2.161 Signal Processing: Continuous and Discrete
Fall 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING

2.161 *Signal Processing - Continuous and Discrete*
Fall Term 2008

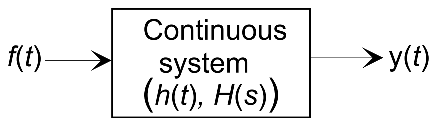
Lecture 13[1]

Reading:

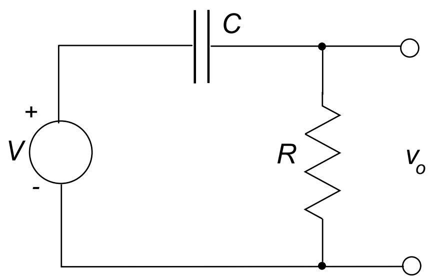
- Proakis & Manolakis, Chapter 3 (The z-transform)
- Oppenheim, Schafer & Buck, Chapter 3 (The z-transform)

1 Introduction to Time-Domain Digital Signal Processing

Consider a continuous-time filter



such as simple first-order RC high-pass filter:



described by a transfer function

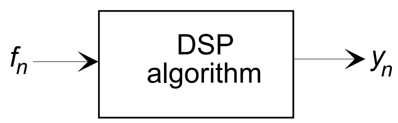
$$H(s) = \frac{RCs}{RCs + 1}.$$

The ODE describing the system is

$$\tau \frac{\mathrm{d}y}{\mathrm{d}t} + y = \tau \frac{\mathrm{d}f}{\mathrm{d}t}$$

where $\tau = RC$ is the time constant.

Our task is to derive a simple discrete-time equivalent of this prototype filter based on samples of the input $f(t)$ taken at intervals ΔT.



If we use a *backwards-difference* numerical approximation to the derivatives, that is

$$\frac{\mathrm{d}x}{\mathrm{d}t} \approx \frac{(x(n\Delta T) - x((n-1)\Delta T)}{\Delta T}$$

and adopt the notation $y_n = y(n\Delta T)$, and let $a = \tau/\Delta T$,

$$a(y_n - y_{n-1}) + y_n = a(f_n - f_{n-1})$$

and solving for y_n

$$\boxed{y_n = \frac{a}{1+a} y_{n-1} + \frac{a}{1+a} f_n - \frac{a}{1+a} f_{n-1}}$$

which is a first-order *difference equation*, and is the computational formula for a sample-by-sample implementation of digital high-pass filter derived from the continuous prototype above. Note that

- The "fidelity" of the approximation depends on ΔT, and becomes more accurate when $\Delta T \ll \tau$.
- At each step the output is a linear combination of the present and/or past samples of the output and input. This is a recursive system because the computation of the current output depends on prior values of the output.

In general, regardless of the design method used, a LTI digital filter implementation will be of a similar form, that is

$$\boxed{y_n = \sum_{i=1}^{N} a_i y_{n-i} + \sum_{i=0}^{M} b_i f_{n-i}}$$

where the a_i and b_i are constant coefficients. Then as in the simple example above, the current output is a weighted combination of past values of the output, and current and past values of the input.

- If $a_i \equiv 0$ for $i = 1 \ldots N$, so that

  $$y_n = \sum_{i=0}^{M} b_i f_{n-i}$$

  The output is simply a weighted sum of the current and prior inputs. Such a filter is a non-recursive filter with a finite-impulse-response (FIR), and is known as a *moving average* (MA) filter, or an *all-zero* filter.

- If $b_i \equiv 0$ for $i = 1 \ldots M$, so that

  $$y_n = \sum_{i=0}^{N} a_i y_{n-i} + b_0 f_n$$

  only the current input value is used. This filter is a recursive filter with an infinite-impulse-response (IIR), and is known as an *auto-regressive* (AR) filter, or an *all-pole* filter.

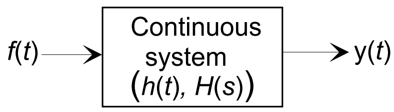
- With the full difference equation

$$y_n = \sum_{i=1}^{N} a_i y_{n-i} + \sum_{i=0}^{M} b_i f_{n-i}$$

the filter is a recursive filter with an infinite-impulse response (IIR), and is known as an *auto-regressive moving-average* (ARMA) filter.

2 The Discrete-time Convolution Sum

For a continuous system



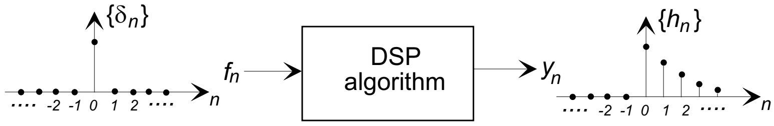
the output $y(t)$, in response to an input $f(t)$, is given by the convolution integral:

$$y(t) = \int_0^{\infty} f(\tau)h(t-\tau)d\tau$$

where $h(t)$ is the system impulse response.

For a LTI discrete-time system, such as defined by a difference equation, we define the *pulse response* sequence $\{h(n)\}$ as the response to a unit-pulse input sequence $\{\delta_n\}$, where

$$\delta_n = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$



If the input sequence $\{f_n\}$ is written as a sum of weighted and shifted pulses, that is

$$f_n = \sum_{k=-\infty}^{\infty} f_k \delta_{n-k}$$

then by superposition the output will be a sequence of similarly weighted and shifted pulse responses

$$\boxed{y_n = \sum_{k=-\infty}^{\infty} f_k h_{n-k}}$$

which defines the *convolution sum*, which is analogous to the convolution integral of the continuous system.

3 The z-Transform

The z-transform in discrete-time system analysis and design serves the same role as the Laplace transform in continuous systems. We begin here with a parallel development of both the z and Laplace transforms from the Fourier transforms.

The Laplace Transform

(1) We begin with causal $f(t)$ and find its Fourier transform (Note that because $f(t)$ is causal, the integral has limits of 0 and ∞):

$$F(j\Omega) = \int_0^\infty f(t)e^{-j\Omega t}dt$$

(2) We note that for some functions $f(t)$ (for example the unit step function), the Fourier integral does not converge.

(3) We introduce a weighted function

$$w(t) = f(t)e^{-\sigma t}$$

and note

$$\lim_{\sigma \to 0} w(t) = f(t)$$

The effect of the exponential weighting by $e^{-\sigma t}$ is to allow convergence of the integral for a much broader range of functions $f(t)$.

(4) We take the Fourier transform of $w(t)$

$$\begin{aligned} W(j\Omega) = \tilde{F}(j\Omega|\sigma) &= \int_0^\infty \left(f(t)e^{-\sigma t}\right) e^{-j\Omega t}dt \\ &= \int_0^\infty f(t)e^{-(\sigma + j\Omega)}dt \end{aligned}$$

and define the complex variable $s = \sigma + j\Omega$ so that we can write

$$F(s) = \tilde{F}(j\omega|\sigma) = \int_0^\infty f(t)e^{-st}dt$$

$F(s)$ is the one-sided Laplace Transform. Note that the Laplace variable $s = \sigma + j\Omega$ is expressed in Cartesian form.

The Z transform

(1) We sample f(t) at intervals ΔT to produce $f^*(t)$. We take its Fourier transform (and use the sifting property of $\delta(t)$) to produce

$$F^*(j\Omega) = \sum_{n=0}^{\infty} f_n e^{-jn\Omega\Delta T}$$

(2) We note that for some sequences f_n (for example the unit step sequence), the summation does not converge.

(3) We introduce a weighted sequence

$$\{w_n\} = \left\{f_n r^{-n}\right\}$$

and note

$$\lim_{r \to 1} \{w_n\} = \{f_n\}$$

The effect of the exponential weighting by r^{-n} is to allow convergence of the summation for a much broader range of sequences f_n.

(4) We take the Fourier transform of w_n

$$\begin{aligned} W^*(j\Omega) = \tilde{F}^*(j\Omega|r) &= \sum_{n=0}^{\infty} \left(f_n r^{-n}\right) e^{-jn\Omega\Delta T} \\ &= \sum_{n=0}^{\infty} f_n \left(re^{j\Omega\Delta T}\right)^{-n} \end{aligned}$$

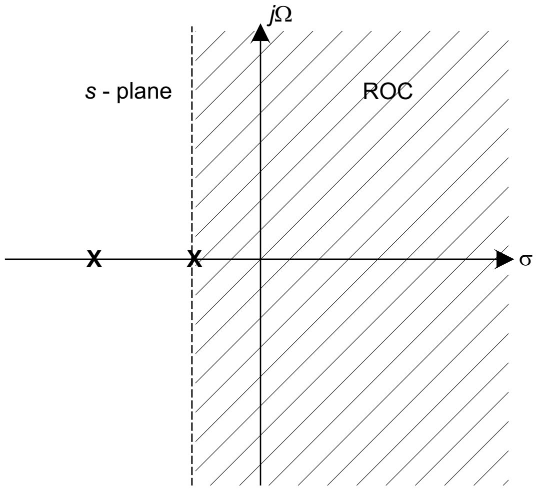
and define the complex variable $z = re^{j\Omega\Delta T}$ so that we can write

$$F(z) = \tilde{F}^*(j\Omega|r) = \sum_{n=0}^{\infty} f_n z^{-n}$$

$F(z)$ is the one-sided Z-transform. Note that $z = re^{j\Omega\Delta T}$ is expressed in polar form.

The Laplace Transform (contd.)

(5) For a causal function $f(t)$, the region of convergence (ROC) includes the s-plane to the right of all poles of $F(j\Omega)$.



(6) If the ROC includes the imaginary axis, the FT of $f(t)$ is $F(j\Omega)$:

$$F(j\Omega) = F(s)\,|_{s=j\Omega}$$

(7) The convolution theorem states

$$f(t)\otimes g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau \stackrel{\mathcal{L}}{\Longleftrightarrow} F(s)G(s)$$

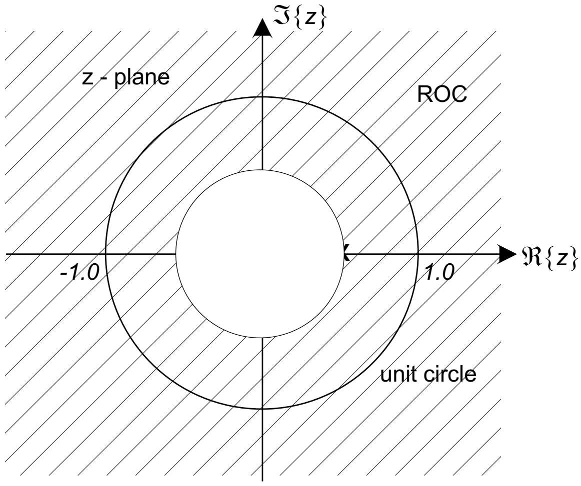
(8) For an LTI system with transfer function $H(s)$, the frequency response is

$$H(s)\,|_{s=j\Omega} = H(j\Omega)$$

if the ROC includes the imaginary axis.

The Z transform (contd.)

(5) For a right-sided (causal) sequence $\{f_n\}$ the region of convergence (ROC) includes the z-plane at a radius greater than all of the poles of $F(z)$.



(6) If the ROC includes the unit circle, the DFT of $\{f_n\}$, $n = 0, 1, \ldots, N-1$. is $\{F_m\}$ where

$$F_m = F(z)\,|_{z=e^{j\omega_m}} = F(e^{j\omega_m}),$$

where $\omega_m = 2\pi m/N$ for $m = 0, 1, \ldots, N-1$.

(7) The convolution theorem states

$$\{f_n\} \otimes \{g_n\} = \sum_{m=-\infty}^{\infty} f_m g_{n-m} \stackrel{\mathcal{Z}}{\Longleftrightarrow} F(z)G(z)$$

(8) For a discrete LSI system with transfer function $H(z)$, the frequency response is

$$H(z)\,|_{z=e^{j\omega}} = H(e^{j\omega}) \quad |\omega| \le \pi$$

if the ROC includes the unit circle.

From the above derivation, the Z-transform of a sequence $\{f_n\}$ is

$$F(z) = \sum_{n=-\infty}^{\infty} f_n z^{-n}$$

where $z = r\,e^{j\,\omega}$ is a complex variable. For a causal sequence $f_n = 0$ for $n < 0$, the transform can be written

$$F(z) = \sum_{n=0}^{\infty} f_n z^{-n}$$

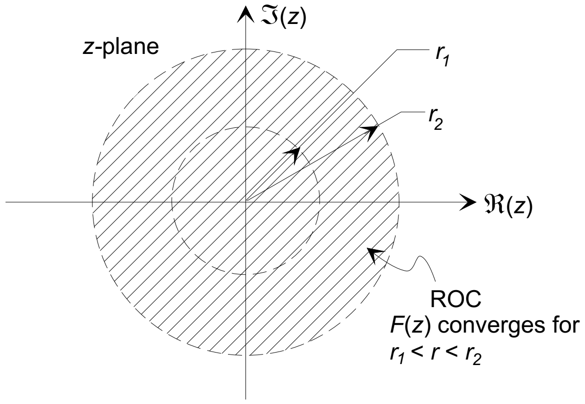
Example: The finite sequence $\{f_0, \ldots, f_3\} = \{5, 3, -1, 4\}$ has the z-transform

$$F(z) = 5z^0 + 3z^{-1} - z^{-2} + 4z^{-3}$$

The Region of Convergence: For a given sequence, the region of the z-plane in which the sum converges is defined as the *region of convergence* (ROC). In general, within the ROC

$$\sum_{n=-\infty}^{\infty} \left| f_n r^{-n} \right| < \infty$$

and the ROC is in general an annular region of the z-plane:



(a) The ROC is a ring or disk in the z-plane.

(b) The ROC cannot contain any poles of $F(z)$.

(c) For a finite sequence, the ROC is the entire z-plane (with the possible exception of $z = 0$ and $z = \infty$.

(d) For a causal sequence, the ROC extends outward from the outermost pole.

(e) for a left-sided sequence, the ROC is a disk, with radius defined by the innermost pole.

(f) For a two sided sequence the ROC is a disk bounded by two poles, but not containing any poles.

(g) The ROC is a connected region.

z-Transform Examples: In the following examples $\{u_n\}$ is the unit step sequence,

$$u_n = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$

and is used to force a causal sequence.

(1) $\{f_n\} = \{\delta_n\}$ (the digital pulse sequence)
From the definition of $F(z)$:

$$\boxed{F(z) = 1z^0 = 1 \qquad \text{for all } z.}$$

(2) $\{f_n\} = \{a^n u_n\}$

$$F(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

$$\boxed{\{a^n\} \overset{\mathcal{Z}}{\longleftrightarrow} F(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a} \qquad \text{for } |z| > a.}$$

since

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{for } x < 1.$$

(3) $\{f_n\} = \{u_n\}$ (the unit step sequence).

$$F(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}} = \frac{z}{z-1} \qquad \text{for } |z| < 1$$

from (2) with $a = 1$.

(4) $\{f_n\} = \left\{ \mathrm{e}^{-bn} u_n \right\}$.

$$F(z) = \sum_{n=0}^{\infty} \mathrm{e}^{-bn} z^{-n} = \sum_{n=0}^{\infty} \left(\mathrm{e}^{-b} z^{-1} \right)^n$$

$$\boxed{\left\{ \mathrm{e}^{-bn} \right\} \overset{\mathcal{Z}}{\longleftrightarrow} F(z) = \frac{1}{1-\mathrm{e}^{-b} z^{-1}} = \frac{z}{z-\mathrm{e}^{-bn}} \qquad \text{for } |z| > \mathrm{e}^{-b}.}$$

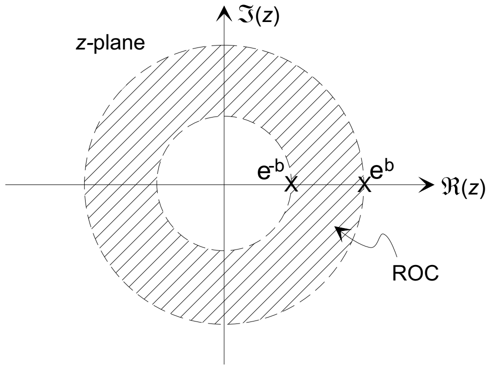
from (2) with $a = \mathrm{e}^{-b}$.

(5) $\{f_n\} = \left\{ \mathrm{e}^{-b|n|} \right\}$.

$$\begin{aligned} F(z) &= \sum_{n=-\infty}^{0} \left(\mathrm{e}^{-b} z \right)^{-n} + \sum_{n=0}^{\infty} \left(\mathrm{e}^{-b} z^{-1} \right)^n - 1 \\ &= \frac{1}{1-\mathrm{e}^{-b} z} + \frac{1}{1-\mathrm{e}^{-b} z^{-1}} - 1 \end{aligned}$$

Note that the item $f_0 = 1$ appears in each sum, therefore it is necessary to subtract 1.

$$\boxed{\left\{ \mathrm{e}^{-b|n|} \right\} \overset{\mathcal{Z}}{\longleftrightarrow} F(z) = \frac{1-\mathrm{e}^{-2b}}{(1-\mathrm{e}^{-b} z)(1-\mathrm{e}^{-b} z^{-1})} \qquad \text{for } \mathrm{e}^{-b} < |z| < \mathrm{e}^{b}.}$$



(6) $\{f_n\} = \{\,\mathrm{e}^{-j\omega_0 n} u_n\} = \{\cos(\omega_0 n)u_n\} - j\,\{\sin(\omega_0 n)u_n\}\;.$

$$F(z) = \mathcal{Z}\,\{\cos(\omega_0 n)u_n\} - j\mathcal{Z}\,\{\sin(\omega_0 n)u_n\}$$

From (1)

$$\begin{aligned} F(z) &= \frac{1}{1-\,\mathrm{e}^{-j\omega_0} z^{-1}} \qquad \text{for } |z|>1 \\ &= \frac{1-\cos(\omega_0)z^{-1} - j\sin(\omega_0)}{1-2\cos(\omega_0)z_*{-1}+z^{-2}} \\ &= \frac{z^2-\cos(\omega_0)z - j\sin(\omega_0)z^2}{z^2-2\cos(\omega_0)z+1} \end{aligned}$$

and therefore

$$\begin{aligned} \mathcal{Z}\,\{\cos(\omega_0 n)u_n\} &= \frac{z^2-\cos(\omega_0)z}{z^2-2\cos(\omega_0)z+1} \qquad \text{for } |z|>1 \\ \mathcal{Z}\,\{\sin(\omega_0 n)u_n\} &= \frac{\sin(\omega_0)z^2}{z^2-2\cos(\omega_0)z+1} \qquad \text{for } |z|>1 \end{aligned}$$

Properties of the z-Transform: Refer to the texts for a full description. We simply summarize some of the more important properties here.

(a) Linearity:

$a\,\{f_n\} + b\,\{g_n\} \stackrel{\mathcal{Z}}{\longleftrightarrow} aF(z) + bG(z)$ ROC: Intersection of ROC_f and ROC_g.

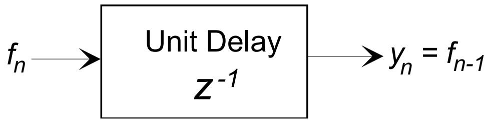
(b) Time Shift:

$\{f_{n-m}\} \stackrel{\mathcal{Z}}{\longleftrightarrow} z^{-m}F(z)$ ROC: ROC_f except for $z=0$ if $k<0$, or $z=\infty$ if $k>0$.

If $g_n = f_{n-m}$,

$$G(z) = \sum_{n=-\infty}^{\infty} f_{n-m} z^{-n} = \sum_{k=-\infty}^{\infty} f_k z^{-(k+m)} = z^{-m}F(z).$$

This is an important property in the analysis and design of discrete-time systems. We will often have recourse to a unit-delay block:



(c) Convolution:

$$\boxed{\{f_n\} \otimes \{g_n\} \stackrel{\mathcal{Z}}{\longleftrightarrow} F(z)G(z) \qquad \text{ROC: Intersection of ROC}_f \text{ and ROC}_g.}$$

where $\{f_n\} \otimes \{g_n\} = \sum_{k=-\infty}^{\infty} f_k g_{n-k}$ is the convolution sum.

Let

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} y_n z^{-n} = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} f_k g_{n-k} \right) z^{-n} \\ &= \sum_{k=-\infty}^{\infty} f_k \left(\sum_{n=-\infty}^{\infty} g_{n-k} z^{-(n-k)} \right) z^{-k} = \sum_{k=-\infty}^{\infty} f_k z^{-k} \sum_{m=-\infty}^{\infty} g_m z^{-m} \\ &= F(z)G(z) \end{aligned}$$

(d) Conjugation of a complex sequence:

$$\boxed{\{\overline{f}_n\} \stackrel{\mathcal{Z}}{\longleftrightarrow} \overline{F}(z) \qquad \text{ROC: ROC}_f}$$

(e) Time reversal:

$$\boxed{\{f_{-n}\} \stackrel{\mathcal{Z}}{\longleftrightarrow} F(1/z) \qquad \text{ROC: } \frac{1}{r_1} < |z| < \frac{1}{r_2}}$$

where the ROC of $F(z)$ lies between r_1 and r_2.

(e) Scaling in the z-domain:

$$\boxed{\{a^n f_n\} \stackrel{\mathcal{Z}}{\longleftrightarrow} F(a^{-1}z) \qquad \text{ROC: } |a|\, r_1 < |z| < |a|\, r_2}$$

where the ROC of $F(z)$ lies between r_1 and r_2.

(e) Differentiation in the z-domain:

$$\boxed{\{nf_n\} \stackrel{\mathcal{Z}}{\longleftrightarrow} -z\frac{\mathrm{d}F(z)}{\mathrm{d}z} \qquad \text{ROC: } r_2 < |z| < r_1}$$

where the ROC of $F(z)$ lies between r_1 and r_2.