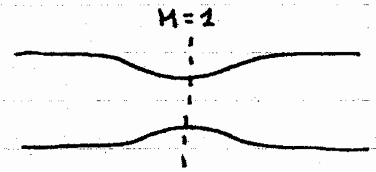
Condensation discontinuities

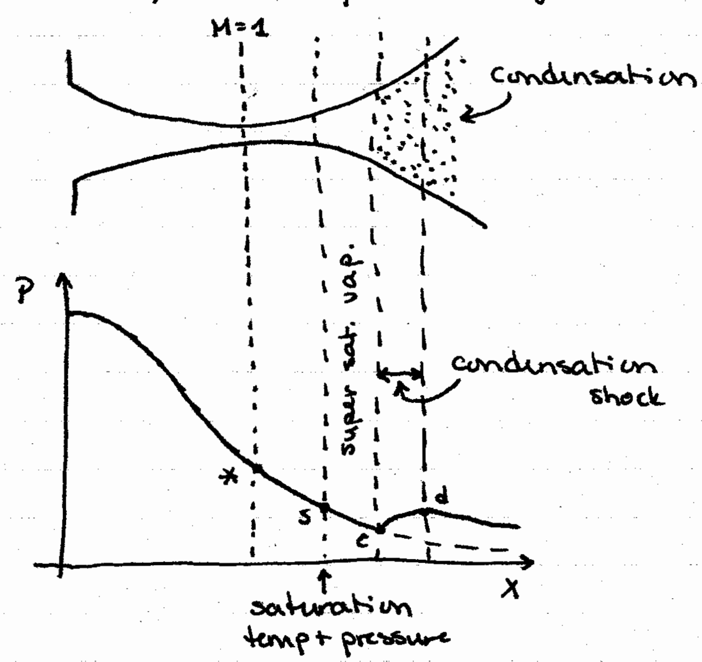


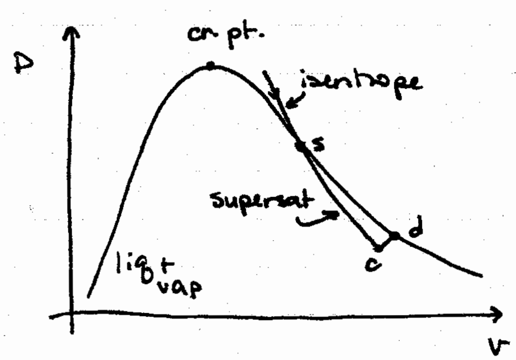
Suppose $s \approx$ const
$P \downarrow$
$T \downarrow$

e.g. $P_0 = 1$ atm $P = 0.027$ atm
$T_0 = 300$ k $T = 107$ k
$M = 3$

Low temp → condensation (e.g. air; oxygen sat. @ 50 k $\approx$ Mach 5)

If air is moist, H_2O condenses much sooner. Look @ simpler case; 1 component system (e.g. steam)





$X_c - X_s$: time for spontaneous nucleation to take place

"shock" :
- "wide"
- pressure jump is small ("weak")
- downstream flow supersonic
- similar to combustion (exothermic; energy release through latent heat).

Enthalpy jump condition:

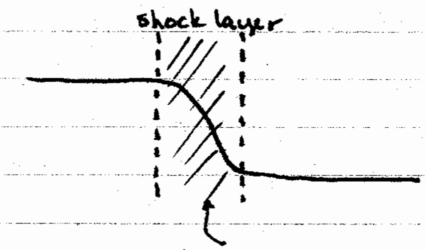
$$h_1 - h_2 = C_{pa}(T_1 - T_2) + \omega_1 L$$

latent heat → L

ω_1 : specific humidity, mass H_2O vap / mass gas

(compare w. combust. $\omega_1 L = \Delta h^\circ$, $\gamma_1 = \gamma_2$; $R_1 = R_2$)

Continuum shock structure



large viscous stresses + heat transfer } (due to large gradients in velocity + temp.)

$\boxed{\text{Estimate shock thickness}}$ Assume:

- shock thickness is small relative to the radius of curvature of the shock front (1D)
- shock is stationary $\left(\frac{\partial}{\partial t} = 0\right)$
- fluid is in equilibrium
- neglect thermal radiation + diffusion

Cons. of mass, momentum + energy:

$$(\rho u)_x = 0$$

$$\rho u u_x + P_x - \left(\tfrac{4}{3}\mu' u_x\right)_x = 0 \qquad \left(\text{Note } \tfrac{4}{3}\mu' = \tfrac{4}{3}\underbrace{\mu}_{\text{shear viscosity}} + \overbrace{\mu_b}^{\text{bulk viscosity}}\right)$$

$$\rho u \left(h + \tfrac{u^2}{2}\right)_x - \left(\tfrac{4}{3}\mu' u u_x\right)_x - (\underbrace{k}_{\text{thermal conductivity}} T_x)_x = 0$$

Integrate once:

$$\rho u = J = \text{mass flux}$$

$$P + \rho u^2 - \tfrac{4}{3}\mu' u_x = k = \text{momentum flux}$$

$$h + \frac{u^2}{2} - \frac{4}{3}\frac{\mu' u_x}{\rho} - \frac{k T_x}{\rho u} = L = \text{specific energy}$$

(note: μ' and k may depend on x). B.c.'s:

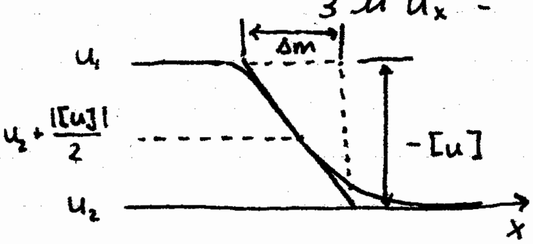
$$\left.\begin{matrix} u = u_1 \\ P = P_1 \\ \mu' = \mu_1' \\ \vdots \end{matrix}\right\} x = -\infty \qquad \left.\begin{matrix} u = u_2 \\ P = P_2 \\ \mu' = \mu_2' \\ \vdots \end{matrix}\right\} x = +\infty \qquad \left(\begin{matrix}\text{gradients} \\ \text{vanish @ } \pm\infty\end{matrix}\right)$$

Weak-shock thickness estimate:

Solve mom. eq. for u_x

$$\tfrac{4}{3}\mu' u_x = P + \rho u^2 - k \qquad k = P_1 + \rho_1 u_1^2$$

$$\tfrac{4}{3}\mu' u_x = P - P_1 + \rho u^2 - \rho_1 u_1^2 = P - P_1 + \rho_1 u_1 (u - u_1) \qquad (*)$$



$$-u_x \approx \frac{-[u]}{\Delta m} \Rightarrow \boxed{\Delta_m \equiv \frac{[u]}{u_x}}$$

(use this to sub for u_x) ... still need P...

a) midpt. $u = u_1 + [u]/2$

$$v = v_1 + [u]/2J \qquad (\text{since } u = Jv)$$

Taylor expand $P(v,s)$ (recall that this is a weak shock)

$$P - P_1 = \left(\frac{\partial P}{\partial v}\right)_s (v - v_1) + \frac{1}{2}\left(\frac{\partial^2 P}{\partial v^2}\right)_s (v-v_1)^2 + \dots \left(\frac{\partial P}{\partial s}\right)_v \underbrace{\cancel{(s-s_1)}}_{\mathcal{O}([v]^3)} + \dots$$

small

$$P - P_1 = \left(\frac{\partial P}{\partial v}\right)_s \left(\frac{[u]}{2J}\right) + \frac{1}{2}\left(\frac{\partial^2 P}{\partial v^2}\right)_s \left(\frac{[u]}{2J}\right)^2$$

$$c^2 = \left(\frac{\partial P}{\partial v}\right)_s \left(-\frac{1}{\rho^2}\right)$$

$$\Gamma \equiv \frac{c^4}{2v^3}\left(\frac{\partial^2 v}{\partial p^2}\right)_s = \frac{v^3}{2c^2}\left(\frac{\partial^2 P}{\partial v^2}\right)_s$$

= fundamental gas dynamic derivative

$$= -\rho_1^2 c_1^2 \frac{[u]}{2J} + \frac{1}{\cancel{2}} \frac{\Gamma \cancel{2} c^2}{v^3}\left(\frac{[u]}{2J}\right)^2$$

$$P - P_1 = -\rho_1 c_1^2 \frac{[u]}{2u_1} + \rho_1 c_1^2 \Gamma_1 \frac{[u]^2}{4u_1^2} + \dots$$

Sub. into $(*)$

$$\frac{\frac{8}{3}\mu' \overset{\frac{[u]}{\Delta m}}{\widetilde{u_x}}}{\rho_1 c_1} = -\cancel{\rho_1} c_1^2 \frac{[u]}{2u_1} + \cancel{\rho_1} c_1^2 \Gamma_1 \frac{[u]^2}{\underset{2}{\cancel{4}} u_1^2} + \cancel{\rho_1} u_1 \frac{(u-u_1)\,2}{c_1 [u]}$$

$$\frac{8\mu'}{3\rho_1 c_1 \Delta m} = -\frac{1}{M_{in}} + \frac{[u]\Gamma_1}{2M_{in}^2 c_1} + M_{in} \frac{\cancel{2}\,\frac{1}{\cancel{2}}\cancel{[u]}}{\cancel{[u]}}$$

For a weak shock: $\frac{-[u]}{c_1} \approx 2(M_{in}-1)/\Gamma_1 + \dots$

$$\frac{8\mu'}{3\rho_1 c_1 \Delta m} = -\frac{1}{M_{in}} - \frac{(M_{in}-1)}{M_{in}^2} + M_{in}$$

$$\boxed{\frac{8\mu'}{3\rho_1 c_1 \Delta m} \approx M_{in} - 1}$$

$\Rightarrow$ estimate for Δm
(use $M_{in} \approx 1 + \Delta M$ here)

mean free path

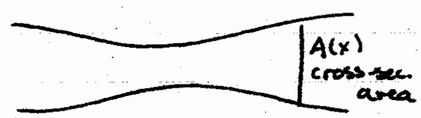
For a dilute gas: $\mu' \approx \rho c \Lambda \approx \rho_1 c_1 \Lambda_1$

$$\Rightarrow \boxed{\frac{\Delta_M}{\Lambda_1} \approx \frac{8}{3(M_{in}-1)}}$$

Can find a more detailed solution for the case of a perfect gas (Taylor's solution for a weak shock). Details in text.

<u>1D unsteady flow</u> (chpt. 8)

Allow waves of finite amplitude (unlike acoustics)
→ method of characteristics



- Flow is quasi-one-dimensional
- homentropic ($\frac{Ds}{Dt} = \nabla s = 0$)

Continuity: $\frac{\partial}{\partial t}(\Delta x A \rho) = -(\rho u A)\Big|_x^{x+\Delta x}$

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} = -\frac{\rho u}{A}\frac{dA}{dx} \qquad (1)$$

Cons. of momentum: $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} = g \qquad (2)$

body force in x direction

Homentropic ⇒ only one indep. thermodynamic variable

Choose P then $\rho = \rho(P)$

$$d\rho = \left(\partial \rho / \partial P\right)_s dP = \frac{1}{c^2} dP$$

From (1): $\frac{1}{c\rho}\frac{\partial P}{\partial t} + \frac{u}{\rho c}\frac{\partial P}{\partial x} + c\frac{\partial u}{\partial x} = \frac{c u A'}{A}$

Add to (2)

$$\left(\frac{\partial u}{\partial t} \pm \frac{1}{\rho c}\frac{\partial P}{\partial t}\right) + (u \pm c)\left(\frac{\partial u}{\partial x} \pm \frac{1}{\rho c}\frac{\partial P}{\partial x}\right) = g \mp \frac{cuA'}{A}$$

Define $\quad F = F(P,s) \Rightarrow F \equiv \int_{P_0}^{P} \frac{dP}{\rho c} \qquad dF = \frac{dP}{\rho c}$

$\uparrow$ fixed reference state

$$\frac{\partial F}{\partial t} = \frac{dF}{dP}\frac{\partial P}{\partial t} = \frac{1}{\rho c}\frac{\partial P}{\partial t} \qquad \frac{\partial F}{\partial x} = \frac{dF}{dP}\frac{\partial P}{\partial x} = \frac{1}{\rho c}\frac{\partial P}{\partial x}$$

$$\left[\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x}\right](u \pm F) = g \mp \frac{cuA'}{A}$$

starting to look like our old wave eq!

Define : $\quad \frac{D^+}{Dt} \equiv \frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x}$

$$\frac{D^-}{Dt} \equiv \frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x}$$

$$\boxed{\begin{aligned} \frac{D^+}{Dt}(u+F) &= g - \frac{cuA'}{A} \\ \frac{D^-}{Dt}(u-F) &= g + \frac{cuA'}{A} \end{aligned}}$$

(In the limit of small disturbances $\rightarrow$ acoustic waves)

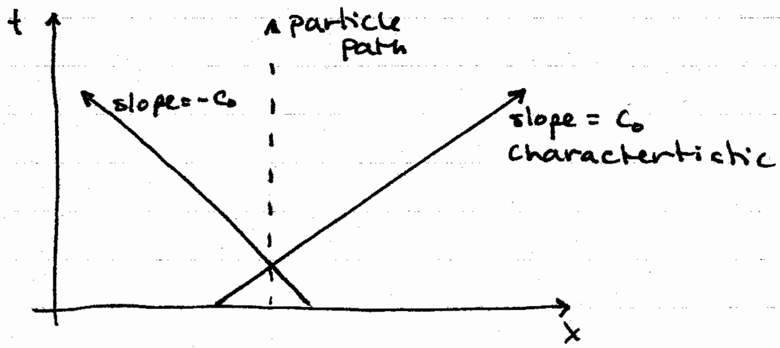
Recall the material derivative: $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$

= time rate of change viewed by an observer moving w. a fluid particle.

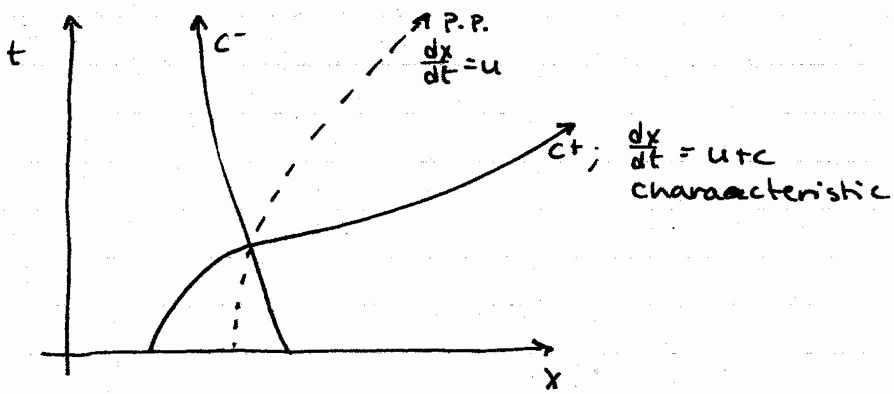
Similarly, $\frac{D^+}{Dt}$ is the time rate of change viewed by an observer moving w. a right moving wave @ velocity $c+u$. (Let $c+u = C^+$; $c-u = C^-$)

Ditto for $\frac{D^-}{Dt}$.

For acoustic waves; $u+c \approx c_0$



For nonlinear waves, $u+c \neq$ const $\therefore$ characteristics are no longer straight lines



Boxed equations give rate of change w.r.t. time along characteristics.

Alternate forms of F:

$$c^2 = \frac{-v^2}{(\partial v/\partial P)_s}$$

Take derive w.r.t. P $\Rightarrow$ $\left(\frac{\partial c^2}{\partial P}\right)_s = 2v(\Gamma - 1)$

$$\rho \cancel{2} c\, dc = dP \cancel{2} (\Gamma - 1)$$

$$\Rightarrow \quad dP = \frac{\rho c\, dc}{\Gamma - 1}$$

$$\Rightarrow \quad \boxed{dF = \frac{dc}{\Gamma - 1}}$$

For a perfect gas, $\Gamma = (\gamma+1)/2$

$$\int dF = \int_{c_0}^{c} \frac{dc}{\left(\frac{\gamma+1}{2}\right)-1}$$ take ~~xxxx~~ $c_0 = 0$ as reference state

$$\boxed{F = \frac{2}{\gamma-1} c}$$

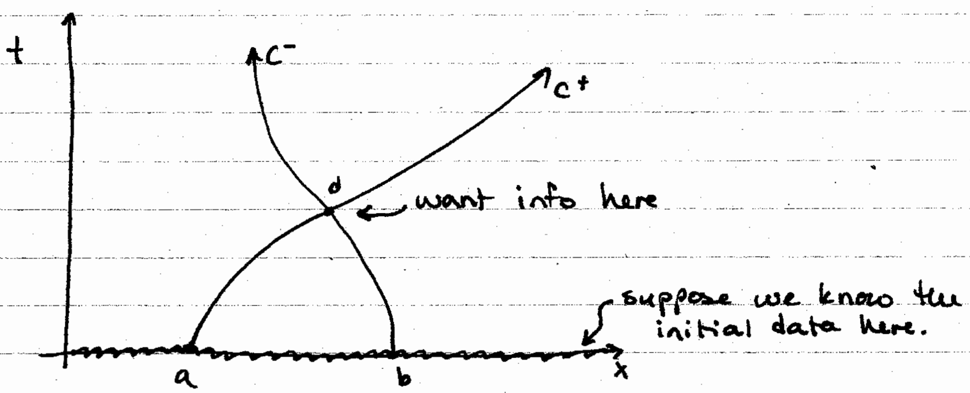
For constant area and $g = 0$
(usually true)

$$\frac{D^+}{Dt}(u+F) = 0 \qquad \frac{D^-}{Dt}(u-F) = 0$$

Thus $u - F \equiv J^-$ and $u + F \equiv J^+$ do <u>not</u> vary along characteristics. These are <u>Riemann</u> <u>Invariants</u>

(For a perfect gas: $J^+ = u + \frac{2}{\gamma-1} c$

$J^- = u - \frac{2}{\gamma-1} c$)



$$J_a^+ = J_d^+ \qquad J_d^- = J_b^-$$

$$u_d + F_d = J_a^+ \qquad u_d - F_d = J_b^-$$

$$\Rightarrow u_d = \tfrac{1}{2}(J_a^+ - J_b^-) \qquad F_d = \tfrac{1}{2}(J_a^+ - J_b^-)$$

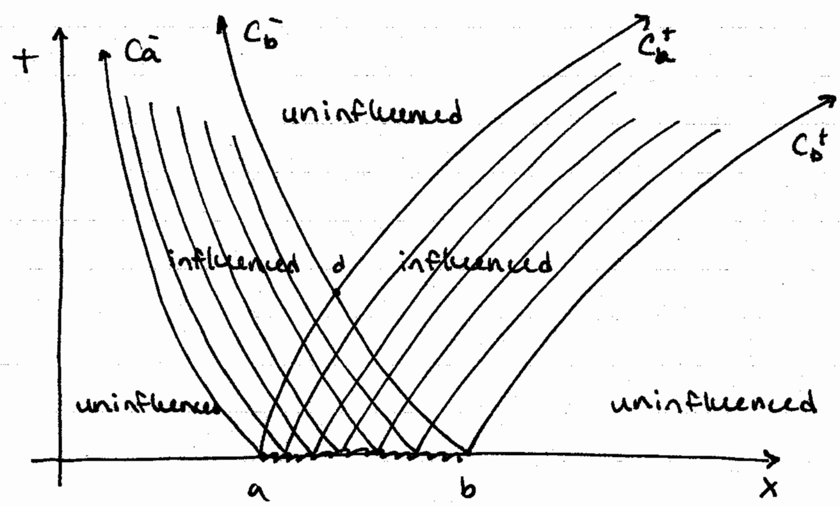
Since $F = F(P)$ all conditions are known @ d.

Furthermore the conditions at any pt. inside abd is known given initial conditions on $\overline{ab}$

A disturbance cannot travel faster than the speed of sound (c^+ and c^-) (neglect radiation)

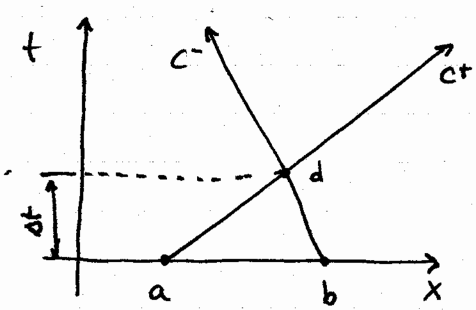
$\Rightarrow$ limited region of influence

" characteristics serve as carriers of information "



In general, we have to find characteristics by integrating numerically.

E.g. $\frac{D^+}{Dt}(u+F) = G^+(x,u,F)$ $\frac{D^-}{Dt}(u-F) = G^-(u,x,F)$



$(u+F)_d - (u+F)_a = G^+ \Delta t$

$(u+F)_d - (u-F)_b = G^- \Delta t$