2.29 Numerical Fluid Mechanics
Spring 2015

Units: (3-0-9, 4-0-8)

Lectures and Recitations:

Lectures: Monday/Wednesday 11:00 a.m. — 12:30 p.m.,
Recitations/Reviews: Wednesday 4:00 p.m. — 5:00 p.m.,

The Wed. afternoon lectures, recitations and review sessions will not be held every week. They are used in response to student's requests (e.g. special topics) or the needs of the course (e.g. make-up lectures). Students will be informed in advance when these sessions are planned..

Prerequisite: 2.006 or 2.016 or 2.20 or 2.25, 18.075

Subject Summary and Objectives:

Introduction to numerical methods and MATLAB: errors, condition numbers and roots of equations. Navier-Stokes. Direct and iterative methods for linear systems. Finite differences for elliptic, parabolic and hyperbolic equations. Fourier decomposition, error analysis and stability. High-order and compact finite-differences. Finite volume methods. Time marching methods. Navier-Stokes solvers. Grid generation. Finite volumes on complex geometries. Finite element methods. Spectral methods. Boundary element and panel methods. Turbulent flows. Boundary layers. Lagrangian Coherent Structures. Subject includes a final research project.

Specific Objectives:

- To introduce and develop the main approaches and techniques which constitute the basis of numerical fluid mechanics for engineers and applied scientists.
- To familiarize students with the numerical implementation of these techniques and numerical schemes, so as provide them with the means to write their own codes and software, and so acquire the knowledge necessary for the skillful utilization of CFD packages or other more complex software.
- To cover a range of modern approaches for numerical and computational fluid dynamics, without entering all these topics in detail, but aiming to provide students with a general knowledge and understanding of the subject, including recommendations for further studies.

This course continues to be a work in progress. New curricular materials are being developed for this course, and feedback from students is always welcome and appreciated during the term. For example, recitations and reviews on specific topics can be provided based on requests from students.

Students are strongly encouraged to attend classes and recitations/reviews. The instructor and teaching assistant are also available for consultation during office hours. Appointments can also be scheduled by emails and/or phone.

Evaluation and Grading:

The final course grade will be weighted as follows:

Homework	**(6 in total, 5% each)**	**30 %**
Quizzes	**(2)**	**40 %**
Final project	**(1)**	**30 %**

Project:

There will be a final project for this class. Students can select the topic of their project in consultation with the instructor and TA. Possible projects include:

i) Comprehensive reviews of material not covered in detail in class, with some numerical examples;

ii) Specific fluid-related problems or questions that are numerically studied or solved by the applications of approaches, methods or schemes covered in class;

iii) A combination of i) and ii).

Projects will be due at the end of term. We plan to have a final session where all students will make a presentation of their projects to the whole class and staff. We have found that such presentations provide an excellent means for additional learning and sharing.

2.29 Numerical Fluid Mechanics
Sample Project Titles (30% of grade)

i) "Comprehensive" Methodological Reviews and Comparisons

- Review of autonomous/adaptive generation of computational grids in complex geometries
- Advanced unstructured grids schemes for numerical fluid mechanics applications in
  - Heat transfer/thermodynamics, Ocean Eng./Science, Civil Engineering, etc.
- Review of Multigrid methods and comparisons of schemes in idealized examples
- Comparisons of solvers for banded/sparse linear systems: theory and idealized examples
- The use of spectral methods for turbulent flows
- Novel advanced computational schemes for reactive/combustion flows: reviews and examples
- Numerical dissipation and dispersion: review and examples of artificial viscosity
- etc.

Sample Project Titles (30% of grade), Cont'd

ii) Computational Fluid Studies and Applications

- Idealized simulations of compressible air flows through pipe systems
- Computational simulations of idealized physical and biogeochemical dynamics in oceanic straits
- Simulations of flow fields around a propeller using a (commercial) CFD software: sensitivity to numerical parameters
  - e.g. sensitivity to numerical scheme, grid resolution, etc
- Simulation of flow dynamics in an idealized porous medium
- Pressure distribution on idealized ship structures: sensitivity to ship shapes and to flow field conditions
- Finite element (or Finite difference) simulations of flows for
  - Idealized capillaries, Laminar duct flows, idealized heat exchangers, etc
- etc.

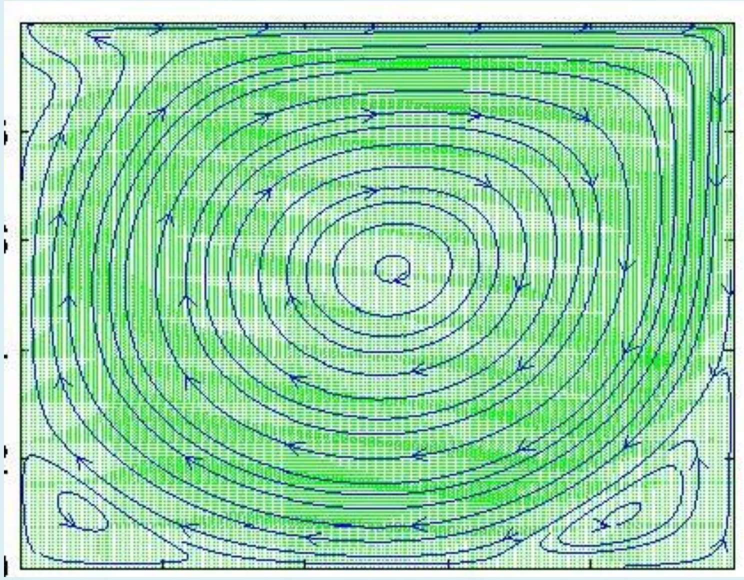
2.29 Numerical Fluid Mechanics
Sample Project Titles (30% of grade), Cont'd

iii) Combination of i) Reviews and ii) Specific computational fluid studies

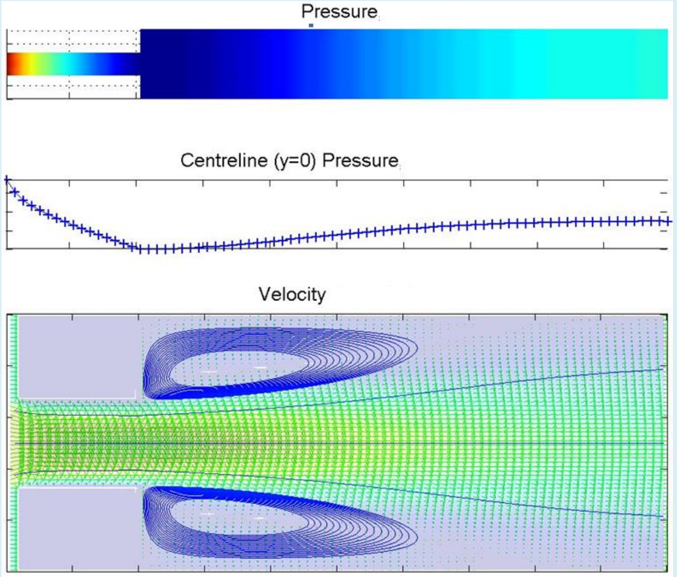
- Review of Panel methods for fluid-flow/structure interactions and preliminary applications to idealized oceanic wind-turbine examples
- Comparisons of finite volume methods of different accuracies in 1D convective problems
- A study of the accuracy of finite volume (or difference or element) methods for two-dimensional fluid mechanics problems over simple domains
- Computational schemes and simulations for chaotic dynamics in nonlinear ODEs
- Stiff ODEs: recent advanced schemes and fluid examples
- High-order schemes for the discretization of the pressure gradient term and their applications to idealized oceanic/atmospheric flows
- etc.

2.29: Numerical Fluid Mechanics

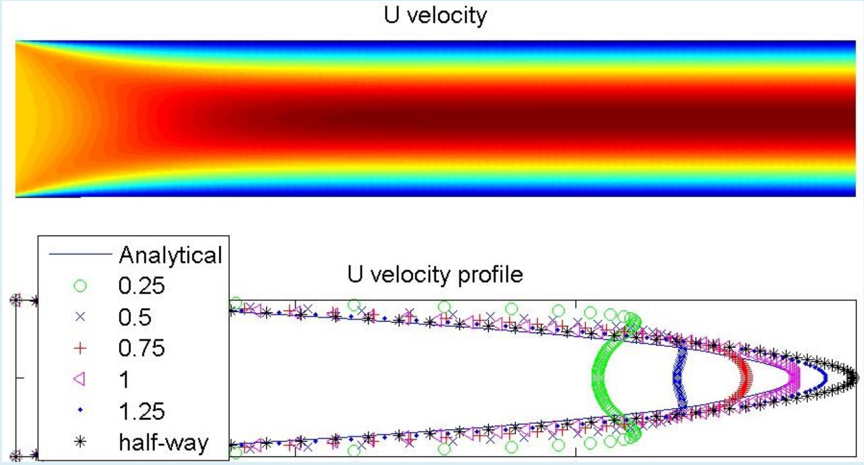
Lid-driven Cavity Flow



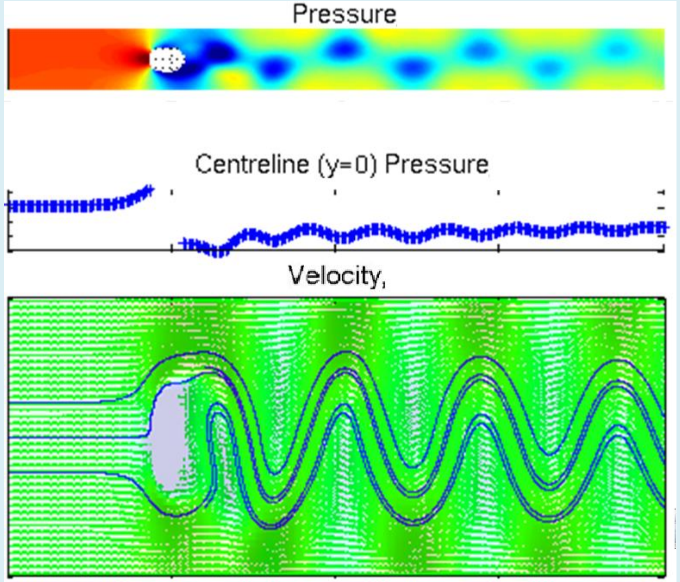
Sudden Expansion



Viscous Flow In A Pipe



Flow Around A Circular Cylinder

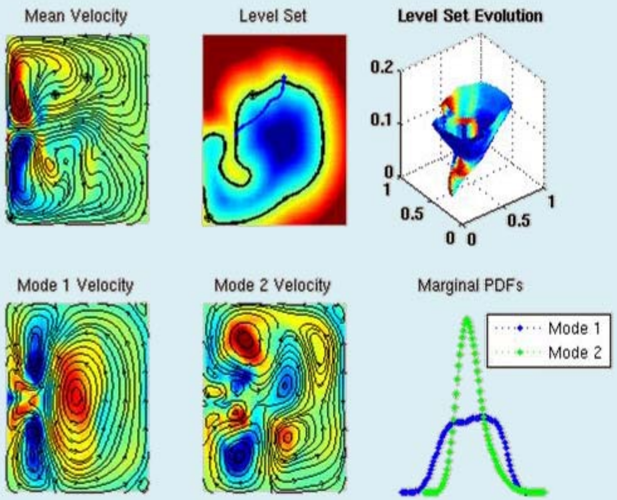


$$\frac{\partial \phi}{\partial t} + F|\nabla \phi| + \mathbf{v} \cdot \nabla \phi = 0$$

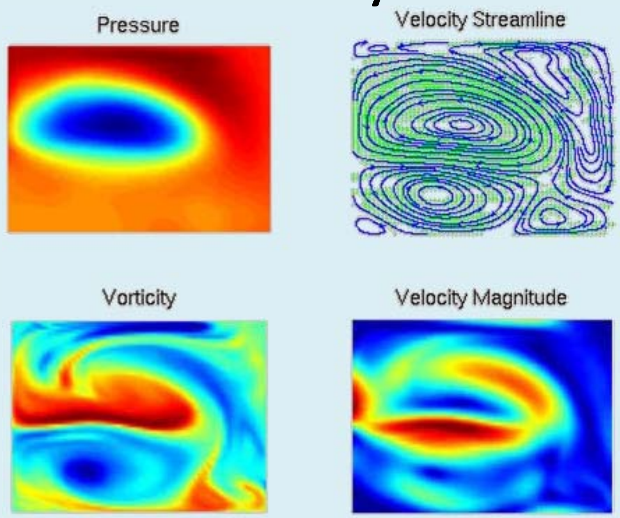
2.29

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}$$

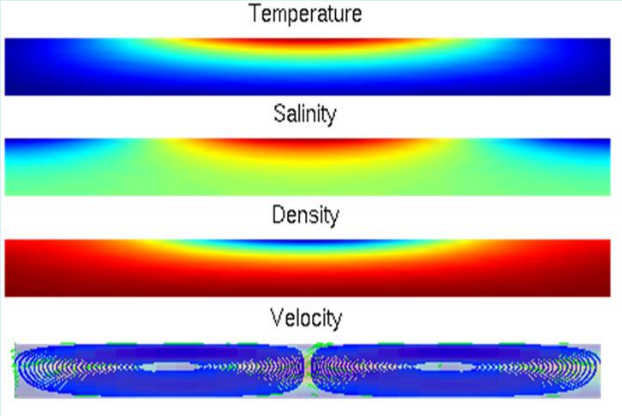
Various Path-Planning



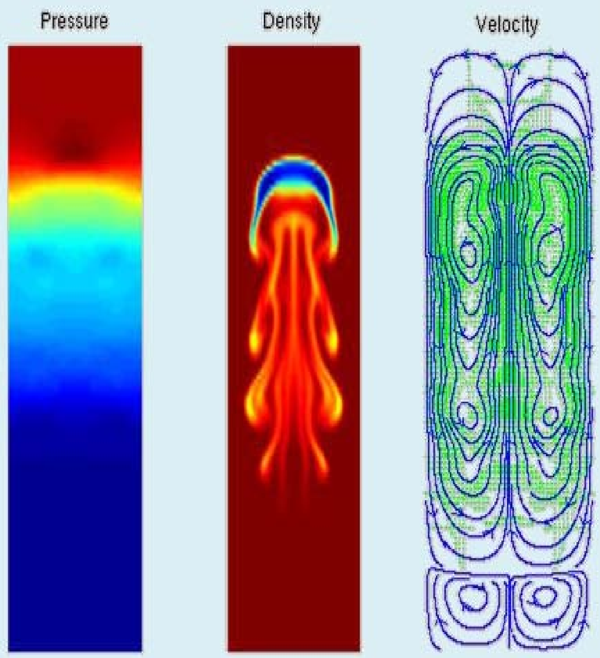
Double-Gyre



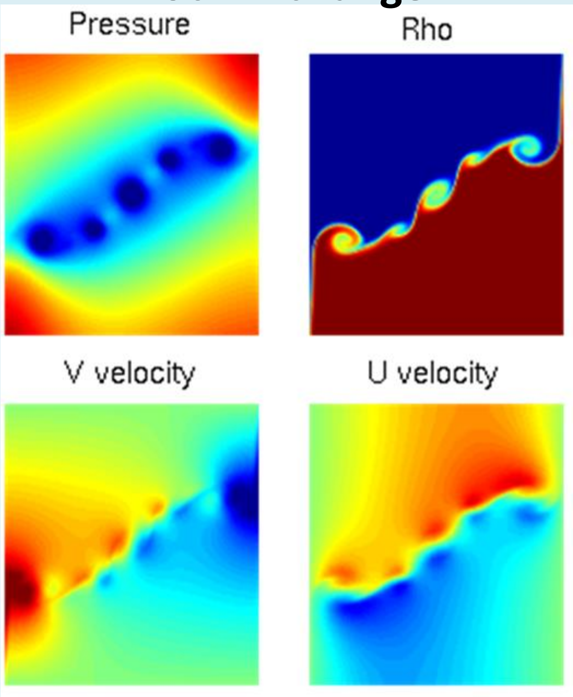
2D Thermohaline Circulation



Warm Rising Bubble



Lock Exchange



2.29 Numerical Fluid Mechanics
Projects completed in Spring 2008

- Analysis of Simple Walking Models: Existence and Stability of Periodic Gaits
- Simulations of Coupled Physics-Biology in Idealized Ocean Straits
- High-resolution Conservative Schemes for Incompressible Advections: The Magic Swirl
- Multigrid Method for Poisson Equations: Towards atom motion simulations
- Stability Analysis for a Two-Phase Flow system at Low Pressure Conditions
- Particle Image Velocimetry and Computations: A Review
- Real-time Updates of Coastal Bathymetry and Flows for Naval Applications

- Simulation of Particles in 2D Incompressible Flows around a Square Block
- Panel Method Simulations for Cylindrical Ocean Structures
- 2D viscous Flow Past Rectangular Shaped Obstacles on Solid Surfaces
- Three-dimensional Acoustic Propagation Modeling: A Review
- Immersed Boundary Methods and Fish Flow Simulations: A Review

2.29 Numerical Fluid Mechanics
Projects completed in Fall 2009

- Lagrangian Coherent Structures and Biological Propulsion
- Fluid Flows and Heat Transfer in Fin Geometries
- Boundary Integral Element Methods and Earthquake Simulations
- Effects of Wind Direction on Street Transports in Cities simulated with FLUENT
- Stochastic Viscid Burgers Equations: Polynomial Chaos and DO equations

- Modeling of Alexandrium fundyense bloom dynamics in the Eastern Maine Coastal Current: Eulerian vs. Lagrangian Approach
- Coupled Neutron Diffusion Studies: Extending Bond Graphs to Field Problems
- CFD Investigation of Air Flow through a Tube-and-Fin Heat Exchanger
- Towards the use of Level-Set Methods for 2D Bubble Dynamics
- Mesh-Free Schemes for Reactive Gas Dynamics Studies
- A review of CFD usage at Bosch Automotive USA

2.29 Numerical Fluid Mechanics
Projects completed in Fall 2011

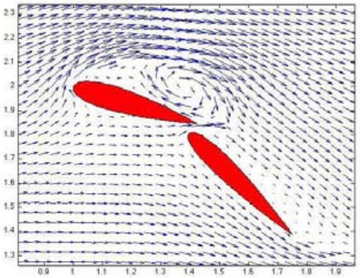
- Dye Hard: An Exploration into 2D Finite Volume Schemes and Flux Limiters
- Sailing and Numerics: 2-D Slotted Wing simulations using the 2.29 Finite Volume Navier-Stokes Code
- Jacobian Free Newton Krylov Methods for solving coupled Neutronic/Thermal Hydraulic Equations
- Cartesian Grid Simulations of High Reynolds Number Flows with Moving Solid Boundaries
- Numerical Predictions of Diffusive Sediment Transport
- Internal Tides Simulated Using the 2.29 Finite Volume Boussinesq Code
- Numerical solution of an open boundary heat diffusion problem with Finite Difference and Lattice-Boltzmann methods
- Predicting Uncertainties with Polynomial Chaos or Dynamically Orthogonal Equations: Who Wins?
- Review of Spectral/hp Methods for Vortex Induced Vibrations of Cylinders
- Direct Numerical Simulation of a Simple 2D Geometry with Heat Transfer at Very Low Reynolds Number
- CFD Methods for Modeling Ducted Propulsors
- Diesel Particle Filter simulations with the 2.29 Finite Volume Navier-Stokes Code
- Comparison of Large Eddy Simulation Sub-grid Models in Jet Flows
- Numerical Simulation of Vortex Induced Vibration

2.29 Numerical Fluid Mechanics
Projects completed in Spring 2013

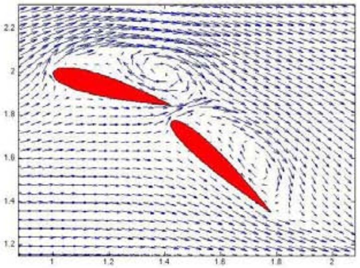
- Molecular Dynamics Simulations of Gas Separation by Nanoporous Graphene Membranes
- Computational Methods for Stirling Engines Simulations
- A Boundary Element Approach to Dolphin Surfing
- High-Order Finite Difference Schemes for Ideal Magnetohydrodynamics
- Implicit Scheme for a Front-tracking/Finite-Volume Navier-Stokes Solver
- Free-Convection around Blinds: Simulations using Fluent
- Advantages and Implementations of Hybrid Discontinuous Galerkin Finite Element Methods with Applications
- Biofilm Growth in Shear Flows: Numerical 2.29 FV Simulations using a Porous Media Model
- Simulations of Two-Phase Flows using a Volume of Fluid (VOF) Approach: Kelvin-Helmholtz Instabilities in Newtonian and Viscoelastic liquids
- Interactions of Non-hydrostatic Internal Tides with Background Flows: 2.29 Finite Volume Simulations
- Numerical 2.29 FV Simulation of Ion Transport in Microchannels through Poisson-Nernst-Plank Eqs.
- A 2-D Finite Volume Framework on Structured Non-Cartesian Grids for a Convection-Diffusion-Reaction Equation
- Steady State Evaporation of a Liquid Microlayer
- Optimal Energy Path Planning using Stochastic Dynamically Orthogonal Level Set Equations
- Interface Tracking Methods for OpenFOAM Simulations of Two-Phase Flows
- High-Order Methods and WENO schemes for Hyperbolic Wave Equations

Pressure Distribution on a Two Dimensional Slotted Wing

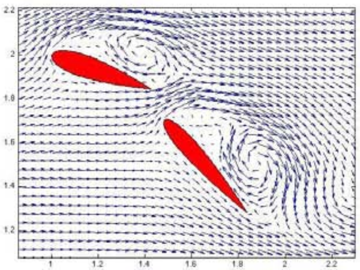
An MIT Student, 2011



(a) Gap=0.2*c1, Alpha=20°, Beta=30°



(b) Gap=0.3*c1, Alpha=20°, Beta=30°



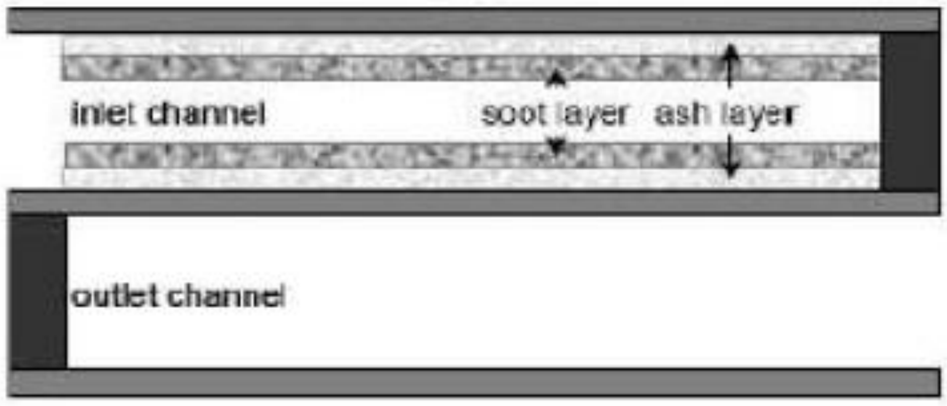
(c) Gap=0.5*c1, Alpha=20°, Beta=30°

Figure 15: The angle of attack of the first element appeared to play an important role in the effectiveness of the element gap.

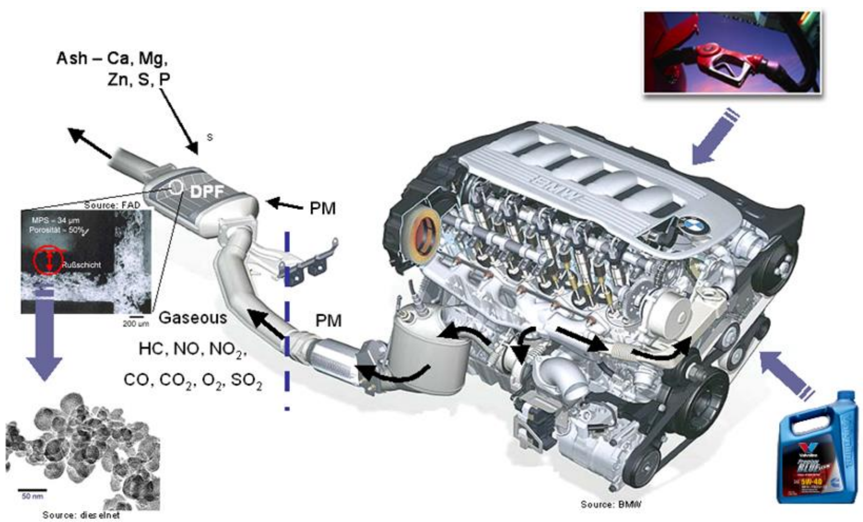
Figure 16: Three element wing sails have been found to be more efficient than two element sails in the C-Class catamaran.

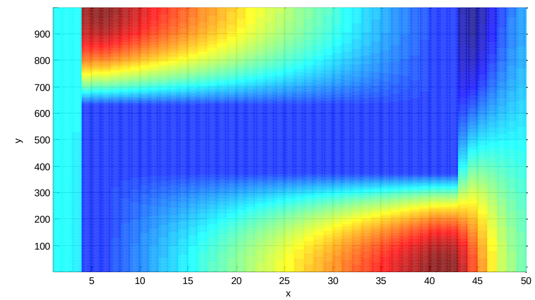
Simulation of Two Dimensional Flow inside Diesel Particulate Filter

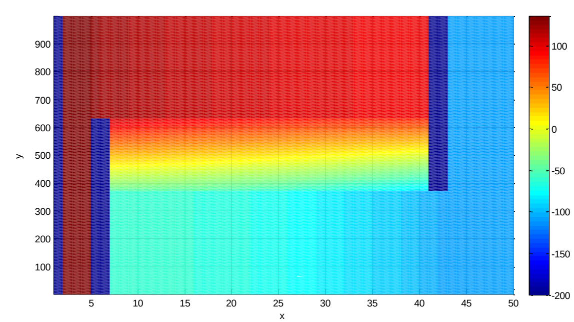
An MIT Student, 2011



Deposition of soot and ash in DPF channels



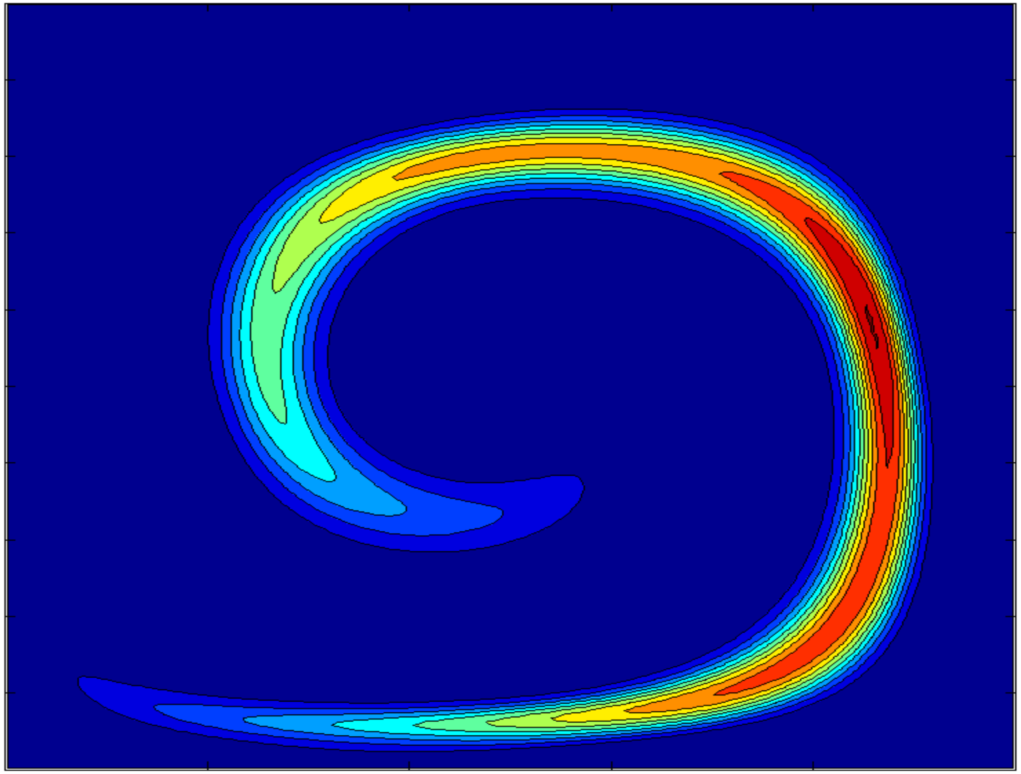




Dye Hard:

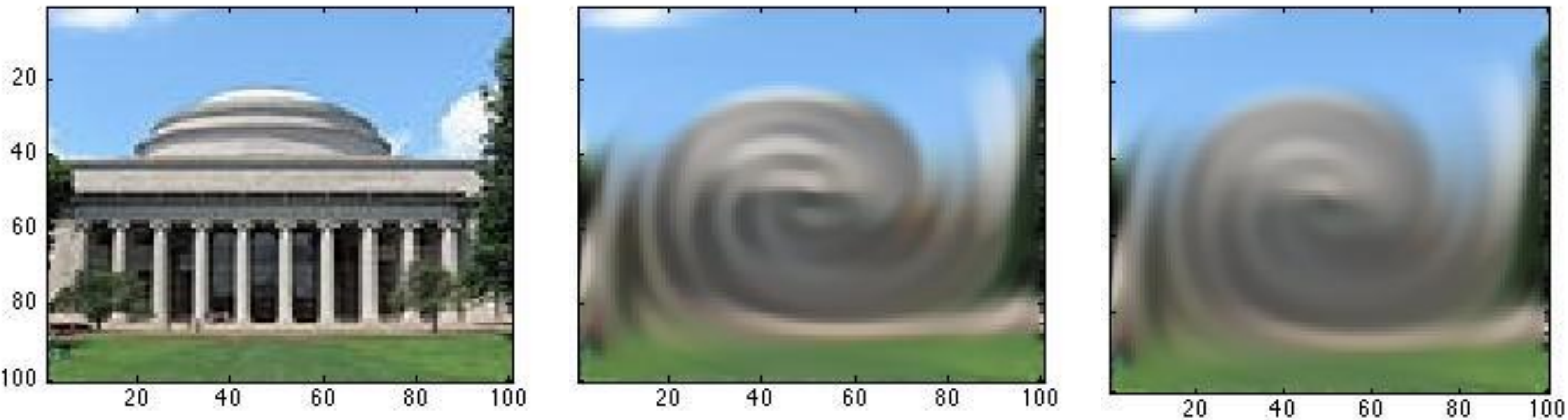
An Exploration into 2D Finite Volume Schemes and Flux Limiters

An MIT Student, 2011



Evaluating Finite-Volume Schemes and Flux Limiters for 2D Advection of Tracers

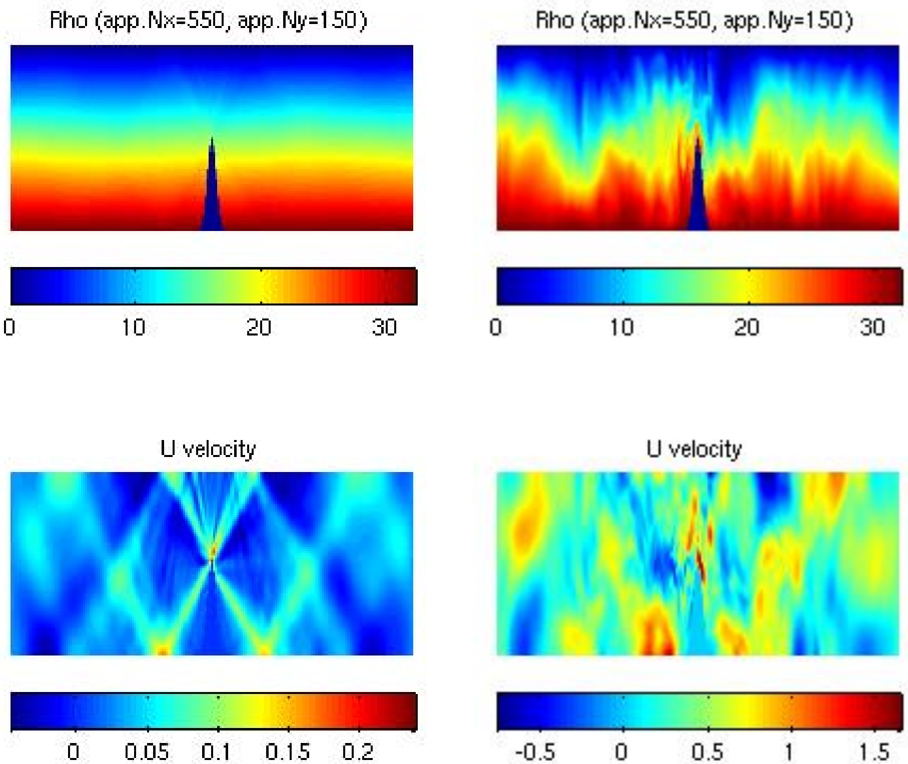
An MIT Student, 2013



Advection scheme applied to (left) an image of MIT's Building 10 using (center) the "superbee" flux limiter and (right) the monotonized center flux limiter. Image is 100 by 100 pixels.

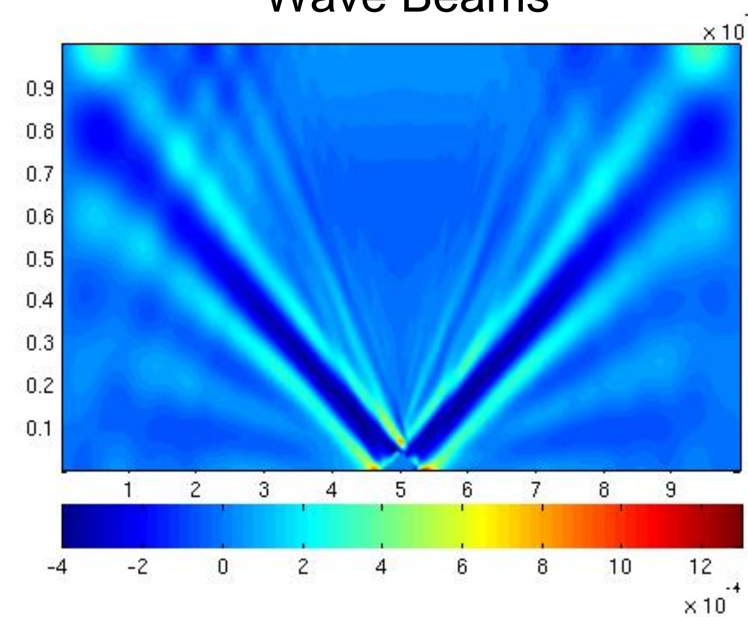
Internal Tides Simulated
Using the 2.29 Finite Volume Boussinesq Code
An MIT Student, 2011

Velocity and Density at Various Speeds



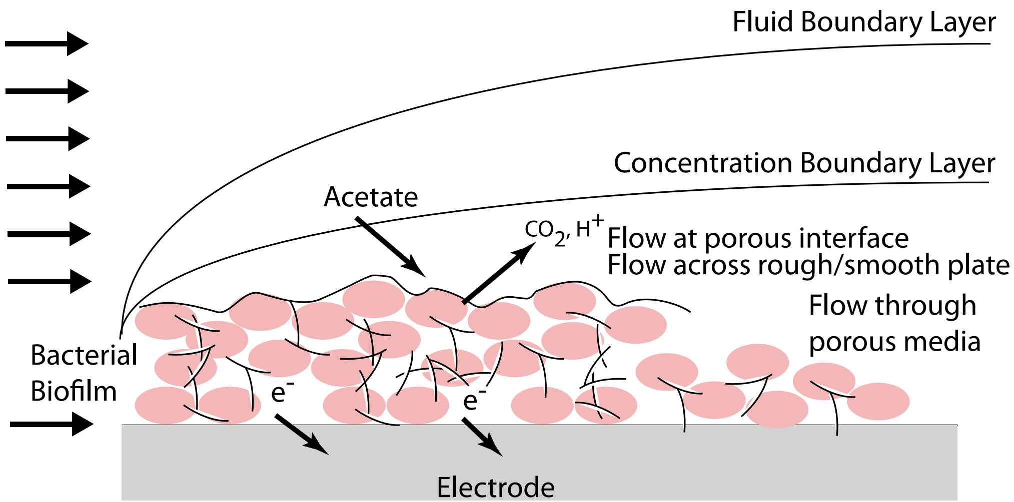
$U_0 = 2$ cm/s $\quad$ $U_0 = 24$ cm/s

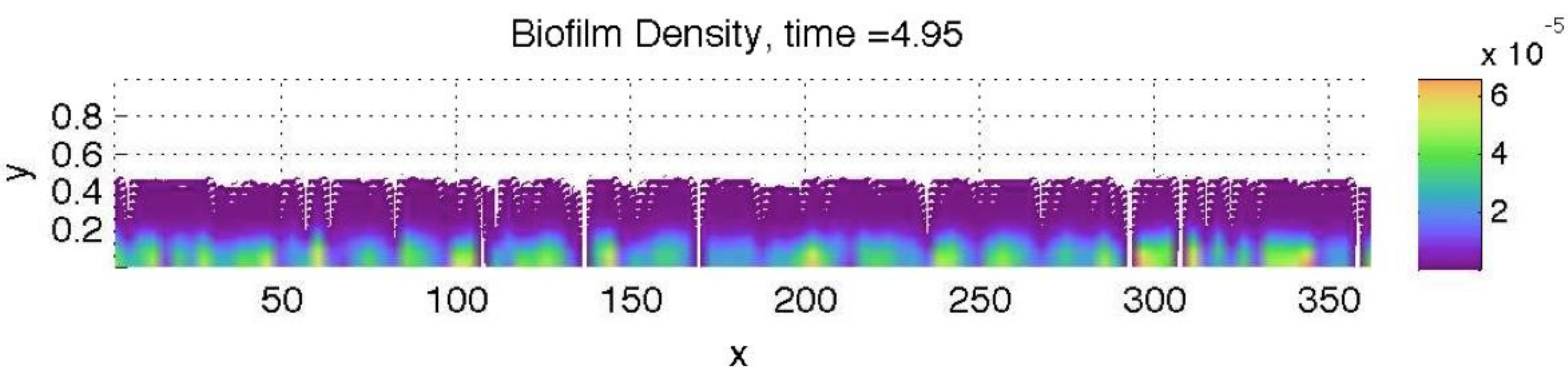
Wave Beams



Biofilm Growth in Shear Flows: Numerical 2.29 FV Simulations using a Porous Media Model

An MIT Student, III, 2013





High viscosity mu = 20

A 2-D Finite Volume Framework on Structured Non-Cartesian Grids for a Convection-Diffusion-Reaction Equation

An MIT Student, 2013

Non-Cartesian Grid:
Quarter Annulus

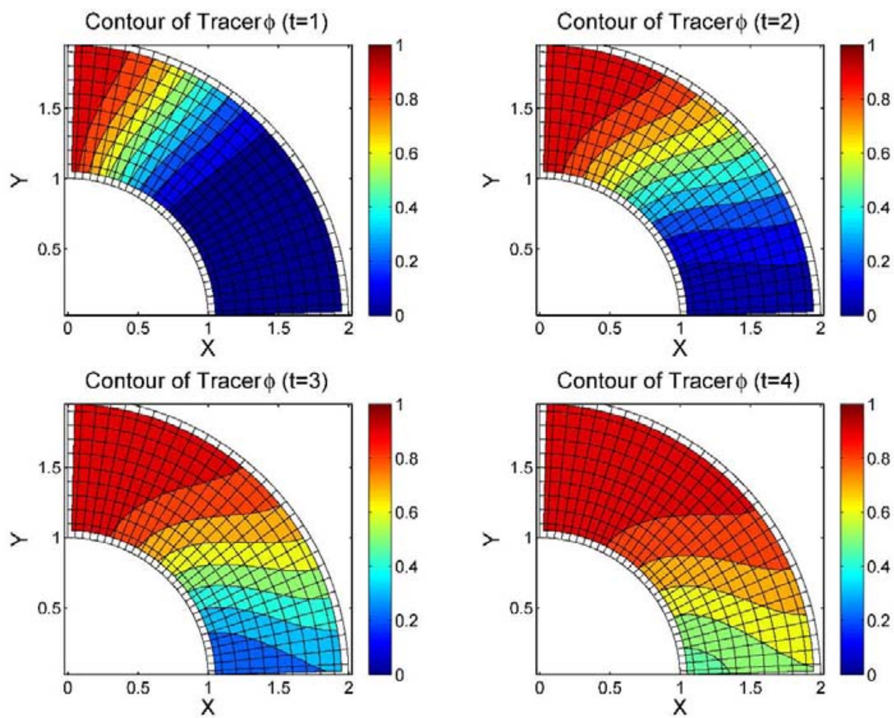


Figure: Time Evolution of ϕ

Numerical Fluid Mechanics – Outline Lectures 1-2

- Introduction to Computational Fluid Dynamics
- Introduction to Numerical Methods in Engineering
  - Digital Computer Models
  - Continuum and Discrete Representation
  - Number representations
  - Arithmetic operations
  - Errors of numerical operations. Recursion algorithms

- Error Analysis
  - Error propagation – numerical stability
  - Error estimation
  - Error cancellation
  - Condition numbers

What is CFD?

Computational Fluid Dynamics is a branch of computer-based science that provides numerical predictions of fluid flows

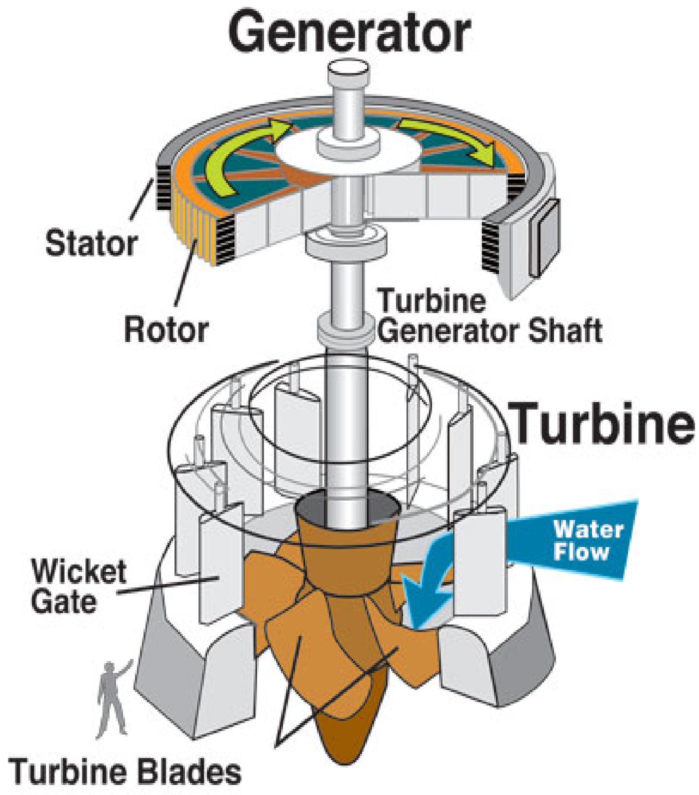
- Mathematical modeling (typically a system of non-linear, coupled PDEs, sometimes linear)
- Numerical methods (discretization and solution techniques)
- Software tools

CFD is used in a growing number of engineering and scientific disciplines

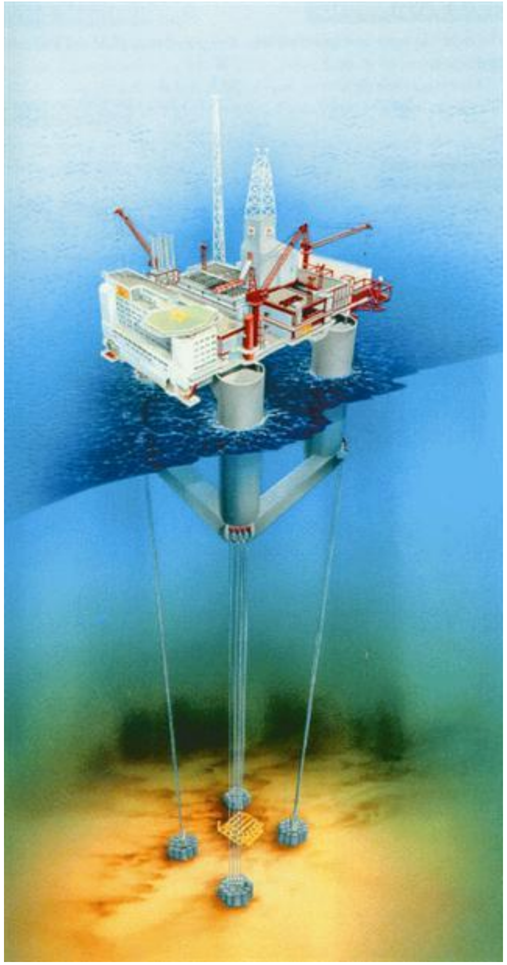
Several CFD software tools are commercially available, but still extensive research and development is ongoing to improve the methods, physical models, etc.

Examples of "Fluid flow" disciplines where CFD is applied

Engineering: aerodynamics, propulsion, Ocean engineering, etc.



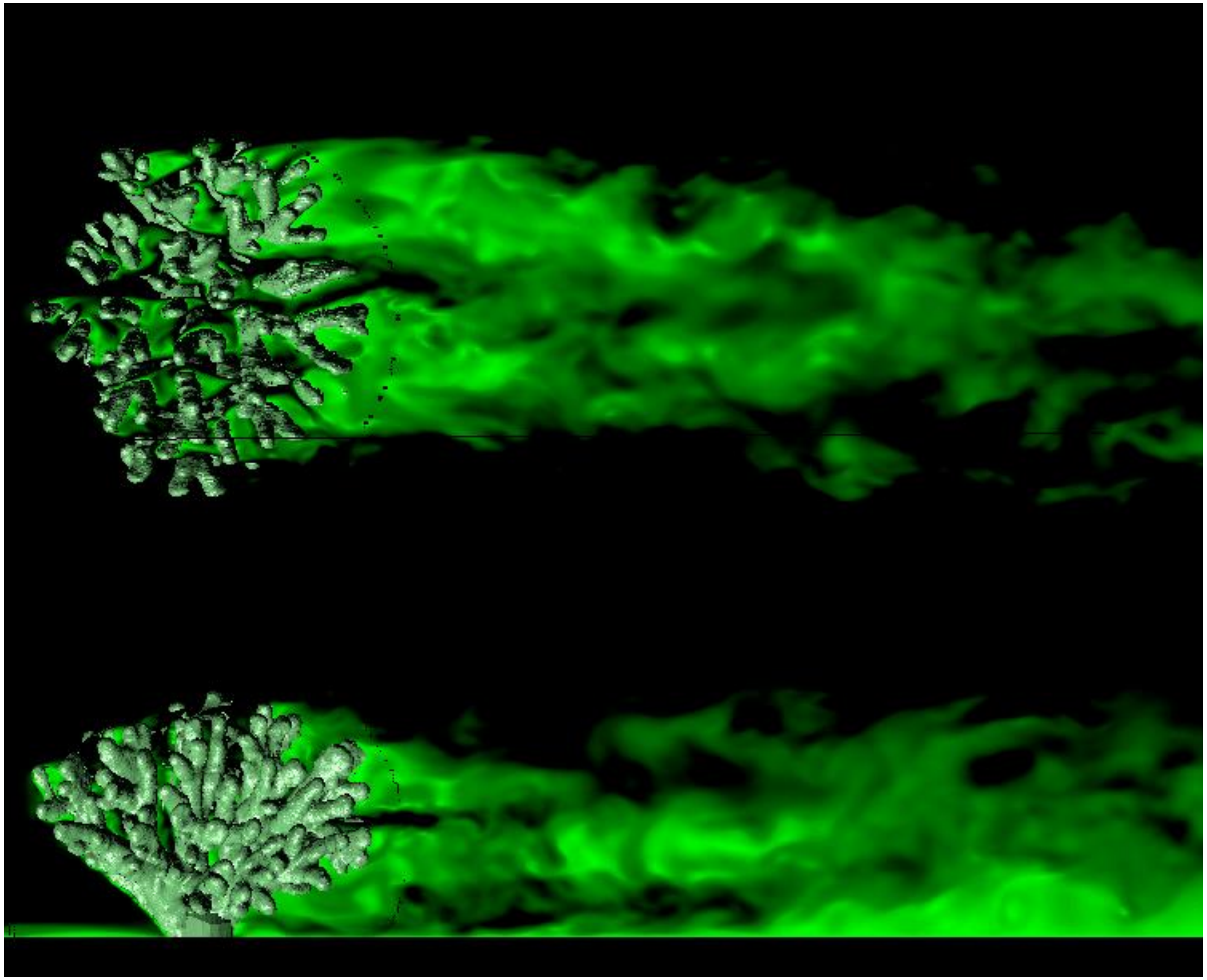
[Image](#) in the public domain.



Courtesy of Paul Sclavounos. Used with permission.

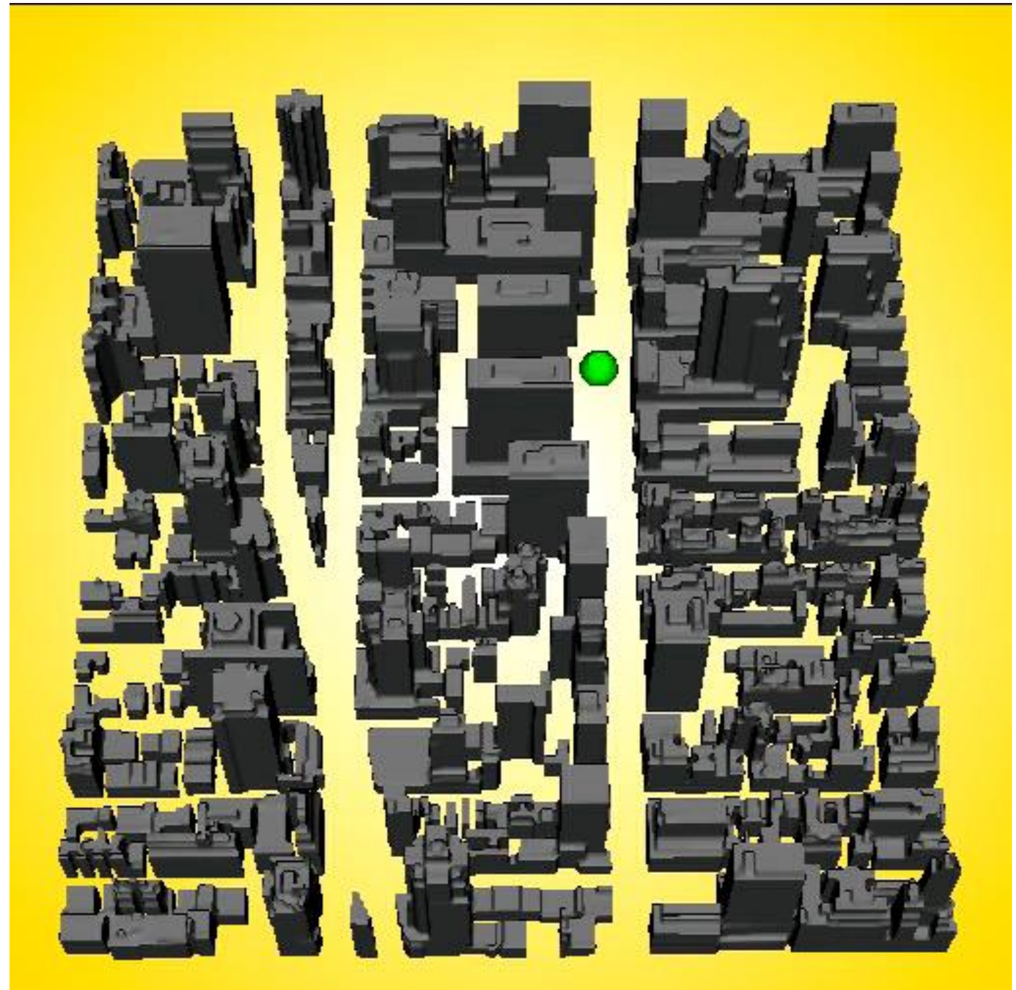
Examples of “Fluid flow” disciplines where CFD is applied

Biological systems: nutrient transport, pollution etc.



Examples of “Fluid flow” disciplines where CFD is applied

Building, City and Homeland security: hazard dispersion,etc.



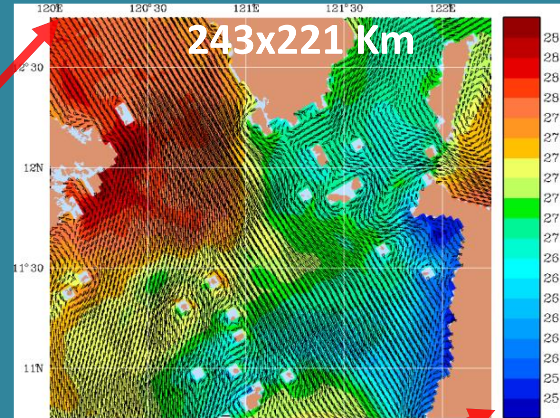
Examples of "Fluid flow" disciplines where CFD is applied

Meteorology, Oceanography and Climate:

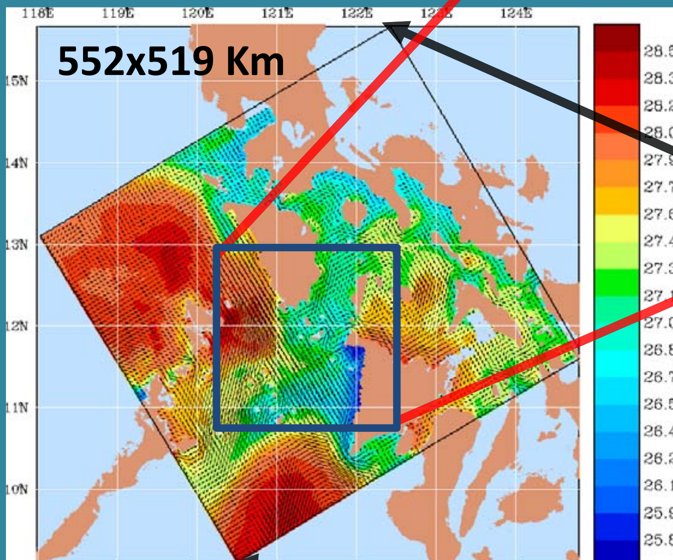
hurricanes, tsunamis, coastal management, climate change, etc.

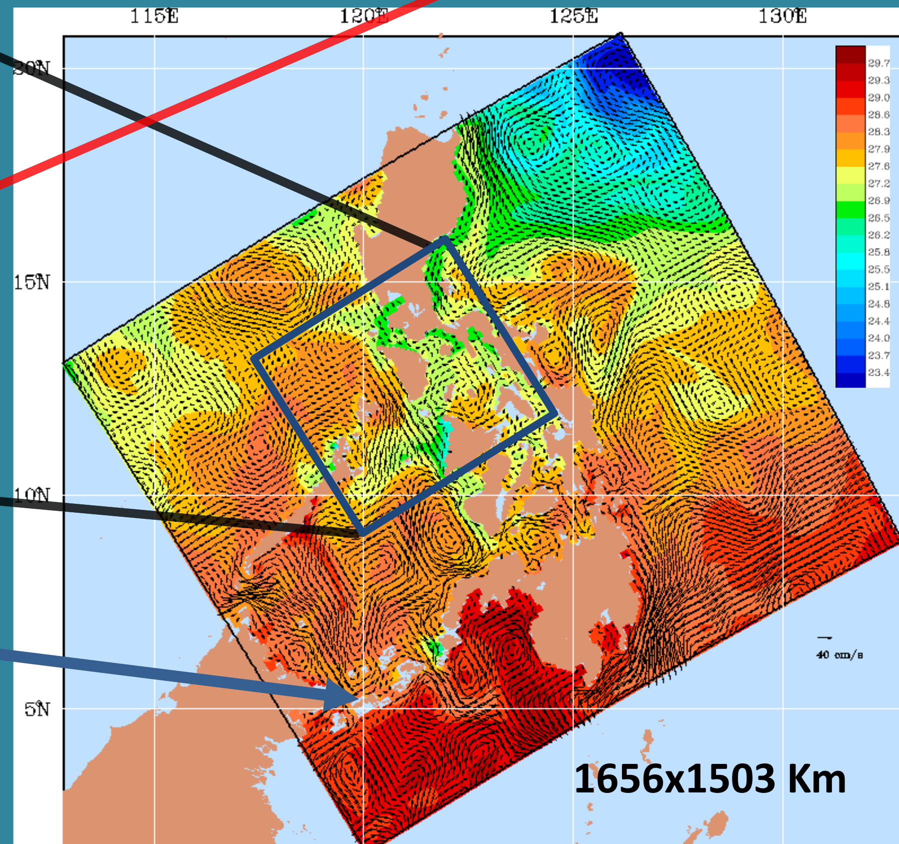
Multiscale Physical and Biological Dynamics in the Philippine Archipelago

(Lermusiaux et al, Oc-2011)

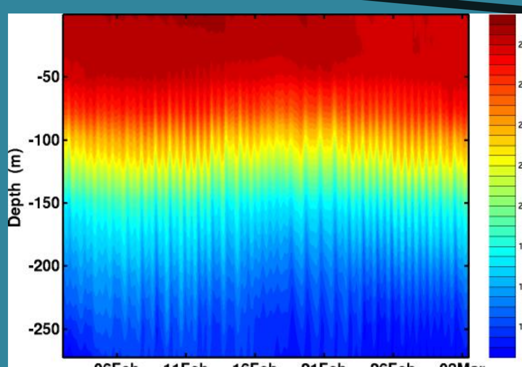


25m temperature from three implicit two-way nested simulations at 1-km 3-km, and 9-km resolutions





Time series of temperature profiles at the Sulu Sea entrance to Sibutu Passage

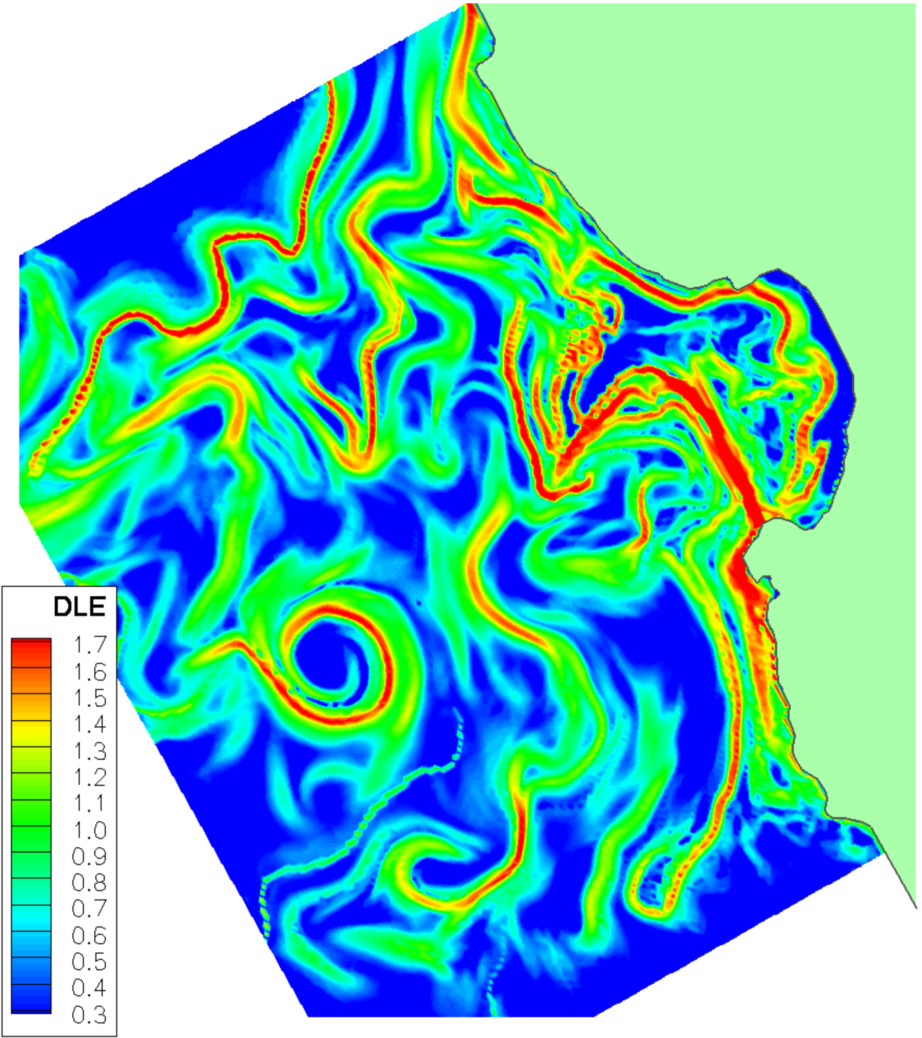


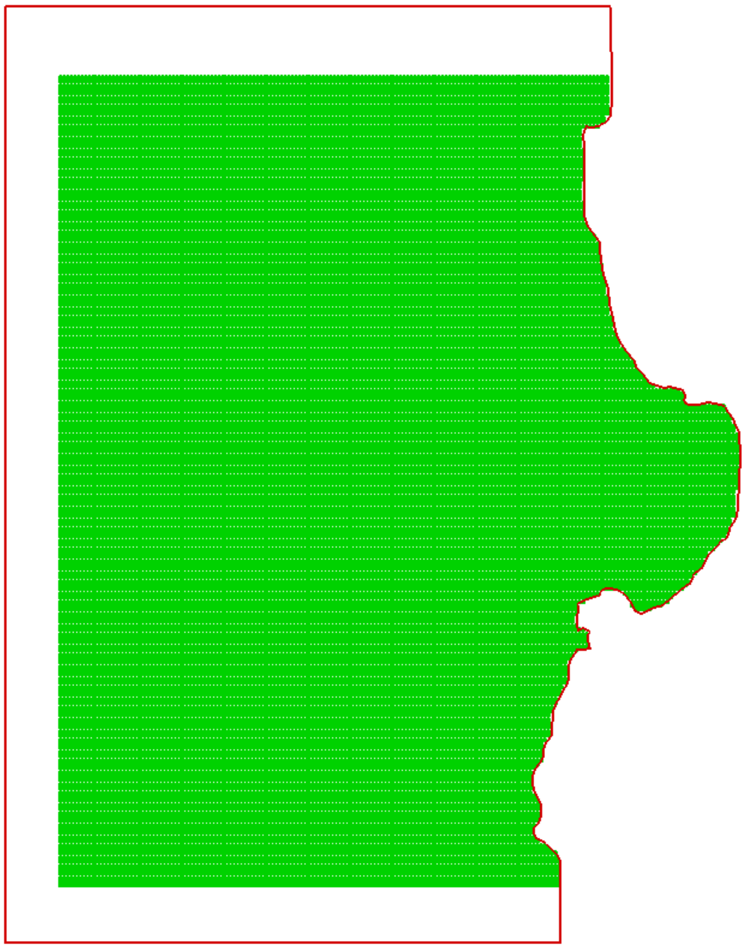
Haley and Lermusiaux, MSEAS, OD-2010

Source: Lermusiaux, P. "Uncertainty Estimation and Prediction for Interdisciplinary Ocean Dynamics." *Journal of Computational Physics* 217 (2006): 176-99.

Monterey Bay & California Current System

Flow field particle evolution (right) & its DLE for T= 3 days (below)





Promotional poster removed due to copyright restrictions; see examples of HD Stereo Theatre simulations at the following URL: http://gladiator.ncsa.illinois.edu/Images/cox/pics.html.

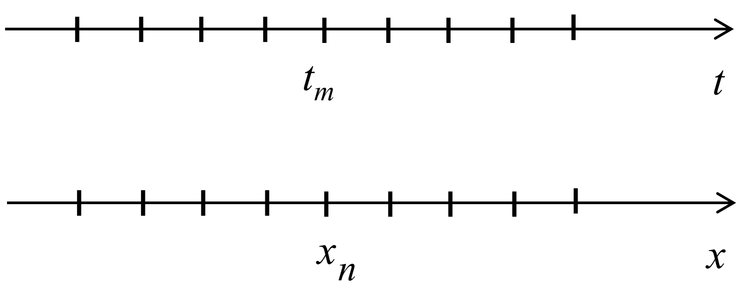
From Mathematical Models to Numerical Simulations

Continuum Model

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0$$

Sommerfeld Wave Equation (c= wave speed). This radiation condition is sometimes used at open boundaries of ocean models.

Discrete Model



$$t_m = t_0 + m\Delta t \ , \quad m = 0, 1, \ldots M-1$$

$$x_n = x_0 + n\,\Delta x \ , \quad n = 0, 1, \ldots N-1$$

$$\frac{\partial w}{\partial t} \simeq \frac{\Delta w}{\Delta t}, \quad \frac{\partial w}{\partial x} \simeq \frac{\Delta w}{\Delta x}$$

p parameters, e.g. variable c

Differential Equation

$$L(p, w, x, t) = 0$$

"Differentiation"
"Integration"

Difference Equation

$$L_{mn}(p_{mn}, w_{mn}, x_n, t_m) = 0$$

System of Equations

$$\sum_{j=0}^{N-1} F_i(w_j) = B_i$$

Linear System of Equations

$$\sum_{j=0}^{N-1} A_{ij} w_j = B_i$$

"Solving linear equations"

Eigenvalue Problems

$$\overline{\overline{\mathbf{A}}}\mathbf{u} = \lambda\mathbf{u} \Leftrightarrow (\overline{\overline{\mathbf{A}}} - \lambda\overline{\overline{\mathbf{I}}})\mathbf{u} = \mathbf{0}$$

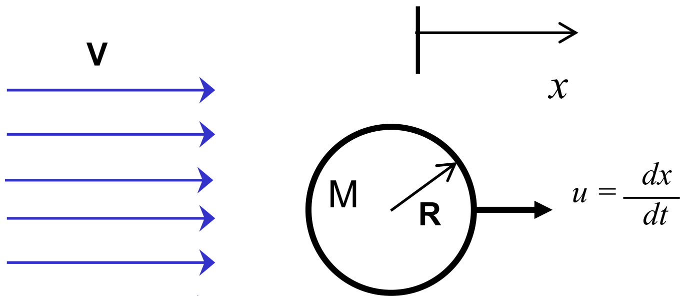
Non-trivial Solutions

$$\det(\overline{\overline{\mathbf{A}}} - \lambda\overline{\overline{\mathbf{I}}}) = 0$$

"Root finding"

Consistency/Accuracy and Stability => Convergence
(Lax equivalence theorem for well-posed linear problems)

Sphere Motion in Fluid Flow



Equation of Motion – 2nd Order Differential Equation

$$M\frac{d^2x}{dt^2} = 1/2\rho C_d \pi R^2 \left(V - \frac{dx}{dt}\right)^2$$

Rewite to 1st Order Differential Equations

$$\frac{dx}{dt} = u$$

$$\frac{du}{dt} = \frac{\rho C_d \pi R^2}{2M}(V^2 - 2uV + u^2)$$

Euler' Method - Difference Equations – First Order scheme

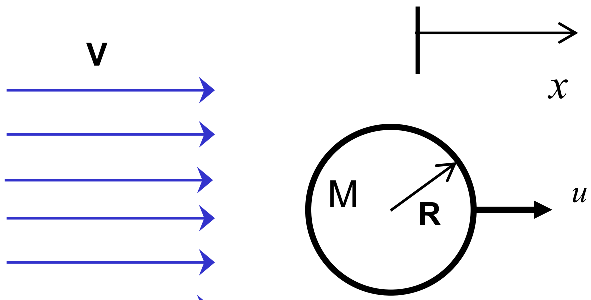
Taylor Series Expansion
(Here forward Euler)

$$u_{i+1} = u_i + \left(\frac{du}{dt}\right)_i \Delta t, \quad u(0) = 0$$

$$x_{i+1} = x_i + u_i \Delta t, \quad x(0) = 0$$

Sphere Motion in Fluid Flow

MATLAB Solutions



$$u = \frac{dx}{dt}$$

dudt.m

```
function [f] = dudt(t,u)
% u(1) = u
% u(2) = x
% f(2) = dx/dt = u
% f(1) = du/dt=rho*Cd*pi*r/(2m)*(v^2-2uv+u^2)
rho=1000;
Cd=1;
m=5;
r=0.05;
fac=rho*Cd*pi*r^2/(2*m);
v=1;

f(1)=fac*(v^2-2*u(1)+u(1)^2);
f(2)=u(1);
f=f';
```

sph_drag_2.m

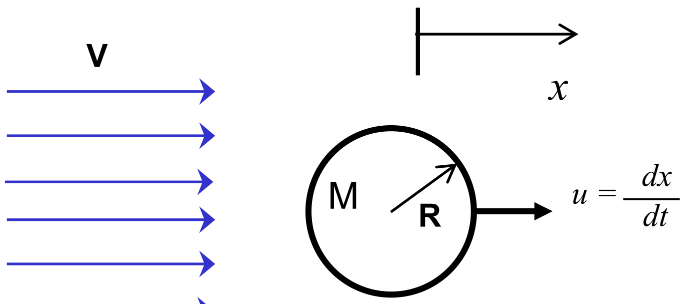
$$u_{i+1} = u_i + \left(\frac{du}{dt}\right)_i \Delta t, \quad u(0) = 0$$

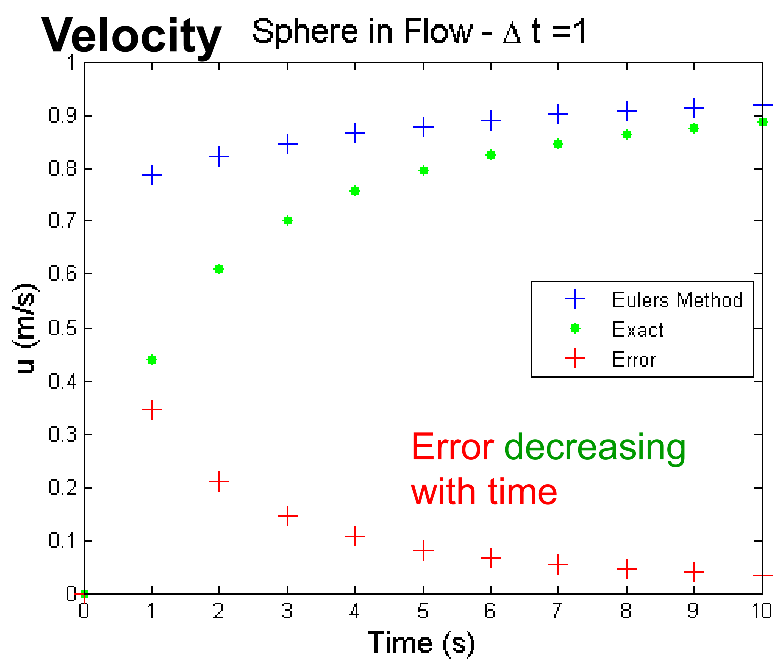
$$x_{i+1} = x_i + u_i \Delta t, \quad x(0) = 0$$

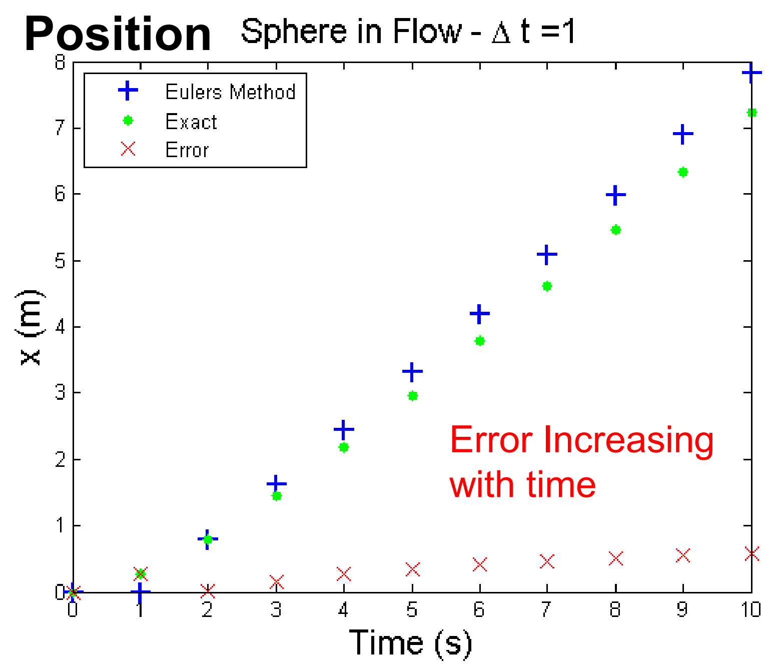
```
x=[0:0.1:10];
%step size
h=1.0;
% Euler's method, forward finite difference
t=[0:h:10];
N=length(t);
u_e=zeros(N,1);
x_e=zeros(N,1);
u_e(1)=0;
x_e(1)=0;
for n=2:N
    u_e(n)=u_e(n-1)+h*fac*(v^2-2*v*u_e(n-1)+u_e(n-1)^2);
    x_e(n)=x_e(n-1)+h*u_e(n-1);
end
% Runge Kutta
u0=[0 0]';
[tt,u]=ode45(@dudt,t,u0);

figure(1)
hold off
a=plot(t,u_e,'+b');
hold on
a=plot(tt,u(:,1),'.g');
a=plot(tt,abs(u(:,1)-u_e),'+r');
...
figure(2)
hold off
a=plot(t,x_e,'+b');
hold on
a=plot(tt,u(:,2),'.g');
a=plot(tt,abs(u(:,2)-x_e),'xr');
...
```

Sphere Motion in Fluid Flow

Error Propagation







2.29 Numerical Fluid Mechanics
Errors

From mathematical models to numerical simulations (e.g. 1D Sphere in 1D flow)

Continuum Model – Differential Equations

=> Difference Equations (often uses Taylor expansion and truncation)

=> Linear/Non-linear System of Equations

=> Numerical Solution (matrix inversion, eigenvalue problem, root finding, etc)

Motivation: What are the uncertainties in our computations and are they tolerable? How do we know?

Error Types

- Round-off error: due to representation by computers of numbers with a finite number of digits
- Truncation error: due to approximation/truncation by numerical methods of "exact" mathematical operations/quantities
- Other errors: model errors, data/parameter input errors, human errors.

Numerical Fluid Mechanics
Error Analysis – Outline

- Approximation and round-off errors
  - Significant digits, true/absolute and relative errors
  - Number representations
  - Arithmetic operations
  - Errors of numerical operations
  - Recursion algorithms

- Truncation Errors, Taylor Series and Error Analysis
  - Error propagation – numerical stability
  - Error estimation
  - Error cancellation
  - Condition numbers

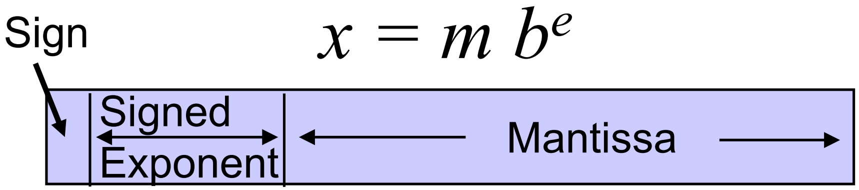
Approximations and Round-off errors

- Significant digits: Numbers that can be used with confidence
  - e.g. 0.001234 and 1.234 $4.56\ 10^3$ and 4,560
  - Omission of significant digits in computers = round-off error
- Accuracy: “how close an estimated value is to the truth”
- Precision: “how closely estimated values agree with each other”
- **True error**: $E_t = \text{Estimate} - \text{Truth} \quad = \hat{x} - x^t$
- **True relative error**: $\varepsilon_t = \dfrac{\text{Estimate} - \text{Truth}}{\text{Truth}} \quad = \dfrac{\hat{x} - x^t}{x^t}$
- In reality, x^t unknown => use best estimate available $\hat{x}_a$
- Hence, what is used is: $\varepsilon_a = \dfrac{\hat{x} - \hat{x}_a}{\hat{x}_a}$
- Iterative schemes, $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$, stop when $\left|\varepsilon_a\right| = \left|\dfrac{\hat{x}_n - \hat{x}_{n-1}}{\hat{x}_n}\right| \leq \varepsilon_s$
- For n digits: $\varepsilon_s = \dfrac{1}{2} 10^{-n}$

Number Representations

- Number Systems:
  - Base-10: $1{,}234_{10} = 1 \times 10^3 + 2 \times 10^2 + 3 \times 10^1 + 4 \times 10^0$
  - Computers (0/1): base-2 $1101_2 = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 13_{10}$
- Integer Representation (signed magnitude method):
  - First bit is the sign (0,1), remaining bits used to store the number
  - For a 16-bits computer:
    - Example: -13_{10} = 1 0 0 0 0 0 0 0 0 0 0 0 1 1 0 1
    - Largest range of numbers: $2^{15}-1$ largest number => -32,768 to 32,767 (from 0 to 111111111111111)
- Floating-point Number Representation

$$x = m\, b^e$$



m	Mantissa/Significand = fractional part
b	Base
e	Exponent

Floating Number Representation

Examples

Decimal $0.00527 = 0.527_{10} \times 10^{-2_{10}}$

Binary $10.1_2 = 0.101_2 \times 2^{2_{10}} = 0.101_2 \times 2^{10_2}$

Convention: Normalization of Mantissa m (so as to have no zeros on the left)

0.01234 => 0.1234 10^{-1}

12.34 => 0.1234 10^{2}

Decimal $0.1 \leq m < 1.0$

Binary $0.1_2 = 0.5_{10} \leq m < 1.0$

=> General $b^{-1} \leq m < b^0$

Example

(Chapra and Canale, pg 61)

Consider hypothetical Floating-Point machine in base-2

7-bits word =

- 1 for sign
- 3 for signed exp.
  (1 sign, 2 for exp.)
- 3 for mantissa

Largest and smallest *positive* number represented are ?

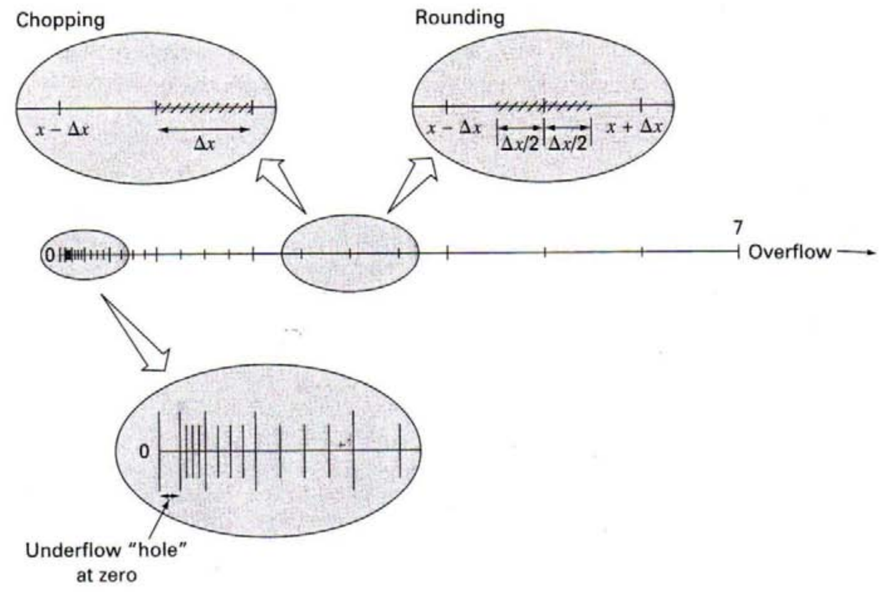


FIGURE 3.7
The hypothetical number system developed in Example 3.4. Each value is indicated by a tick mark. Only the positive numbers are shown. An identical set would also extend in the negative direction.

The mantissa is decreased back to its smallest value of 100. Therefore, the next number is

$$0110100 = (1 \times 2^{-1} + 0 \times 2^{-2} + 0 \times 2^{-3}) \times 2^{-2} = (0.125000)_{10}$$

This still represents a gap of 0.125000 − 0.109375 = 0.015625. However, now when higher numbers are generated by increasing the mantissa, the gap is lengthened to 0.03125,

$$0110101 = (1 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3}) \times 2^{-2} = (0.156250)_{10}$$

$$0110110 = (1 \times 2^{-1} + 1 \times 2^{-2} + 0 \times 2^{-3}) \times 2^{-2} = (0.187500)_{10}$$

$$0110111 = (1 \times 2^{-1} + 1 \times 2^{-2} + 1 \times 2^{-3}) \times 2^{-2} = (0.218750)_{10}$$

This pattern is repeated as each larger quantity is formulated until a maximum number is reached,

$$0011111 = (1 \times 2^{-1} + 1 \times 2^{-2} + 1 \times 2^{-3}) \times 2^{3} = (7)_{10}$$

The final number set is depicted graphically in Fig. 3.7.

Source: Chapra, Steven C., and Raymond P. Canale. *Numerical Methods for Engineers*. Vol. 2. McGraw-Hill, 2012.

Example

(Chapra and Canale, pg 61)

Consider hypothetical Floating-Point machine in base-2

7-bits word =

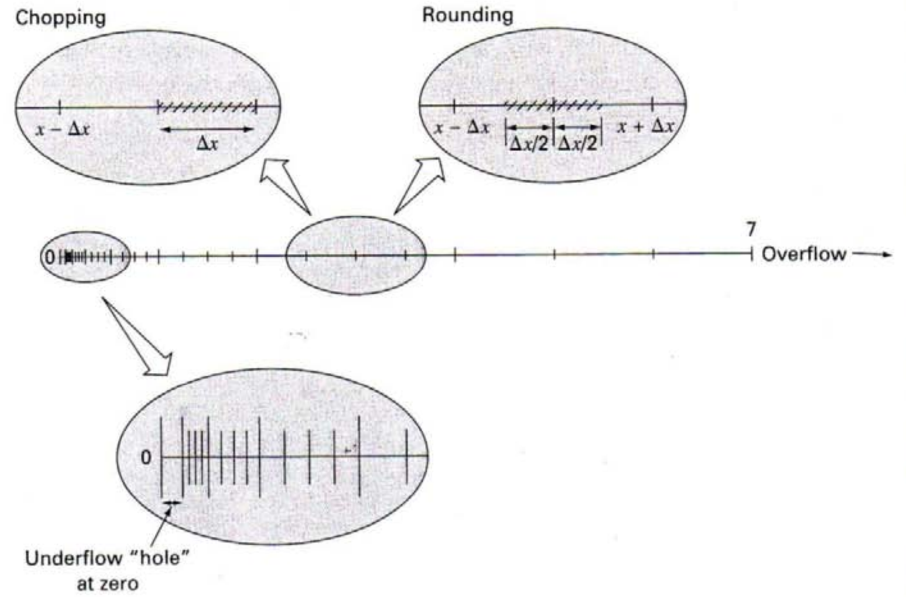
- 1 for sign
- 3 for signed exp.
- 3 for mantissa

Largest number is: $7 = 2^{(2+1)} (2^{-1} + 2^{-2} + 2^{-3})$

Sign nb	Sign exp	2^1	2^0	2^{-1}	2^{-2}	2^{-3}
0	0	1	1	1	1	1

Smallest positive number is: $0.5\ 2^{-3}$

Sign	Sign exp	2^1	2^0	2^{-1}	2^{-2}	2^{-3}
0	1	1	1	1	0	0



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Consequence of Floating Point Reals

- Limited range of quantities can be represented
  - Min number (Underflow Error) and Max number (Overflow)
- Finite number of quantities can be represented within the range (**limited precision**) => "**Quantizing errors**"
  - Quantizing errors treated either by round-off or chopping.
- Interval Δx between numbers increases as numbers grow in magnitude
  - For t = number of significant digits in mantissa and rounding,

$\varepsilon = b^{1-t} =$ Machine Epsilon

Relative Error $$\frac{|\Delta x|}{|x|} \leq \frac{\varepsilon}{2}$$

Absolute Error $$|\Delta x| \leq \frac{\varepsilon}{2}|x|$$

```
%Determine machine epsilon in matlab
%
eps=1;
while (eps+1>1)
  eps=eps/2;
end
eps*2
```

2.29 Numerical Fluid Mechanics
Spring 2015