2.25 – Fluid Mechanics – Fall 2013

Solutions to quiz 1, problem 1

Part a (verbal interpretation of Bernoulli's equation)

- **Along a streamline**: Bernoulli's equation is essentially a special case of the balance of energy for a moving fluid element. For steady, inviscid, incompressible flow along a streamline, the work done on a fluid element by pressure forces and gravity causes a change in the kinetic energy of the element.
- **Normal to a streamline**: This form of Bernoulli's equation is a force balance across streamlines. Fluid elements moving along curved streamlines experience a centripetal force due either to gradients in pressure or gravitational body forces, both of which act toward the local center of curvature.

Part b (show that B is a true constant for irrotational flow)

There are several ways to do this part. Below we show two possible methods.

"Standard" solution: Let the velocity vector $\underline{v} = u\underline{e}_x + v\underline{e}_y$. Under the assumptions we have made, the flow is described by the Euler equations in the x and y directions:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x},$$
$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} - \rho g.$$

The vorticity is zero by definition of an irrotational flow:

$$\underline{\omega} = \underline{e}_z\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \underline{0} \Rightarrow \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}.$$

Substituting $\partial v / \partial x = \partial u / \partial y$ into the Euler equations (and dividing through by ρ)

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$
$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g.$$

Now, consider an infinitesimal displacement vector in an arbitrary direction, $d\underline{r} = dx\underline{e}_x + dy\underline{e}_y$. Multiplying the Euler equations in x and y by dx and dy respectively,

$$u\frac{\partial u}{\partial x}dx + v\frac{\partial v}{\partial x}dx = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx,$$
$$u\frac{\partial u}{\partial y}dy + v\frac{\partial v}{\partial y}dy = -\frac{1}{\rho}\frac{\partial p}{\partial y}dy - gdy.$$

Adding these equations gives

$$u\underbrace{\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right)}_{=du} + v\underbrace{\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)}_{=dv} = -\frac{1}{\rho}\underbrace{\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy\right)}_{=dp} - gdy.$$

Since the terms in parentheses are the total differentials of u, v, and p along $d\underline{r}$, this equation becomes

$$udu + vdv = -\frac{dp}{\rho} - gdy,$$

or

$$d\left(\frac{u^2}{2}\right) + d\left(\frac{v^2}{2}\right) + \frac{dp}{\rho} + gdy = 0.$$

Integrating indefinitely, and assuming the flow is incompressible, we obtain Bernoulli's equation,

$$\boxed{\frac{u^2+v^2}{2} + \frac{p}{\rho} + gy = B,}$$

where B is a constant. Since we have made no assumptions about the direction of the displacement vector $d\underline{r}$, we conclude that the constant B must be the same for *any* $d\underline{r}$, that is, it must be the same everywhere in the flow.

"Elegant" solution:

Using the vector identity $\underline{v}\cdot\underline{\nabla}\underline{v} = \underline{\nabla}(\underline{v}\cdot\underline{v})/2 - \underline{v}\times(\underline{\nabla}\times\underline{v})$, one can rewrite the general vector Euler equation in the form

$$\frac{\partial \underline{v}}{\partial t} + \underline{\nabla}\left(\frac{\|\underline{v}\|^2}{2} + \frac{p}{\rho} + gy\right) = \underline{v}\times(\underline{\nabla}\times\underline{v}) = \underline{v}\times\underline{\omega}.$$

Here $\|\underline{v}\|$ is the velocity magnitude. In this case the right-hand side is identically zero because the flow is irrotational. If we assume a steady flow and integrate the resulting equation, we are left with

$$\frac{\|\underline{v}\|^2}{2}+\frac{p}{\rho}+gy=B$$

This equation applies everywhere in the flow field; therefore, the Bernoulli constant is the same everywhere in an arbitrary inviscid, irrotational ("potential") flow.

Part c (show that velocity is independent of θ and z)

There are several ways to do both parts. Below we show the most straightforward way for each.

Independent of θ: Consider the differential statement of the conservation of mass for this steady, incompressible flow in cylindrical coordinates:

$$\underline{\nabla}\cdot\underline{v}=\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(rv_r\right)}_{=0}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\underbrace{\frac{\partial v_z}{\partial z}}_{=0}=0.$$

Since it was given that the streamlines in the bend are circular arcs, this implies that $v_r=v_z=0$. Thus, the first and third terms vanish, and we are left with $\partial v_\theta/\partial\theta=0$. Clearly, this implies that velocity in the bend is independent of θ. This part is also relatively straightforward to show with the control-volume statement of mass conservation.

Independent of *z* (actually shows independent of θ as well): Consider Euler's equation in *z*:

$$\frac{\partial}{\partial z}\left(p+\rho gz\right)=0.$$

Integrating this equation gives $p+\rho gz=C+f\left(r\right)$, where *C* is a constant. Noting that $p=p_{atm}$ at $z=h\left(r\right)$, we see that at any value of *r*,

$$p+\rho gz=p_{atm}+\rho gh\left(r\right).$$

We can substitute this result into Bernoulli's equation normal to a streamline in the bend (which is really Euler's equation in the *r* direction):

$$\frac{\partial}{\partial r}\left(p+\rho gz\right)=\frac{\partial}{\partial r}\left(p_{atm}+\rho gh\left(r\right)\right)=\frac{\rho v_\theta^2}{r}.$$

Simplifying,

$$v_\theta^2 = rg\frac{dh}{dr}.$$

This equation depends on r only, thus v_θ does not depend on z or θ.

Part d (argue that the Bernoulli constant is the same for all streamlines; show that $v_\theta r=K$)

The key to this part is to recognize that under the assumptions we have made, any streamline in the bend originates somewhere in the inlet, where the velocity is uniform (station 1).

If we apply the z-direction Euler equation to the inlet, we can write the Bernoulli constant in the entire inlet region as

$$B_{inlet} = p_{atm} + \rho g h_1 + v_1^2$$

since the fluid height at the inlet is equal to h_1 everywhere. Since all streamlines in the bend originate at the inlet, and since B is constant along a streamline by definition, the value of B for *any streamline* is equal to B_{inlet}:

$$\left(p + \frac{\rho v_\theta^2}{2} + \rho g z\right)_{bend} = p_{atm} + \frac{\rho v_1^2}{2} + \rho g h_1 = B_{inlet}.$$

Thus, **the Bernoulli constant is the same for all streamlines in this flow.**

There are many ways to determine the velocity profile. One possible way is to take the radial derivative of the Bernoulli equation along a streamline to obtain:

$$\frac{\partial}{\partial r}\left(p + \frac{\rho v_\theta^2}{2} + \rho g z\right) = \frac{\partial B}{\partial r} = 0 \Rightarrow \frac{\partial}{\partial r}\left(p + \rho g z\right) = -\frac{\partial}{\partial r}\left(\frac{\rho v_\theta^2}{2}\right) = -\rho v_\theta \frac{\partial v_\theta}{\partial r},$$

where $\partial B / \partial r = 0$ because B is constant everywhere in the flow. We also know from Bernoulli's equation in the radial direction that

$$\frac{\partial}{\partial r}\left(p + \rho g z\right) = \frac{\rho v_\theta^2}{r}.$$

Thus we can set

$$\frac{\rho v_\theta^2}{r} = -\rho v_\theta \frac{\partial v_\theta}{\partial r}.$$

Simplifying and rearranging,

$$\frac{dv_\theta}{v_\theta} + \frac{dr}{r} = 0 \Rightarrow d\left(\ln v_\theta\right) + d\left(\ln r\right) = 0.$$

Integrating,

$$\ln\left(v_\theta\right) + \ln\left(r\right) = C,$$

where C is a constant of integration. Exponentiating both sides,

$$\boxed{v_\theta r = e^C = K,}$$

where K is another constant.

Part e (derive the shape of the free surface in the bend)

Consider a streamline along the free surface extending from the inlet to somewhere in the bend. Bernoulli's equation along this streamline is

$$B = B_{inlet} = p_{atm} + \rho g h_1 + \frac{\rho v_1^2}{2} = p_{atm} + \rho g h\left(r\right) + \frac{\rho v_\theta^2}{2}.$$

Equation (24) is valid for any streamline (i.e., at any value of r or z) because B is the same for all streamlines. Simplifying,

$$\rho g h_1 + \frac{\rho v_1^2}{2} = \rho g h\left(r\right) + \frac{\rho v_\theta^2}{2} = \rho g h\left(r\right) + \frac{\rho\left(K/r\right)^2}{2}.$$

Solving for h,

$$\boxed{h\left(r\right) = h_1 + \frac{v_1^2}{2g} - \frac{K^2}{2gr^2}.}$$

Part f (derive a condition on K from a control-volume theorem)

Consider a control volume that surrounds the liquid in the bend, from the inlet to some arbitrary cross section in the bend. Applying form A of the control-volume statement of conservation of mass,

$$\underbrace{\frac{d}{dt}\int_{CV} \rho d\cancel{V}}_{=0,\text{ steady}} + \int_{CS} \rho\left(\underline{v} - \underline{v}_c\right)\cdot \underline{n} dA = 0.$$

Noting that the control surface is not moving ($\underline{v}_c = \underline{0}$), the flux integral over the inlet is

$$\int_{Inlet} \rho \underline{v} \cdot \underline{n} dA = -\rho v_1 h_1 \left(r_o - r_i \right)$$

and at the outlet (which is at an arbitrary cross-section in the bend) it is

$$\int_{Outlet} \rho \underline{v} \cdot \underline{n} dA = \rho \int_{r_i}^{r_o} v_\theta \left(r \right) h\left(r \right) dr.$$

Thus the conservation of mass equation becomes

$$\begin{aligned} v_1 h_1 \left(r_o - r_i \right) &= \int_{r_i}^{r_o} \frac{K}{r} \left(h_1 + \frac{v_1^2}{2g} - \frac{K^2}{2gr^2} \right) dr \\ &= K h_1 \left(1 + \frac{v_1^2}{2gh_1} \right) \int_{r_i}^{r_o} \frac{dr}{r} - \frac{K^3}{2g} \int_{r_i}^{r_o} \frac{dr}{r^3} \\ &= K h_1 \left(1 + \frac{v_1^2}{2gh_1} \right) \ln\left(\frac{r_o}{r_i} \right) - \frac{K^3}{4g} \left(\frac{1}{r_i^2} - \frac{1}{r_o^2} \right) \end{aligned}$$

Thus, the constraint on K is

$$\boxed{v_1 h_1 \left(r_o - r_i \right) = K h_1 \left(1 + \frac{v_1^2}{2gh_1} \right) \ln\left(\frac{r_o}{r_i} \right) - \frac{K^3}{4gr_i^2} \left(1 - \frac{r_i^2}{r_o^2} \right).}$$

Part g (explain why Bernoulli is valid for rigid-body rotation)

Although the fluid is viscous, there is no viscous stress in steady rigid-body rotation since there is no relative motion between fluid elements, by definition. Thus, there are no viscous losses to consider. Since the flow is also incompressible, steady, and without energy input, Bernoulli's equation may be applied along a streamline.

If one wishes to make a more rigorous argument, one can look at the velocity gradient tensor, $\underline{\nabla} \underline{v}$. All components of this tensor vanish except for two:

$$\underline{\nabla} \underline{v} = \begin{bmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{1}{r} \dfrac{\partial v_r}{\partial \theta} - \dfrac{v_\theta}{r} \\ \dfrac{\partial v_\theta}{\partial r} & \dfrac{1}{r} \dfrac{\partial v_\theta}{\partial \theta} + \dfrac{v_r}{r} \end{bmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}.$$

The symmetric (deformation) tensor then reads

$$\underline{\underline{e}} = \frac{1}{2} \left(\underline{\nabla} \underline{v} + \left(\underline{\nabla} \underline{v} \right)^T \right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

As expected, fluid elements experience no deformation, only pure rotation. Thus, if we wish to calculate the viscous stress tensor, e.g. $\underline{\underline{\sigma}} = 2\mu \underline{\underline{e}}$ for a Newtonian fluid, we see that there is no viscous stress in the flow (at steady state) regardless of how viscous the fluid is. Thus, there are no viscous losses, and Bernoulli's equation is applicable.

Part h (show that the Bernoulli constant does not depend on z; show that it does depend on r and determine its value for a given r; explain why it varies in this case)

The Bernoulli equation for this case is

$$p + \frac{\rho v_\theta^2}{2} + \rho g z = p + \frac{\rho r^2 \omega^2}{2} + \rho g z = B_{sb} = const.$$

Again, we can integrate Euler's equation in the *z* direction to find

$$\frac{\partial}{\partial z}\left(p + \rho g z\right) = 0 \Rightarrow p + \rho g z = p_{atm} + \rho g h(r).$$

Here $h(r)$ is the concave, parabolic shape for the free surface that we derived in class for rigid-body rotation. Substituting this relation into the Bernoulli equation,

$$\boxed{B_{sb} = p_{atm} + \rho g h(r) + \frac{\rho r^2 \omega^2}{2}.}$$

This shows both that the Bernoulli constant **does not depend on *z*** and that it **does depend on *r***. This equation is an acceptable final answer for the Bernoulli constant, but one can also substitute the height profile from the class notes (not required for credit) to obtain

$$\boxed{B_{sb} = p_{atm} + \frac{\rho r^2 \omega^2}{2} + \rho g\left(H + \frac{r^2 \omega^2}{2g}\right) = p_{atm} + \rho g H + \rho r^2 \omega^2.}$$

Since the flow is rotational, the Bernoulli constant is not the same everywhere in the flow field, and varies across streamlines as indicated above.

Interpretation

Recall that the Bernoulli constant *B* is roughly a measure of the total energy of a fluid element. As one looks outward in *r*, the fluid elements are moving faster and thus have greater kinetic energy. Thus it makes sense that the total energy of fluid elements increases as *r* increases, i.e. that *B* varies with *r*.

Later in the class, we will show that *B* is actually constant along both streamlines and vortex lines. In this flow, the vortex lines point in the *z* direction, so it makes sense that the Bernoulli constant does not depend on *z*.

Solution to Problem 2-Quiz 1 2013
Part I:
(a):
The pressure distribution on line AB follows the hydrostatic rule. It is true that the flow is not static but by picking an arbitrary control volume at any point on line AB (green dashed control volume in Figure 1) one can see that the balance of forces in the $y-$direction will tell us that the difference between the pressure at the bottom and the ambient pressure should balance the weight of the liquid inside the control volume. This simply implies that the static pressure on line AB should be equal to $P_a + \rho g h(x)$. This result is shown in Figure 1.

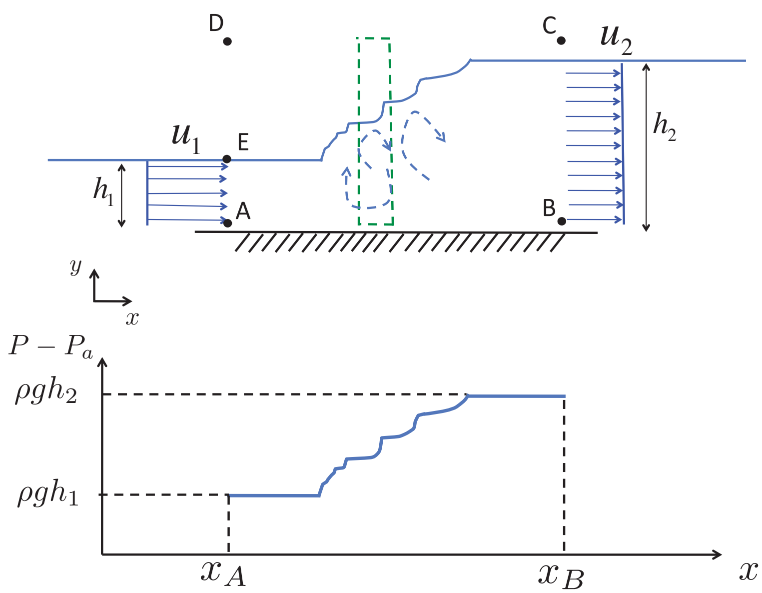


Figure 1: Pressure distribution on line AB.

The pressure distribution on line DEA also follows the hydrostatic change merely due to the fact that there is no curvature in the streamlines as one integrates the Euler equation normal to them and thus the only change in pressure when one moves from E to A will be the hydrostatic part. Ignoring the density of air one can see that the pressure is constant from D to E and then start to grow linearly with height as we move from E to A. The result is shown in Figure 2.

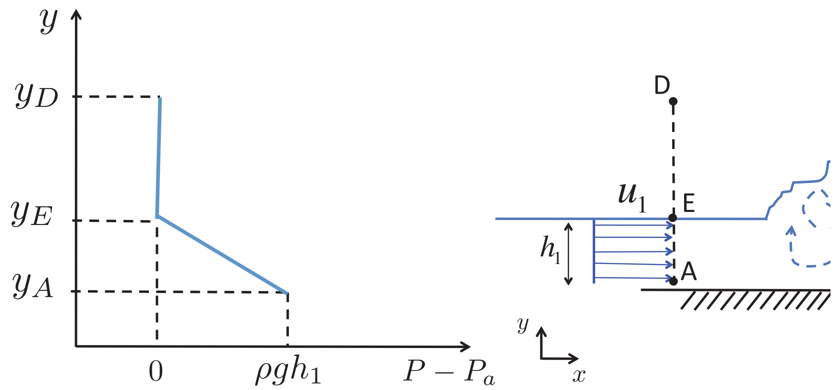


Figure 2: Pressure distribution on line DEA.

(b) and (c):
The selected control volume is shown in Figure 3 (dashed green line). One can subtract the ambient pressure from the entire problem and knowing that the net effect of uniform P_a acting on the control volume is zero then there will be no change in the problem analysis if we only deal with gauge pressures ($P(x, y) - P_a$).

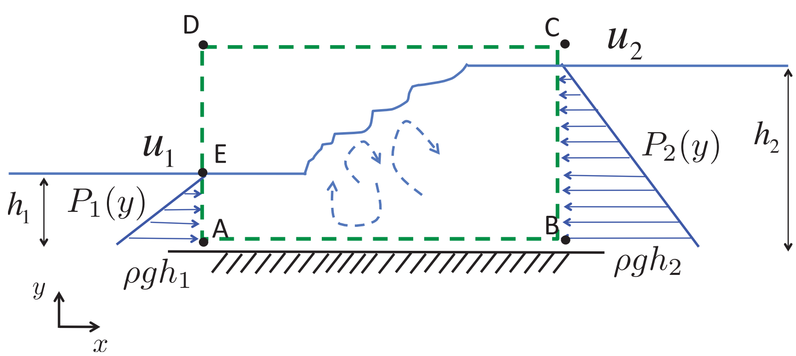


Figure 3: A schematic of the selected control volume for the hydraulic jump problem.

Table 1 summarizes all the important parameters acting on different control surfaces for the selected control volume:

Table 1: Bookkeeping of all the related properties at different control surfaces in the control volume.

	$\underline{n}$	$\underline{v}$	$\underline{v_c}$	$(\underline{v}-\underline{v_c}).\underline{n}$	A	$P-P_a$
AD	$-\underline{e_x}$	$u_1\underline{e_x}$	0	$-u_1$	h_1W	$\rho g y$
AB	$-\underline{e_y}$	$?\underline{e_x}$	0	0	?	$\rho g h(x)$
BC	$\underline{e_x}$	$u_2\underline{e_x}$	0	u_2	h_2W	$\rho g y$
CD	$\underline{e_y}$	0	0	0	?	0

Now we can start by writing the conservation rules using the RTT. It is important to notice that due to the turbulent mixing happening in the region of the hydraulic jump, energy will not be conserved and thus either applying the conservation of energy or the Bernoulli equation will not be the right approach. If we write the conservation of mass for the selected control volume then we will have:

$$\textit{C.O.Mass: } 0 = \frac{d}{dt}\int_{c.v.} \rho dV + \int_{c.s.} \rho(\underline{v}-\underline{v_c}).\underline{n}dA$$

Knowing that the problem is steady state and using the tabulated quantities, conservation of mass can be simplified to:

$$\rho u_1 h_1 = \rho u_2 h_2 \Rightarrow \boxed{u_1 h_1 = u_2 h_2} \tag{1}$$

The conservation of linear momentum in the x direction can also be written in the RTT form:

$$\textit{C.O.Momentum: } \frac{1}{W}\sum F_x = \frac{d}{dt}\int_{c.v.} \rho v_x dV + \int_{c.s.} \rho v_x(\underline{v}-\underline{v_c}).\underline{n}dA$$

where W is the width into the page.
The net of external forces acting in the x-direction on the control volume neglecting the wall shear effect is a result of pressure forces acting on the AD and BC control surfaces:

$$\frac{1}{W}\sum F_x = \int_{AD}(P-P_a)dy - \int_{BC}(P-P_a)dy = \int_0^{h_1} \rho g y dy - \int_0^{h_2} \rho g y dy = \rho g\left(\frac{h_1^2}{2}-\frac{h_2^2}{2}\right)$$

The right hand side of the RTT for the conservation of linear momentum can also be simplified to (knowing that the problem is steady and using the tabulated identities):

$$\textit{R.H.S. of RTT for C.O. Momentum} = \rho u_2^2 h_2 - \rho u_1^2 h_1$$

thus the conservation of linear momentum implies that:

$$\rho g\left(\frac{h_1^2}{2}-\frac{h_2^2}{2}\right) = \rho u_2^2 h_2 - \rho u_1^2 h_1 \Rightarrow \boxed{\frac{g}{2}\left(h_1^2-h_2^2\right) = u_2^2 h_2 - u_1^2 h_1} \tag{2}$$

using the result from conservation of mass (equation (1)) one can eliminate u_2 from equation (2) to give:

$$\frac{g}{2}\left(h_1^2 - h_2^2\right) = h_1 u_1^2 \left(h_1/h_2 - 1\right) \Rightarrow \boxed{u_1 = \sqrt{\frac{gh_2}{2h_1}(h_1 + h_2)}} \tag{3}$$

where we have used the identity $h_1^2 - h_2^2 = (h_1 - h_2)(h_1 + h_2)$.

Part II:

(d):

For the selected control volume (Figure 4) one can easily write the conservation of mass using Taylor series to obtain expressions for $u(x + \Delta x)$ and $h(x + \Delta x)$:

$$u(x)h(x) = u(x + \Delta x)h(x + \Delta x) \rightarrow u(x)h(x) = (u(x) + \frac{du}{dx}\Delta x)(h(x) + \frac{dh}{dx}\Delta x)$$

which after ignoring the second order terms such (Δx^2) it can be rewritten as:

$$\Delta x \left(u(x)\frac{dh}{dx} + h(x)\frac{du}{dx}\right) = 0 \Rightarrow \boxed{u\frac{dh}{dx} + h\frac{du}{dx} = 0} \tag{4}$$

Another way to reach the same result is to say that since the flow is incompressible then the volumetric flow rate should remain unchanged thus $d(uh)/dx = 0$ which will lead to the same result we just derived in equation (4).

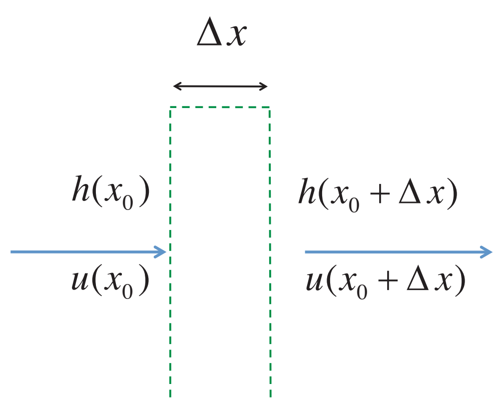


Figure 4: An arbitrary control volume selected to derive the conservation of mass in the differential form.

(e) and (f):

One plausible answer is that $h(x)$ decreases since the flow starts to accelerate as it reaches the bump and thus due to conservation of mass $hu = const.$ the value of h should decrease (Figure 5(b)). It is also easy to show that the case of h remaining constant (Figure 5(c)) will be wrong since in that case at constant velocity we are gaining height which is similar to generating potential energy from nowhere and is thus unphysical[1]. For further explanations see the appendix.

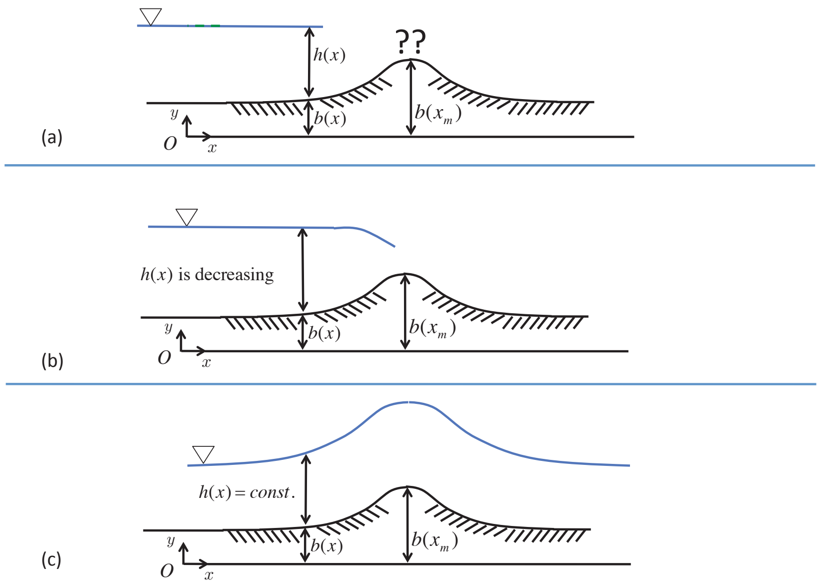


Figure 5: (a) Flow approaching the bump (b) The case for low speed flows in which $Fr_1 < 1$ (c) The non-physical case in which h remains constant.

Since the incoming flow is smooth and has not undergone tany mixing and also viscous effects are negligible we can think of writing Bernoulli equation on a streamline very close to the water surface:

$$P_a + \frac{1}{2}\rho u(x)^2 + \rho g(h(x) + b(x)) = const. \rightarrow \frac{d}{dx}\left[P_a + \frac{1}{2}\rho u(x)^2 + \rho g(h(x) + b(x)) = const.\right] = 0$$

which can be simplified to:

$$\boxed{u(x)\frac{du}{dx} + g\frac{dh}{dx} + g\frac{db}{dx} = 0} \tag{5}$$

From conservation of mass (equation (4)) we have:

$$u\frac{dh}{dx} + h\frac{du}{dx} = 0 \Rightarrow \boxed{\frac{dh}{dx} = -\frac{h(x)}{u(x)}\frac{du(x)}{dx}} \tag{6}$$

[1] The mentioned argument holds for cases in which the incoming kinetic energy of the flow before the bump is small compared to the potential energy of the fluid (i.e. $u_1^2 < gh_1$ or $Fr_1 < 1$). If the initial flow has a high kinetic energy compared to its potential energy (i.e. $u_1^2 > gh_1$ or $Fr_1 > 1$) then it is possible to see a different solution in which $h(x)$ does increase as the liquid goes over the bump. Later in the solution we will see that combining the conservation of mass and Bernoulli equation it can be shown that: $dh/dx = db/dx(Fr^2 - 1)^{-1}$ and consequently the rise or decrease in $h(x)$ depends on the Froude number of the entering flow (Fr_1).

plugging the result from (6) into (5) will give the following result:

$$\boxed{\frac{1}{u(x)}\frac{du(x)}{dx}\left[u(x^2) - gh(x)\right] + g\frac{db(x)}{dx} = 0} \tag{7}$$

(g): As the flow reaches x_m, the slope db/dx becomes zero. Slightly after the bump (x_m^+) we will have a negative value for db/dx (i.e. $db/dx < 0$ at x_m^+) and thus the sign of the quantity du/dx depends on the value of the quantity $\left[u(x)^2 - gh(x)\right]$. Using equation (7) one can see that it means that at x_m^+ we will have the following:

$$\frac{1}{u(x)}\frac{du}{dx}\left[u^2 - gh\right]|_{x_m^+} > 0$$

thus depending on the sign of $u(x_m^+)^2 - gh(x_m^+)$ there will be two situations:
(i):

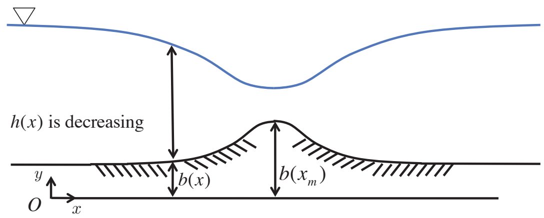
$$\left[u^2 - gh\right]\Big|_{x_m^+} < 0 \Rightarrow at\ x_m^+ : \boxed{\frac{1}{u(x)}\frac{du}{dx} < 0}$$

which means that the flow will decelerate after the bump and with decrease in velocity the height will increase due to the conservation of mass, equation (6) as shown in figure 6(a).

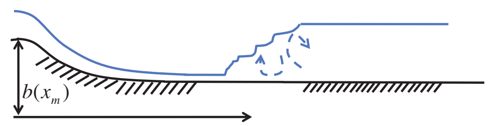
(ii):

$$\left[u^2 - gh\right]\Big|_{x_m^+} > 0 \Rightarrow at\ x_m^+ : \boxed{\frac{1}{u(x)}\frac{du}{dx} > 0}$$

which means that the flow will accelerate after the bump and with the increase in velocity the height will decrease due to the conservation of mass (Figure 6(b)). This will lead to a state that is called a "super-critical" flow ($Fr > 1$) and slightly after the bump the viscous effects become important since the speed is increasing and the height is decreasing (remember that $\dot{\gamma} \sim v/h$) and the flow ultimately reaches a point at which it has no more kinetic energy to continue its acceleration into the super-critical zone. As a result, hydraulic jump will occur with a lot of turbulent energy dissipation and the flow returns into a sub-critical stage ($Fr < 1$). Figure 6(b) shows a sketch of this flow.



(a)



(b)

Figure 6: (a) Physical image for the (i) case. (b) Physical image for the (ii) case.

(extra additional part of the problem:)
From equation (7) we have the following:

$$\frac{1}{u(x)}\frac{du(x)}{dx}\left[u(x^2) - gh(x)\right] + g\frac{db(x)}{dx} = 0$$

using the following:

$$u(x)\frac{du(x)}{dx} - g\frac{du}{dx}\frac{h(x)}{u(x)} = \frac{1}{2}\frac{d(u^2(x))}{dx} - gQ\frac{1}{u^2(x)}\frac{du}{dx} = \frac{1}{2}\frac{d(u^2(x))}{dx} + gQ\frac{d}{dx}\left(\frac{1}{u(x)}\right)$$

we can expand equation (7) into:

$$\boxed{\frac{1}{2}\frac{d(u^2(x))}{dx} + gQ\frac{d}{dx}\left(\frac{1}{u(x)}\right) + g\frac{db(x)}{dx} = 0} \tag{8}$$

Integrating equation (8) from point 1 to point 2 (far after the bump at which point the surface becomes flat again, $b(x) = 0$) will give the following:

$$\frac{1}{2}u(x)^2 + gQ\left(\frac{1}{u(x)}\right) + gb(x) = const. \rightarrow \frac{1}{2}u_1^2 + gQ\left(\frac{1}{u_1}\right) = \frac{1}{2}u_2^2 + gQ\left(\frac{1}{u_2}\right) \tag{9}$$

Equation (9) is nothing but a reformulation of Bernoulli equation or conservation of energy between any two arbitrary points on the water surface. Looking at equation (9) it is easy to see that one possible solution is $u_2 = u_1$ which is the case (i) studied in the previous part. Also it is worthy to mention that there is another root which satisfies the following:

$$gQ = \frac{1}{2}u_1u_2(u_1+u_2) \rightarrow \frac{1}{2}u_1^2+gQ\frac{1}{u_1} = \frac{1}{2}u_1^2+\frac{1}{2}(u_2u_1+u_2^2) = \frac{1}{2}u_2^2+\frac{1}{2}(u_2u_1+u_1^2) = \frac{1}{2}u_2^2+gQ\frac{1}{u_2}$$

This second solution or root happens in the case in which the flow is sub-critical before the bump, becomes critical at the bump point and then becomes super-critical after the bump. Using the definition of the Froude number ($Fr \equiv u^2/gh$) one can manipulate the derived result in equation (9) in the following way (using the hint that $Fr = u^3/gQ$):

$$
\begin{aligned}
& u_1 u_2 (\frac{gQ}{u_1} + \frac{1}{2}u_1^2 = \frac{gQ}{u_2} + \frac{1}{2}u_2^2) \rightarrow u_2(gQ + \frac{1}{2}u_1^3) = u_1(gQ + \frac{1}{2}u_2^3) \\
\rightarrow & \frac{u_2}{gQ}(1 + \frac{1}{2}Fr_1) = \frac{u_1}{gQ}(1 + \frac{1}{2}Fr_2) \rightarrow (gQFr_2)^{1/3}(1 + \frac{1}{2}Fr_1) = (gQFr_1)^{1/3}(1 + \frac{1}{2}Fr_2) \\
\rightarrow & 2Fr_2^{1/3} + Fr_2^{1/3}Fr_1 = 2Fr_1^{1/3} + Fr_1^{1/3}Fr_2 \rightarrow 2(Fr_2^{1/3} - Fr_1^{1/3}) = Fr_1^{1/3}Fr_2^{1/3}(Fr_2^{2/3} - Fr_1^{2/3}) \\
\Rightarrow & \boxed{2 = Fr_1^{1/3}Fr_2^{1/3}\left(Fr_1^{1/3} + Fr_2^{1/3}\right)}
\end{aligned}
\tag{10}
$$

Appendix:
The flow that has just been studied in this problem is a well known subject in the hydraulic literature. It is worthwhile to mention a few words about different physical aspects of these flows which are generally named "open channel flows" in hydraulics. One way to study these problems is to introduce two identities called the "specific energy head" (E) and "momentum of the flow" (M):

$$\begin{aligned} E &\equiv \frac{1}{2}\frac{q^2}{gh^2} + h = \frac{1}{2g}u^2 + h \\ M &\equiv \frac{q^2}{gh} + \frac{h^2}{2} \end{aligned} \tag{11}$$

knowing that $q \equiv uh$ one can easily see that E is the total specific energy and has similarities to Bernoulli constant whereas M is the linear momentum of the flow and when defined in this way will have dimensions of length squared. At a constant flow rate (q) one can plot the relationship between h and E (Figure 7) or h and M (Figure 8) using the relationships in equation (11). One can easily see that in both equations dE/dh and dM/dh will be zero at a critical height $h_c = (q^2/g)^{1/3}$. It is also possible to show that at $h = h_c$ the local Froude number which is defined as $Fr \equiv u^2/gh$ becomes equal to 1 and in fact another way to define the Froude number is to define it as the ratio of local height (h) over the critical height (h_c) (i.e. $Fr \equiv h/h_c$). Based on this we can categorize open channel flows in three different types:

- Sub-critical flow in which the Froude number is lower than one and the flow is dominated by gravity rather than inertia. This is a characteristic for low speed, deep river flows.
- Critical flow in which the Froude number is one and the inertia and gravity are equally important.
- Super-critical flow in which the Froude number is higher than one and inertia dominates over the gravity.

Looking to Figures (7) and (8) it is worthwhile to notice that at a constant flow rate for any given specific energy (E) or momentum (M) value there are two possible heights: one in the sub-critical zone and the other one in the super-critical zone. Also it is noteworthy that for specific energies lower than the value at E_c at h_c there is no possible physical solution.
One benefit of these diagrams is enabling us to find solutions for hydraulics problems just by visual inspection of the curves. For example in the bump problem we know that the Bernoulli equation between points on the free surface is equivalent to:

$$\frac{1}{2}u_1^2 + gh_1 = \frac{1}{2}u_2^2 + gh_2 + gb(x) \rightarrow g(E_1) = g(E_2 + b(x)) \Rightarrow \boxed{\frac{1}{h_c}E_2 = \frac{1}{h_c}E_1 - \frac{1}{h_c}b(x)} \tag{12}$$

where $b(x)$ is the height of the bed of the river.
What equation (12) is showing is the fact that knowing the value of E_1 we can easily find the solution for E_2 and h_2 by following the constant q curve and subtracting the $b(x)$ from the value of $E1$ (Figure 9).

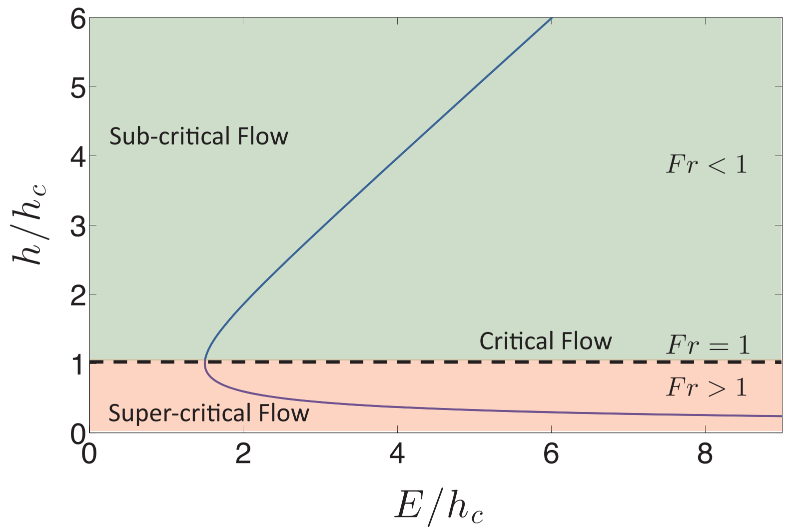


Figure 7: The specific energy (E)-height (h) diagram for a flow with constant flow rate. In which hc is defined as $(q^2/g)^{1/3}$.

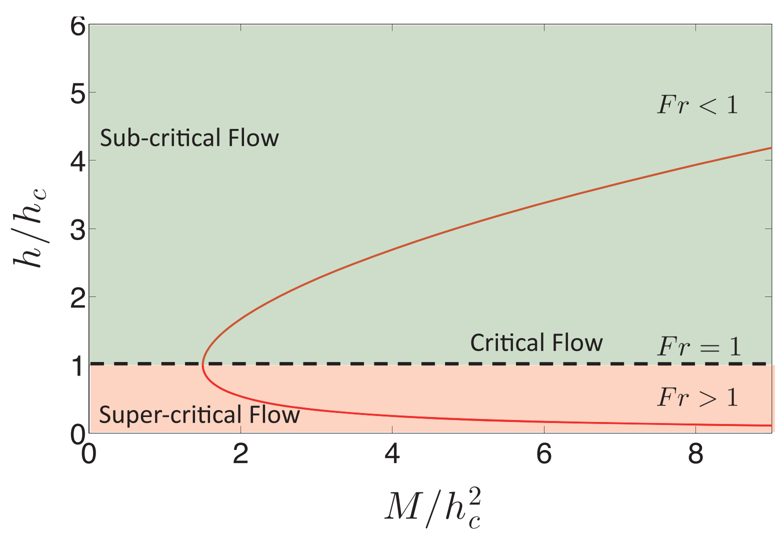


Figure 8: The momentum of the flow (M)-height (h) diagram for a flow with constant flow rate. In which hc is defined as $(q^2/g)^{1/3}$.

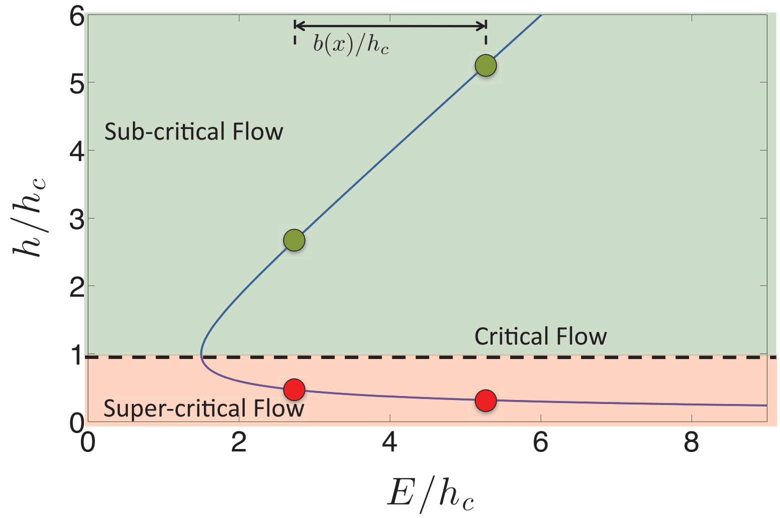


Figure 9: Solution to the bump problem using the constant q curve in the specific energy-height diagram. The green points show the sub-critical solution while the red points show the super-critical solution.

Using the visual solution from Figure 9 one can easily detect that there are two possible branches of solution for the flow over a bump based on the initial Froude number. The sub-critical solution ($Fr < 1$) predicts that as the flow goes over the bump the value of h will decrease and the flow will speed up while the super-critical solution ($Fr > 1$) predicts the opposite. If the flow starts from sub-critical branch in the upstream as it reaches the bump it is possible to become very close to the critical point in the specific energy-height diagram (Figure 10(b)) but right after the bump with decrease in $b(x)$ it will return to the original point and thus all through the process the flow will remain sub-critical. The solution for the change in height of the flow over the bump for this case is shown in Figure 10(a).

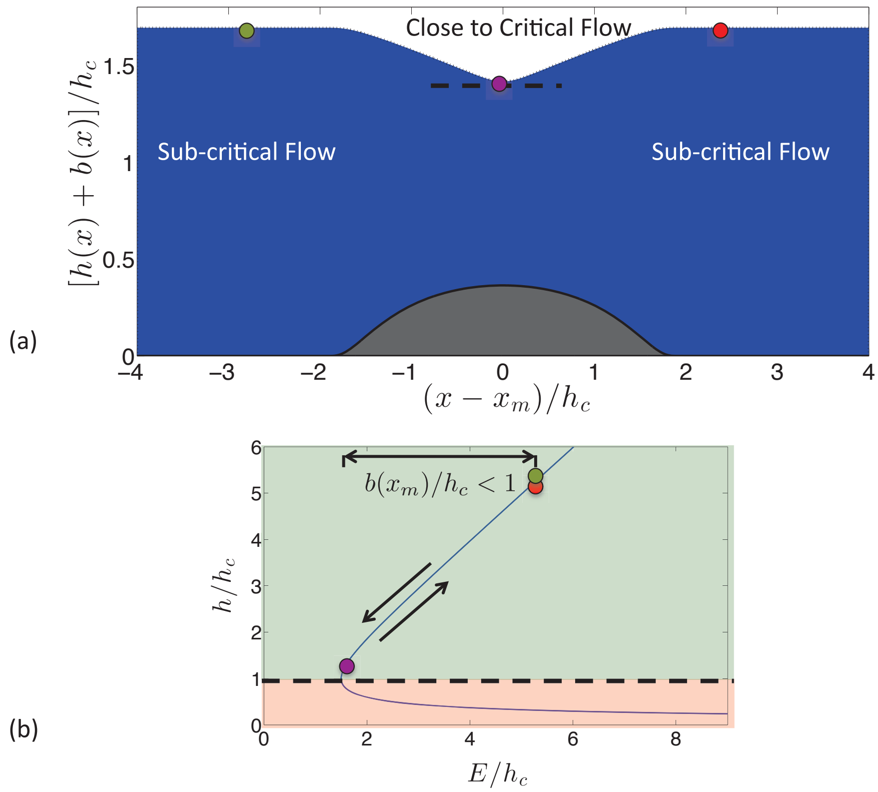


Figure 10: Flow over the bump. Sub-critical solution for the entire flow.

If the bump is high enough then it is possible that the flow initially is sub-critical but as it passes over the bump (where $x = x_m$ and $db/dx = 0$) it becomes critical and with a small perturbation after ($x = x_m$) the solution will follow the super-critical branch (Figure 11 (b)). This means that before the bump $h(x)$ will decrease as it reaches the bump and after x_m this decrease will continue since the flow has become super-critical (Figure 11). In most hydraulics labs the transition from sub-critical to critical and super-critical flow is demonstrated in a water tank experiment of the flow over the bump (link on youtube: http://www.youtube.com/watch?v=cRnIsqSTX7Q).

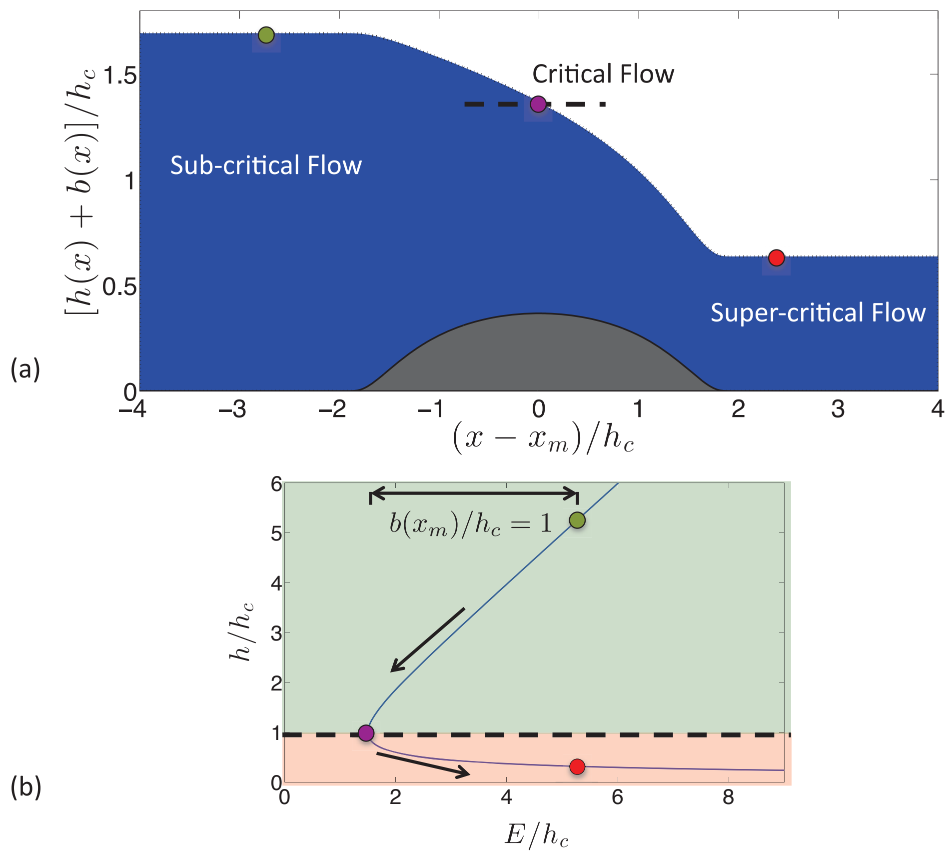


Figure 11: Flow over the bump. Sub-critical solution for the flow before the bump, critical at x_m and super-critical after the bump.

Another possibility is to have an upstream flow that is already in the super-critical regime. The flow this time stays entirely on the super-critical branch of the solution (Figure 12). This time the height initially increases and after x_m it starts to decrease back to the original height. Achieving this flow in the lab is not easy since a hydraulic jump can easily occur either before or after the bump.

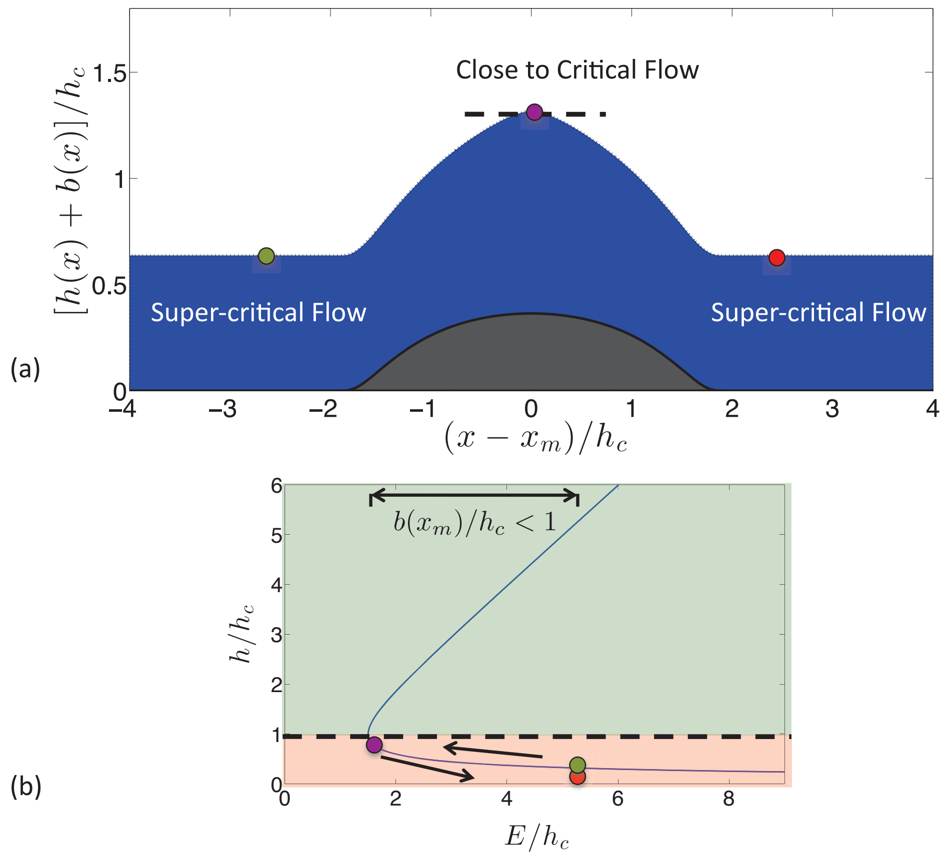


Figure 12: Flow over the bump. Super-critical solution for the entire flow.

It is also important to notice that although the application of specific energy-height diagrams is quite useful in the case of the flow over the bump, special caution should be taken in interpreting the diagram. As we saw in this problem the hydraulic jump does not conserve the total energy (but in fact dissipates a lot). The conserved identity is the linear momentum. Thus a good approach would be to find the point on the super-critical branch before the jump and find a corresponding point on the sub-critical branch with an equal value of momentum and see where this new point sits on the energy diagram (as shown in Figure 13). It is noticeable that the jump will lead to a considerable drop of the energy head due to viscous dissipation involved in the jump region.

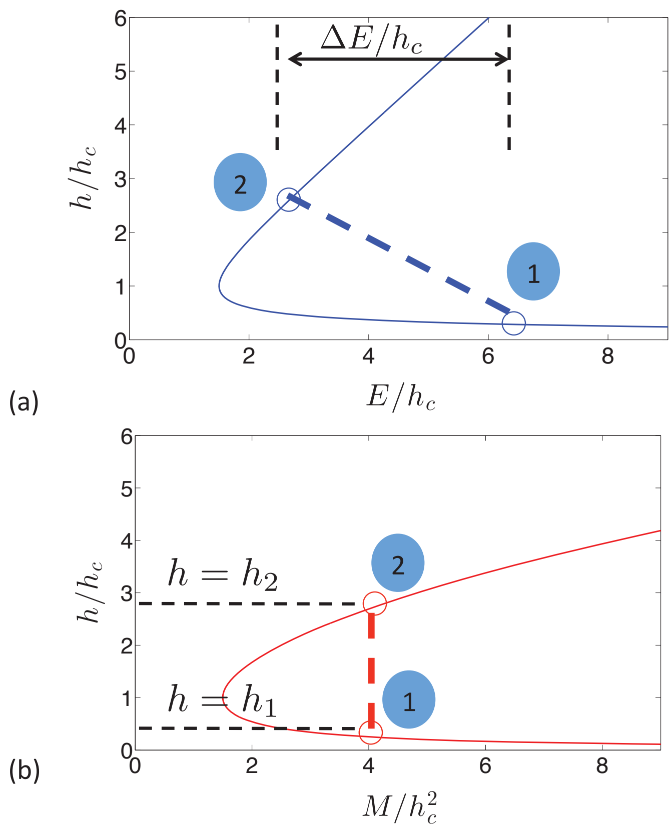


Figure 13: (a) Blue curve showing h/h_c as a function of E/h_c for constant flow rate. (b) Red curve showing h/h_c as a function of M/h_c^2 for constant flow rate. The flow starts at point 1 on each curve and after the hydraulic jump transitions to point 2 for downstream of the jump.