) \text{ on } S \\ 2\pi\phi(x,y) & (x,y) \text{ inside } S \end{cases}$$

FMB8c(2)

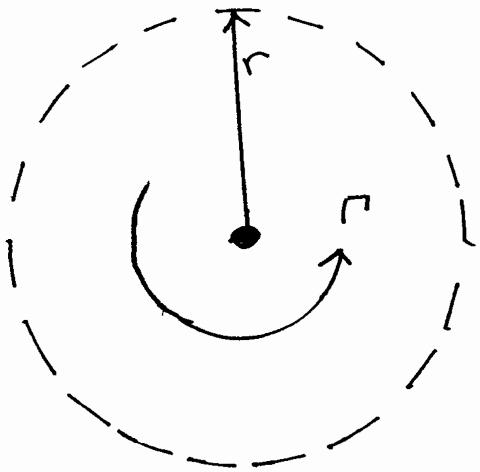
Sometimes, the two-dimensional Green function is taken as:

$$G(x,y,\xi,\eta) = -\ln r + \psi_r(x,y,\xi,\eta) \qquad \text{where: } \nabla^2\psi_r = 0$$

Then,

$$\int_S \left[\phi\frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n}\right] ds = \begin{cases} 0 & (x,y) \text{ outside } S \\ -\pi\phi(x,y) & (x,y) \text{ on } S \\ -2\pi\phi(x,y) & (x,y) \text{ inside } S \end{cases}$$

Force on a Vortex



$$\mathbf{n} = -\hat{\imath}\cos\theta - \hat{k}\sin\theta$$

$$u_v = -\frac{\Gamma}{2\pi r}\sin\theta \qquad w_v = \frac{\Gamma}{2\pi r}\cos\theta$$

$$P = -\frac{\rho}{2}\left\{\left(U - \frac{\Gamma}{2\pi r}\sin\theta\right)^2 + \left(\frac{\Gamma}{2\pi r}\cos\theta\right)^2 - U^2\right\}$$

$$P = -\frac{\rho}{2}\left\{-\frac{\Gamma U}{\pi r}\sin\theta + \left(\frac{\Gamma}{2\pi r}\right)^2\right\}$$

$$\begin{aligned} F_P &= \int_0^{2\pi} P\mathbf{n}\,ds \\ &= -\int_0^{2\pi} \frac{\rho\Gamma}{2\pi r}\left(-U\sin\theta + \frac{\Gamma}{4\pi r}\right)\left(-\hat{\imath}\cos\theta - \hat{k}\sin\theta\right) r d\theta \\ &= \hat{k}\int_0^{2\pi} -\frac{\rho U\Gamma}{2\pi r}\sin^2\theta\, r\, d\theta = -\hat{k}\frac{\rho U\Gamma}{2} \end{aligned}$$

Momentum influx $\equiv M_{in}$ $\qquad U_{in} = \hat{\imath}U \cdot \mathbf{n} = -U\ \cos\theta$

$$F_M = M_{in} = \rho\int_0^{2\pi} \hat{k} w_v U_{in} r\, d\theta = -\hat{k}\frac{\rho U\Gamma}{2}$$

$$F_{total} = F_P + F_M = -\hat{k}\rho U\Gamma$$

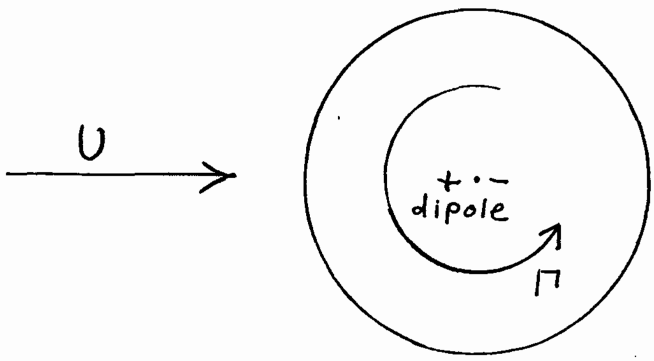
FMB72(1)

Lift on a Vortex in a Cylinder

When a vortex is in a uniform stream, to determine the lift force both the pressure force and the momentum influx into a circular cylinder must be considered.

If the vortex is in a flow whose streamlines form a cylinder around it, there is no momentum influx so the pressure force is the complete force.

A closed circle in a stream can be represented by a dipole.



The velocity potential of a 2D dipole is $\phi_d = A\frac{x}{x^2+z^2}$.

For the flow to make a circle of radius equal to 1 in a stream of speed U, $A = U$. the $x-$ and $z-$directed speeds on the circle of radius 1 due to the dipole are:

$$u_d = U(z^2 - x^2) = U(\sin^2\theta - \cos^2\theta) = -U\cos 2\theta$$

$$w_d = -2Uzx = -2U\sin\theta\cos\theta = -U\sin 2\theta$$

The speeds obn the circle due to the vortex are:

$$u_v = -\frac{\Gamma}{2\pi}\sin\theta \qquad\qquad w_v = \frac{\Gamma}{2\pi}\cos\theta$$

The pressure on the circle is:

$$\begin{aligned} P &= -\frac{\rho}{2}\left[\left(U - U\cos 2\theta - \frac{\Gamma}{2\pi}\sin\theta\right)^2 + \left(-U\sin 2\theta + \frac{\Gamma}{2\pi}\cos\theta\right)^2 - U^2\right] \\ &= -\frac{\rho}{2}\left[U^2 + \left(\frac{\Gamma}{2\pi}\right)^2 - 2U^2\cos 2\theta - \frac{U\Gamma}{\pi}\sin\theta + \frac{U\Gamma}{\pi}\sin\theta\cos 2\theta - \frac{U\Gamma}{\pi}\cos\theta\sin 2\theta\right] \end{aligned}$$

FMBV7(M)

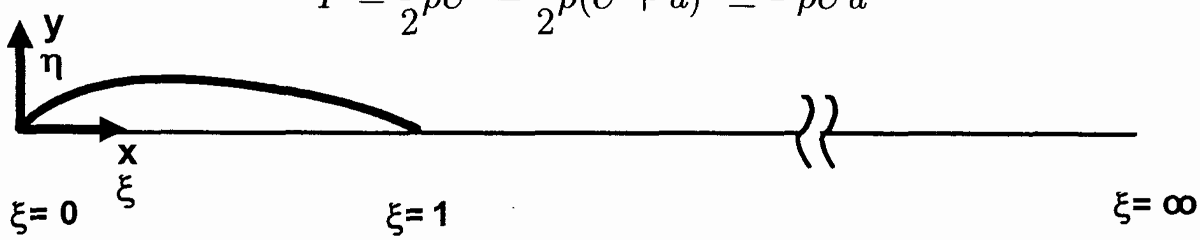
The vertical force, F_w is:

$$F_w = \int_0^{2\pi} P\mathbf{n} \cdot \hat{k} d\theta = \int_0^{2\pi} -P \sin\theta d\theta$$

$$\begin{aligned}
F_w &= \frac{\rho U \Gamma}{2\pi} \int_0^{2\pi} \left(-\sin^2\theta + \sin^2\theta \cos 2\theta - \sin\theta \cos\theta \sin 2\theta \right) d\theta \\
&= \frac{\rho U \Gamma}{2\pi} \int_0^{2\pi} \left(-\sin^2\theta - \frac{1}{2}\cos^2 2\theta - \frac{1}{2}\sin^2 2\theta \right) d\theta \\
&= \frac{\rho U \Gamma}{2\pi} \left(-\pi - \frac{\pi}{2} - \frac{\pi}{2} \right) = -\rho U \Gamma
\end{aligned}$$

Example: Design of 2D Airfoil Mean Line using Dipoles and Vortices

$$P = \frac{1}{2}\rho U^2 - \frac{1}{2}\rho(U+u)^2 \simeq -\rho U u$$



$$\xi = x/c \qquad x = c\xi \qquad dx = c\,d\xi \qquad \eta = y/c$$

<u>Design Condition:</u> $P_{\text{top}} = -1.0\rho U^2 \xi(1-\xi) \qquad P_{\text{bottom}} = 1.0\rho U^2 \xi(1-\xi)$

$$u_t = 1.0U\xi(1-\xi) \qquad u_b = -1.0U\xi(1-\xi)$$

$$\phi_t \simeq \int_0^x u dx' = c\int_0^\xi u d\xi = 1.0Uc\int_0^\xi (\xi - \xi^2)d\xi = 1.0Uc\left(\frac{\xi^2}{2} - \frac{\xi^3}{3}\right)$$

$$\phi_b = -1.0Uc\left(\frac{\xi^2}{2} - \frac{\xi^3}{3}\right)$$

$$\phi_t(\xi=1) = \frac{1.0}{6}Uc \qquad \phi_b(\xi=1) = -\frac{1.0}{6}Uc \qquad (\phi_t - \phi_b)_{\xi=1} = \frac{1.0}{3}Uc$$

$$[\text{Dipole Strength}]_{\text{foil}} = \mu = 2.0Uc\left(\frac{\xi^2}{2} - \frac{\xi^3}{3}\right) \qquad \mu_{\text{wake}} = \frac{1.0}{3}Uc$$

$$G = \ln r = \ln\left[(x-x_o)^2 + (y-y_o)^2\right]^{1/2} = \frac{1}{2}\ln\left[(x-x_o)^2 + (y-y_o)^2\right]$$

$$\left(\frac{\partial G}{\partial n}\right)_t = -\frac{\partial G}{\partial y_o} \qquad \left(\frac{\partial G}{\partial n}\right)_b = \frac{\partial G}{\partial y_o}$$

Consider the upper surface:

$$\left(\frac{\partial G}{\partial n}\right)_t = -\frac{\partial G}{\partial y_o} = \frac{y-y_o}{(x-x_o)^2+(y-y_o)^2}$$

$$\phi(x,y) = \frac{1}{\pi}\int_0^\infty \mu(x_o)\frac{y-y_o}{(x-x_o)^2+(y-y_o)^2}dx_o$$

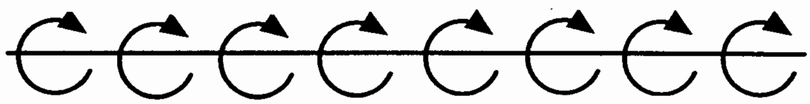
$$\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{1}{\pi}\int_0^c \mu(x_o)\frac{(x-x_o)^2+(y-y_o)^2-2(y-y_o)^2}{\left[(x-x_o)^2+(y-y_o)^2\right]^2}dx_o \\
&\quad +\mu(c)\int_c^\infty \frac{(x-x_o)^2+(y-y_o)^2-2(y-y_o)^2}{\left[(x-x_o)^2+(y-y_o)^2\right]^2}dx_o \\
&= \frac{1}{\pi}\int_0^c \mu(x_o)\frac{(x-x_o)^2-(y-y_o)^2}{\left[(x-x_o)^2+(y-y_o)^2\right]^2}dx_o + \frac{1}{\pi}\mu(c)\int_c^\infty \frac{(x-x_o)^2-(y-y_o)^2}{\left[(x-x_o)^2+(y-y_o)^2\right]^2}dx_o
\end{aligned}$$

$$\left[\frac{\partial \phi}{\partial y}\right]_{y=y_o=0} = \frac{1}{\pi}\int_0^c \mu(x_o)\frac{1}{(x-x_o)^2}dx_o + \frac{1}{\pi}\mu(c)\int_c^\infty \frac{1}{(x-x_o)^2}dx_o$$

The above analysis has an incorrect non-integrable singularity at $x = x_o$ because a careful limiting analysis requiring $\nabla^2\phi = 0$ was not done.

However, another, and simpler, approach exists.

A dipole represents a jump in the potential. Another way to achieve a potential jump is a vortex distribution.



In length dx_o, vortex strength $=\gamma(x_o)dx_o$. $\gamma(x_o)$ is vorticity/unit-length.

$$u_t(x) = U + \frac{\gamma(x)}{2} \qquad u_b(x) = U - \frac{\gamma(x)}{2} \qquad \gamma(x) = u_t(x) - u_b(x)$$

$$v(x) = -\int_0^c \frac{\gamma(x_o)}{2\pi(x-x_o)}dx_o$$

$$\text{slope} = \frac{v(x)}{U} = \int_0^1 \frac{\gamma'(\xi_o)}{2\pi(\xi-\xi_o)}d\xi_o$$

$$\text{where: } \gamma'(\xi_o) = \frac{\gamma(c\xi_o)}{U}$$

Now, we can solve for the mean line shape of the airfoil

For an arbitrarily defined pressure distribution, the integral for the slope can be done numerically. Here, for the particular pressure distribution given, we will solve for the slope analytically.

Then the shape is found by integrating the slope, $s(\xi)$. This will be done numerically.

$$\gamma'(\xi) = 2.0\xi(1.0 - \xi)$$

$$s(\xi) = -\int_0^1 \frac{\gamma'(\xi_o)}{(2\pi(\xi - \xi_o)} d\xi_o = -\frac{1.0}{\pi} \int_0^1 \frac{\xi_o(1 - \xi_o)}{\xi - \xi_o} d\xi_o$$

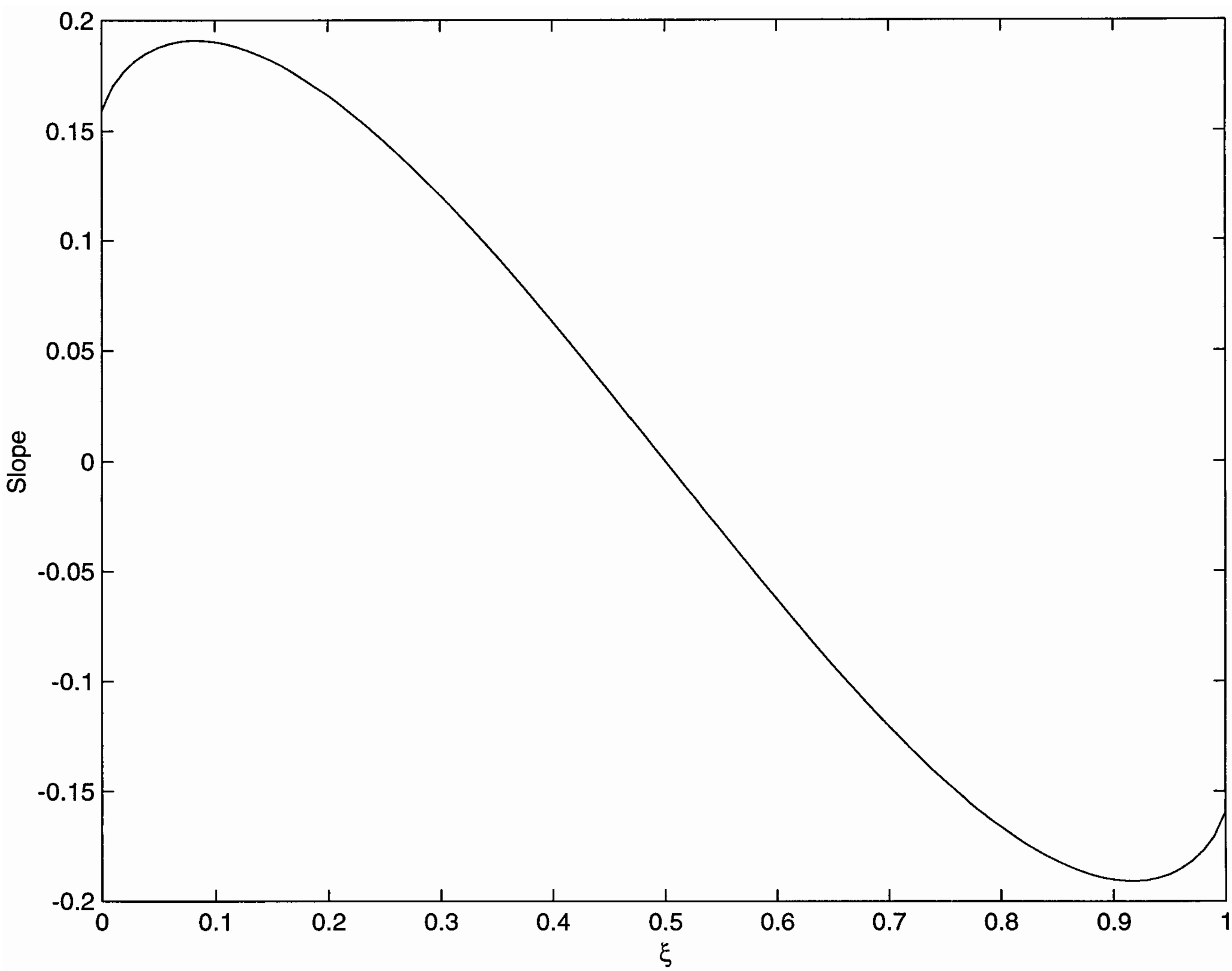
$$s(\xi) = -\frac{1}{\pi} \left\{ \frac{1}{2} - (1 - \xi) \left[\xi \ln \frac{1 - \xi}{\xi} + 1 \right] \right\}$$

$$\text{Non-Dimensional Height } = \eta(\xi) = \int_0^\xi s(\xi_o) d\xi_o$$

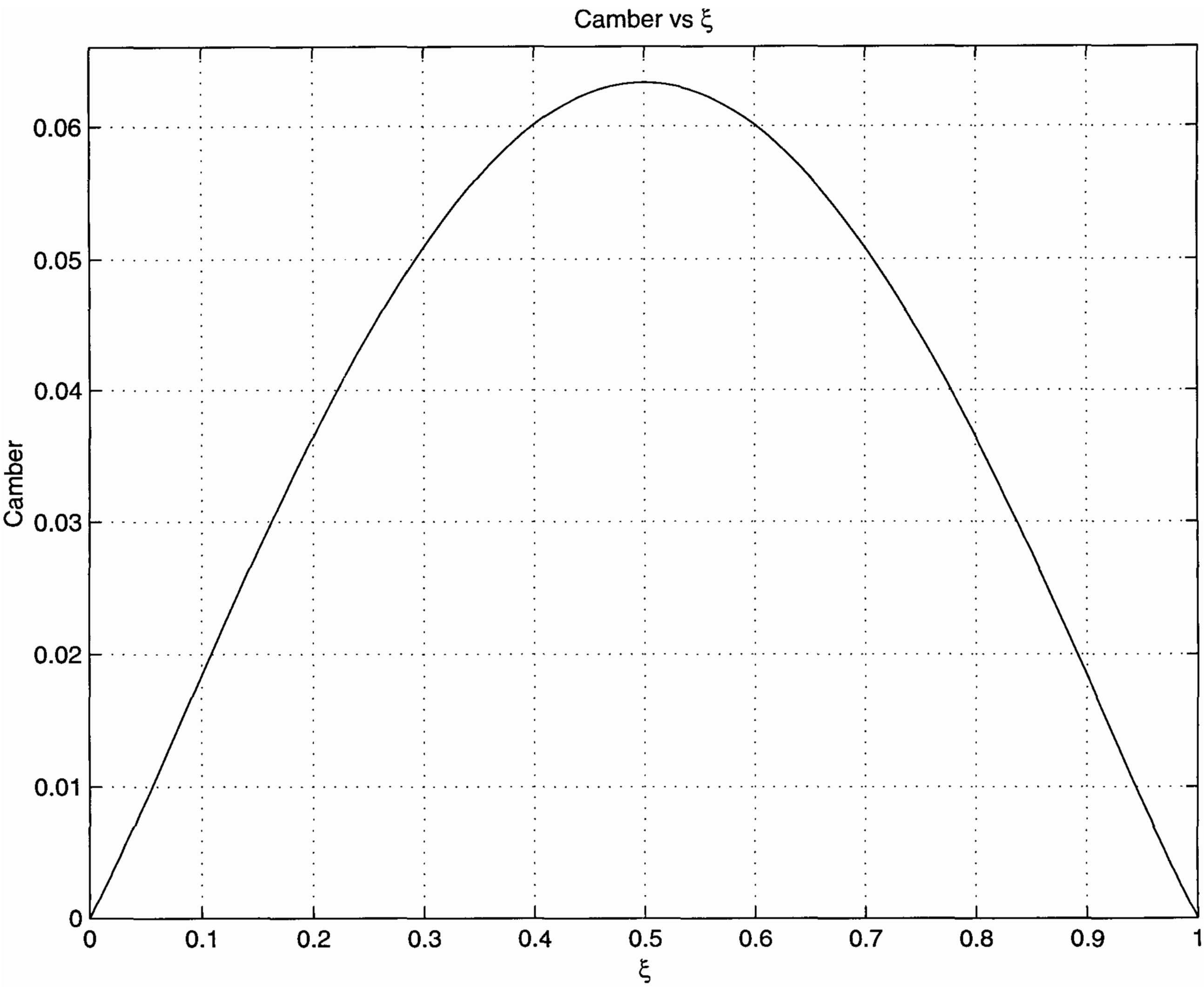
```
format compact
x = 0  : 0.01 : 1.0;
fac = -1.0/pi;
s = fac.*(0.5 - (1.0-x).*( x.*log((1.0-x+eps)./(x+eps))+1.0));
h(1) = 0.0;
for i = 2:101
h(i) = h(i-1) + (x(i)-x(i-1))*0.5*(s(i)+s(i-1));
end;
fid = fopen('ht.dat','w');
for m = 1:101
fprintf(fid,'%6.2f  %7.4f  %7.4f\n',x(m), s(m), h(m));
end;
fclose (fid);
plot(x,s)
ylabel('Slope')
xlabel('\xi')
pause

plot(x,h)
ylabel('Camber')
xlabel('\xi')
title('Camber vs \xi')
axis([0 1 0 0.066])
grid
```

X	SLOPE	HEIGHT
0.00	0.1592	0.0000
0.01	0.1705	0.0016
0.02	0.1771	0.0034
0.03	0.1818	0.0052
0.04	0.1853	0.0070
0.05	0.1878	0.0089
0.06	0.1895	0.0108
0.07	0.1905	0.0127
0.08	0.1909	0.0146
0.09	0.1908	0.0165
0.10	0.1903	0.0184
0.11	0.1893	0.0203
0.12	0.1879	0.0222
0.13	0.1862	0.0240
0.14	0.1842	0.0259
0.15	0.1818	0.0277
0.16	0.1792	0.0295
0.17	0.1763	0.0313
0.18	0.1731	0.0331
0.19	0.1697	0.0348
0.20	0.1661	0.0364
0.21	0.1623	0.0381
0.22	0.1583	0.0397
0.23	0.1541	0.0413
0.24	0.1497	0.0428
0.25	0.1451	0.0442
0.26	0.1405	0.0457
0.27	0.1356	0.0471
0.28	0.1306	0.0484
0.29	0.1255	0.0497
0.30	0.1203	0.0509
0.31	0.1150	0.0521
0.32	0.1095	0.0532
0.33	0.1040	0.0543
0.34	0.0983	0.0553
0.35	0.0926	0.0562
0.36	0.0868	0.0571
0.37	0.0809	0.0580
0.38	0.0749	0.0587
0.39	0.0689	0.0595
0.40	0.0628	0.0601
0.41	0.0567	0.0607
0.42	0.0505	0.0612
0.43	0.0443	0.0617
0.44	0.0380	0.0621
0.45	0.0317	0.0625
0.46	0.0254	0.0628
0.47	0.0191	0.0630
0.48	0.0127	0.0632
0.49	0.0064	0.0632
0.50	0.0000	0.0633
0.51	-0.0064	0.0632
0.52	-0.0127	0.0632
0.53	-0.0191	0.0630
0.54	-0.0254	0.0628
0.55	-0.0317	0.0625
0.56	-0.0380	0.0621

0.57	-0.0443	0.0617
0.58	-0.0505	0.0612
0.59	-0.0567	0.0607
0.60	-0.0628	0.0601
0.61	-0.0689	0.0595
0.62	-0.0749	0.0587
0.63	-0.0809	0.0580
0.64	-0.0868	0.0571
0.65	-0.0926	0.0562
0.66	-0.0983	0.0553
0.67	-0.1040	0.0543
0.68	-0.1095	0.0532
0.69	-0.1150	0.0521
0.70	-0.1203	0.0509
0.71	-0.1255	0.0497
0.72	-0.1306	0.0484
0.73	-0.1356	0.0471
0.74	-0.1405	0.0457
0.75	-0.1451	0.0442
0.76	-0.1497	0.0428
0.77	-0.1541	0.0413
0.78	-0.1583	0.0397
0.79	-0.1623	0.0381
0.80	-0.1661	0.0364
0.81	-0.1697	0.0348
0.82	-0.1731	0.0331
0.83	-0.1763	0.0313
0.84	-0.1792	0.0295
0.85	-0.1818	0.0277
0.86	-0.1842	0.0259
0.87	-0.1862	0.0240
0.88	-0.1879	0.0222
0.89	-0.1893	0.0203
0.90	-0.1903	0.0184
0.91	-0.1908	0.0165
0.92	-0.1909	0.0146
0.93	-0.1905	0.0127
0.94	-0.1895	0.0108
0.95	-0.1878	0.0089
0.96	-0.1853	0.0070
0.97	-0.1818	0.0052
0.98	-0.1771	0.0034
0.99	-0.1705	0.0016
1.00	-0.1592	-0.0000

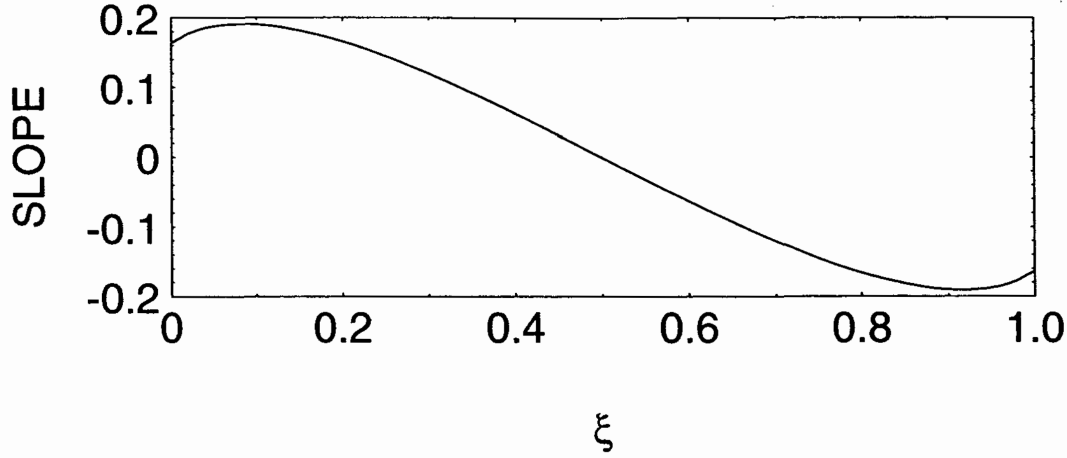


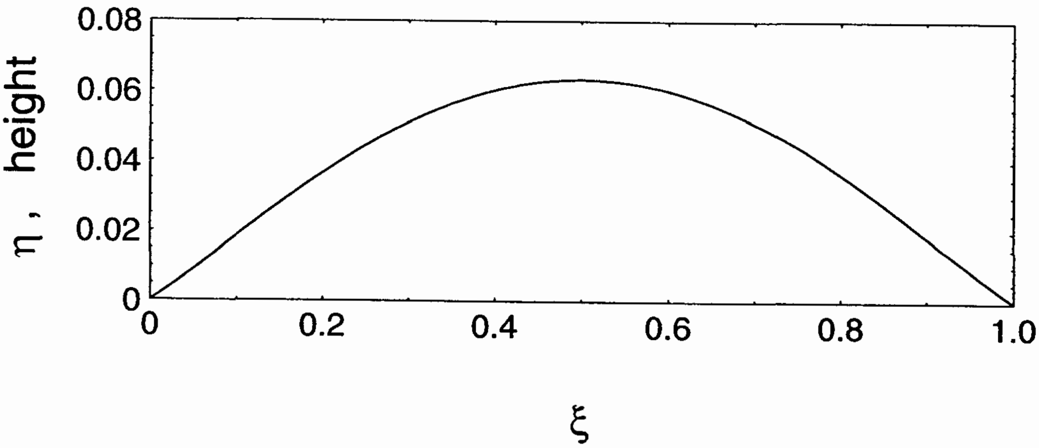
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
Slope
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
ξ



Camber vs ξ
0.06
0.05
0.04
0.03
0.02
0.01
0
Camber
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
ξ

foiltdq - results





foiltd

```
format compact
x = 0  : 0.01 : 1.0;
fac = -1.0/pi;
s = fac.*(0.5 - (1.0-x).*( x.*log((1.0-x+eps)./(x+eps))+1.0));
h(1) = 0.0;
for i = 2:101
h(i) = h(i-1) + (x(i)-x(i-1))*0.5*(s(i)+s(i-1));
end;
fid = fopen('ht.dat','w');
for m = 1:101
fprintf(fid,'%6.2f  %7.4f  %7.4f\n',x(m), s(m), h(m));
end;
fclose (fid);
plot(x,s)
ylabel('Slope')
xlabel('\xi')
pause

plot(x,h)
ylabel('Camber')
xlabel('\xi')
title('Camber vs \xi')
axis([0 1 0 0.066])
grid
```

foiltda

```
% Version of foiltd with one less loop for computing speed improvement

x = 0  : 0.01 : 1.0;
fac = -1.0/pi;
s = fac.*(0.5 - (1.0-x).*( x.*log((1.0-x+eps)./(x+eps))+1.0));
h(1) = 0.0;
for i = 2:101
h(i) = h(i-1) + (x(i)-x(i-1))*0.5*(s(i)+s(i-1));
end;
fid = fopen('hta.dat','w');
q = [x;s;h];
fprintf(fid,'%6.2f  %7.4f  %7.4f\n',q);
fclose (fid);
plot(x,s)
ylabel('Slope')
xlabel('\xi')
pause

plot(x,h)
ylabel('Camber')
xlabel('\xi')
title('Camber vs \xi')
axis([0 1 0 0.066])
grid
```

foiltdb

```
% This version uses even more vectorization and no "for" loops at all.

% Version of foiltd with one less loop for computing speed improvement

x = 0  : 0.01 : 1.0;
fac = -1.0/pi;
s = fac.*(0.5 - (1.0-x).*( x.*log((1.0-x+eps)./(x+eps))+1.0));
h(1) = 0.0;
xd = [0 diff(x)]; % This is [0 x(2)-x(1)  x(3)-x(2) ...]
ss = [0 s(1:end-1) + s(2:end) ] % This is [0 s(2)+s(1)  s(3)+s(2)
h = 0.5*xd .* ss;
h = cumsum(h); % Each element is the sum of the ones bvefore it.
  fid = fopen('htb.dat','w');
q = [x;s;h];
fprintf(fid,'%6.2f  %7.4f  %7.4f\n',q);
fclose (fid);
plot(x,s)
ylabel('Slope')
xlabel('\xi')
pause

plot(x,h)
ylabel('Camber')
xlabel('\xi')
title('Camber vs \xi')
axis([0 1 0 0.066])
grid
```

```
% Version of foiltd with one less loop for computing speed improvement

x = 0  : 0.01 : 1.0;
fac = -1.0/pi;
s = fac.*(0.5 - (1.0-x).*( x.*log((1.0-x+eps)./(x+eps))+1.0));
h(1) = 0.0;
for i = 2:101
h(i) = h(i-1) + (x(i)-x(i-1))*0.5*(s(i)+s(i-1));
end;
fid = fopen('hta.dat','w');
q = [x;s;h];
fprintf(fid,'%6.2f  %7.4f  %7.4f\n',q);
fclose (fid);
plot(x,s)
ylabel('Slope')
xlabel('\xi')
pause

plot(x,h)
ylabel('Camber')
xlabel('\xi')
title('Camber vs \xi')
axis([0 1 0 0.066])
grid
```

```
% This version uses even more vectorization and no "for" loops at all.
% Version of foiltd with one less loop for computing speed improvement

x = 0  : 0.01 : 1.0;
fac = -1.0/pi;
s = fac.*(0.5 - (1.0-x).*( x.*log((1.0-x+eps)./(x+eps))+1.0));
h(1) = 0.0;
xd = [0 diff(x)]; % This is [0 x(2)-x(1)  x(3)-x(2) ...]
ss = [0 s(1:end-1) + s(2:end) ] % This is [0 s(2)+s(1)  s(3)+s(2)
h = 0.5*xd .* ss;
h = cumsum(h); % Each element is the sum of the ones bvefore it.
  fid = fopen('htb.dat','w');
q = [x;s;h];
fprintf(fid,'%6.2f  %7.4f  %7.4f\n',q);
fclose (fid);
plot(x,s)
ylabel('Slope')
xlabel('\xi')
pause

plot(x,h)
ylabel('Camber')
xlabel('\xi')
title('Camber vs \xi')
axis([0 1 0 0.066])
grid
```