2.161 Signal Processing: Continuous and Discrete
Fall 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING

2.161 *Signal Processing - Continuous and Discrete*
Fall Term 2008

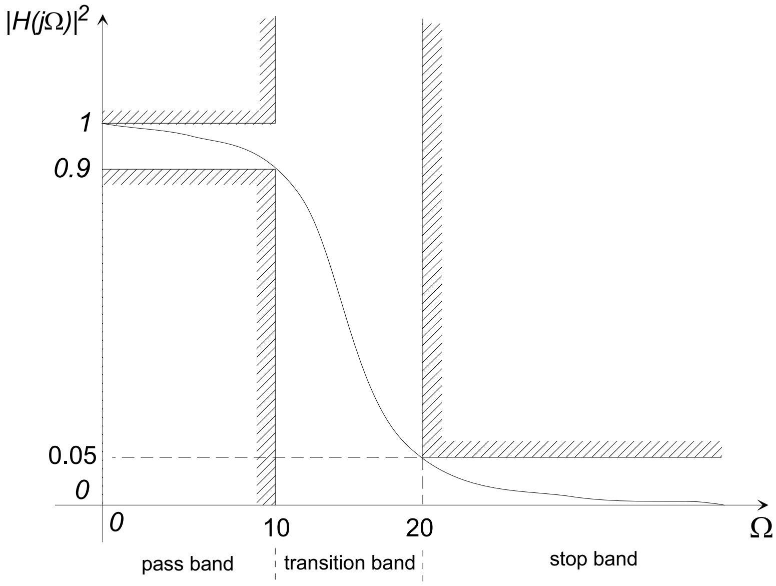
Lecture 7[1]

Reading:

- Class handout: *Introduction to Continuous Time Filter Design.*

1 Butterworth Filter Design Example

(Same problem as in the Class Handout). Design a Butterworth low-pass filter to meet the power gain specifications shown below:



At the two critical frequencies

$$\frac{1}{1+\epsilon^2} = 0.9 \quad \longrightarrow \quad \epsilon = 0.3333$$
$$\frac{1}{1+\lambda^2} = 0.05 \quad \longrightarrow \quad \lambda = 4.358$$

Then

$$N \geq \frac{\log(\lambda/\epsilon)}{\log(\Omega_r/\Omega_c)} = 3.70$$

we therefore select N=4. The 4 poles ($p_1 \ldots p_4$) lie on a circle of radius $r = \Omega_c \epsilon^{-1/N} = 13.16$ and are given by

$$\begin{aligned} |p_n| &= 13.16 \\ \angle p_n &= \pi(2n+3)/8 \end{aligned}$$

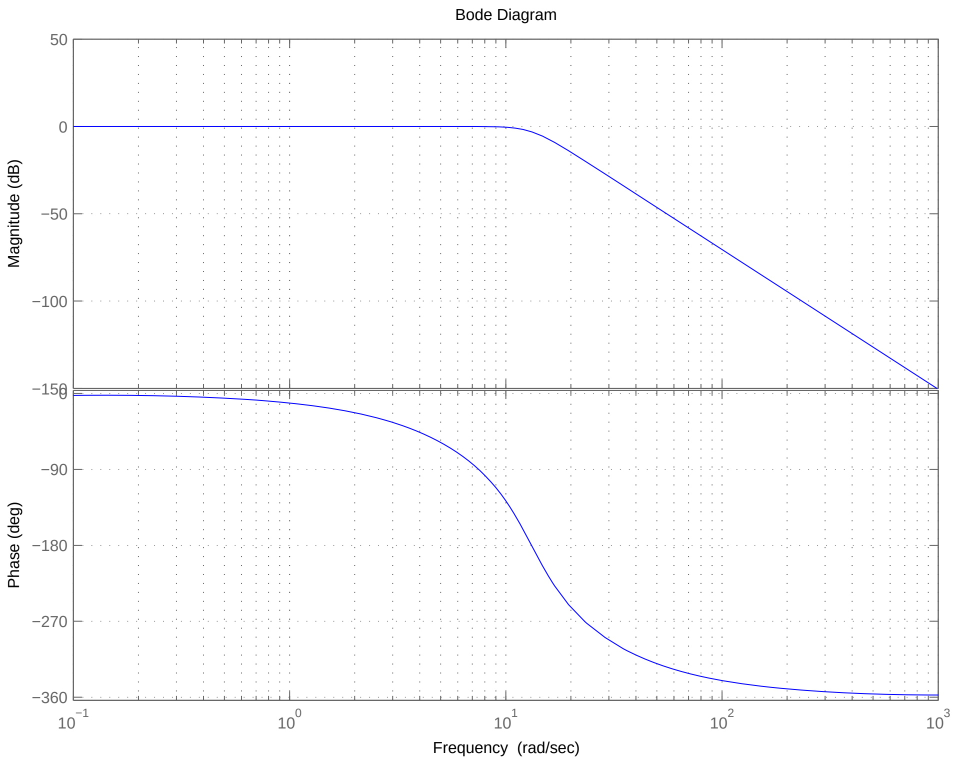
for $n = 1 \ldots 4$, giving a pair of complex conjugate pole pairs

$$\begin{aligned} p_{1,4} &= -5.04 \pm j12.16 \\ p_{2,3} &= -12.16 \pm j5.04 \end{aligned}$$

The transfer function, normalized to unity gain, is

$$H(s) = \frac{29993}{(s^2 + 10.07s + 173.2)(s^2 + 24.32s + 173.2)}$$

and the filter Bode plots are shown below.



2 Chebyshev Filters

The order of a filter required to met a low-pass specification may often be reduced by relaxing the requirement of a monotonically decreasing power gain with frequency, and allowing "ripple" to occur in either the pass-band or the stop-band. The Chebyshev filters allow these conditions:

$$\text{Type 1} \qquad |H(j\Omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\Omega/\Omega_c)} \tag{1}$$

$$\text{Type 2} \qquad |H(j\Omega)|^2 = \frac{1}{1+\epsilon^2 \left(T_N^2(\Omega_r/\Omega_c)/T_N^2(\Omega_r/\Omega)\right)} \tag{2}$$

Where $T_N(x)$ is the Chebyshev polynomial of degree N. Note the similarity of the form of the Type 1 power gain (Eq. (1)) to that of the Butterworth filter, where the function $T_N(\Omega/\Omega_c)$ has replaced $(\Omega/\Omega_c)^N$. The Type 1 filter produces an all-pole design with slightly different pole placement from the Butterworth filters, allowing resonant peaks in the pass-band to introduce ripple, while the Type 2 filter introduces a set of zeros on the imaginary axis above Ω_r, causing a ripple in the stop-band.

The Chebyshev polynomials are defined recursively as follows

$$\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
&\vdots \\
T_N(x) &= 2xT_{N-1}(x) - T_{N-2}(x), \qquad N > 1 \qquad (3)
\end{aligned}$$

with alternate definitions

$$T_N(x) = \cos(N\cos^{-1}(x)) \tag{4}$$

$$= \cosh(N\cosh^{-1}(x)) \tag{5}$$

The Chebyshev polynomials have the *min-max* property:

> *Of all polynomials of degree N with leading coefficient equal to one, the polynomial*
>
> $$T_N(x)/2^{N-1}$$
>
> *has the smallest magnitude in the interval* $|x| \le 1$. *This "minimum maximum" amplitude is* 2^{1-N}.

In low-pass filters given by Eqs. (13) and (14), this property translates to the following characteristics:

Filter	**Pass-Band Characteristic**	**Stop-Band Characteristic**
Butterworth	Maximally flat	Maximally flat
Chebyshev Type 1	Ripple between 1 and $1/(1+\epsilon^2)$	Maximally flat
Chebyshev Type 2	Maximally flat	Ripple between 1 and $1/(1+\lambda^2)$

2.1 The Chebyshev Type 1 Filter

With the power response from Eq. (13)

$$|H(j\Omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\Omega/\Omega_c)}$$

and the filter specification from Fig. 1, the required filter order may be found as follows. At the edge of the stop-band $\Omega = \Omega_r$

$$|H(j\Omega_r|^2 = \frac{1}{1+\epsilon^2 T_N^2(\Omega_r/\Omega_c)} \leq \frac{1}{1+\lambda^2}$$

so that

$$\lambda \leq \epsilon T_N(\Omega_r/\Omega_c) = \epsilon \cosh\left(N \cosh^{-1}(\Omega_r/\Omega_c)\right)$$

and solving for N

$$N \geq \frac{\cosh^{-1}(\lambda/\epsilon)}{\cosh^{-1}(\Omega_r/\Omega_c)} \tag{6}$$

The characteristic equation of the power transfer function is

$$1+\epsilon^2 T_N^2\left(\frac{s}{j\Omega_c}\right) = 0 \quad \text{or} \quad T_N\left(\frac{s}{j\Omega_c}\right) = \pm\frac{j}{\epsilon}$$

Now $T_N(x) = \cos(N\cos^{-1}(x))$, so that

$$\cos\left(N\cos^{-1}\left(\frac{s}{j\Omega_c}\right)\right) = \pm\frac{j}{\epsilon} \tag{7}$$

If we write $\cos^{-1}(s/j\Omega_c) = \gamma + j\alpha$, then

$$\begin{aligned} s &= \Omega_c\left(j\cos(\gamma+j\alpha)\right) \\ &= \Omega_c\left(\sinh\alpha\sin\gamma + j\cosh\alpha\cos\gamma\right) \end{aligned} \tag{8}$$

which defines an ellipse of width $2\Omega_c\sinh(\alpha)$ and height $2\Omega_c\cosh(\alpha)$ in the s-plane. The poles will lie on this ellipse. Substituting into Eq. (16)

$$\begin{aligned} T_N\left(\frac{s}{j\Omega_c}\right) &= \cos\left(N(\gamma+j\alpha)\right) \\ &= \cos N\gamma\cosh N\alpha - j\sin N\gamma\sinh N\alpha, \end{aligned}$$

the characteristic equation becomes

$$\cos N\gamma\cosh N\alpha - j\sin N\gamma\sinh N\alpha = \pm\frac{j}{\epsilon}. \tag{9}$$

Equating the real and imaginary parts in Eq. (21), (1) since $\cosh x \neq 0$ for real x we require $\cos N\gamma = 0$, or

$$\gamma_n = \frac{(2n-1)\pi}{2N} \qquad n = 1,\ldots,2N \tag{10}$$

and, (2) since at these values of γ, $\sin N\gamma = \pm 1$ we have

$$\alpha = \pm \frac{1}{N} \sinh^{-1} \frac{1}{\epsilon} \tag{11}$$

As in the Butterworth design procedure, we select the left half-plane poles as the poles of the filter frequency response.

Design Procedure:

1. Determine the filter order

$$N \geq \frac{\cosh^{-1}(\lambda/\epsilon)}{\cosh^{-1}(\Omega_r/\Omega_c)}$$

2. Determine α

$$\alpha = \pm \frac{1}{N} \sinh^{-1} \frac{1}{\epsilon}$$

3. Determine γ_n, $n = 1 \ldots N$

$$\gamma_n = \frac{(2n-1)\pi}{2N} \qquad n = 1, \ldots, N$$

4. Determine the N left half-plane poles

$$p_n = \Omega_c (\sinh\alpha \sin\gamma_n + j \cosh\alpha \cos\gamma_n) \qquad n = 1, \ldots, N$$

5. Form the transfer function

(a) If N is odd

$$H(s) = \frac{-p_1 p_2 \ldots p_N}{(s-p_1)(s-p_2)\ldots(s-p_N)}$$

(b) If N is even

$$H(s) = \frac{1}{1+\epsilon^2} \frac{p_1 p_2 \ldots p_N}{(s-p_1)(s-p_2)\ldots(s-p_N)}$$

The difference in the gain constants in the two cases arises because of the ripple in the pass-band. When N is odd, the response $|H(j0)|^2 = 1$, whereas if N is even the value of $|H(j0)|^2 = 1/(1+\epsilon^2)$.

■ Example 1

Repeat the previous Butterworth design example using a Chebyshev Type 1 design.
From the previous example we have $\Omega_c = 10$ rad/s., $\Omega_r = 20$ rad/s., $\epsilon = 0.3333$, $\lambda = 4.358$. The required order is

$$N \geq \frac{\cosh^{-1}(\lambda/\epsilon)}{\cosh^{-1}(\Omega_r/\Omega_c)} = \frac{\cosh^{-1} 13.07}{\cosh^{-1} 2} = 2.47$$

Therefore take $N = 3$. Determine α:

$$\alpha = \frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right) = \frac{1}{3}\sinh^{-1}(3) = 0.6061$$

and $\sinh\alpha = 0.6438$, and $\cosh\alpha = 1.189$. Also, $\gamma_n = (2n-1)\pi/6$ for $n = 1 \ldots 6$ as follows:

n:	1	2	3	4	5	6
γ_n:	$\pi/6$	$\pi/2$	$5\pi/6$	$7\pi/6$	$3\pi/2$	$11\pi/6$
$\sin\gamma_n$:	$1/2$	1	$1/2$	$-1/2$	-1	$-1/2$
$\cos\gamma_n$:	$\sqrt{3}/2$	0	$-\sqrt{3}/2$	$-\sqrt{3}/2$	0	$\sqrt{3}/2$

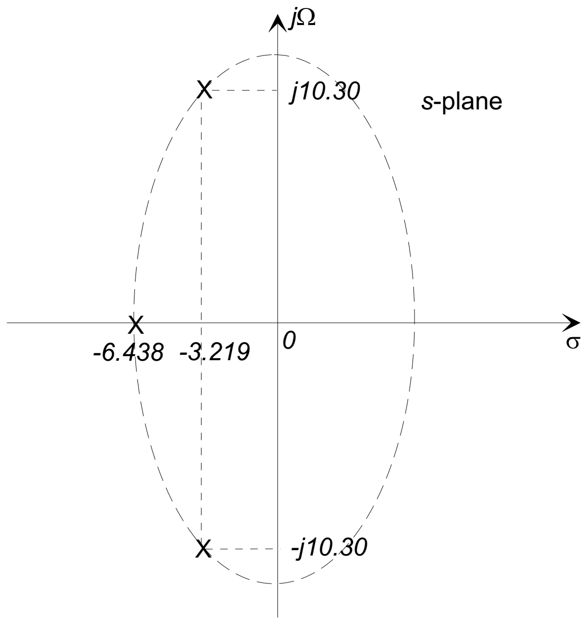
Then the poles are

$$\begin{aligned}
p_n &= \Omega_c\left(\sinh\alpha\sin\gamma_n + j\cosh\alpha\cos\gamma_n\right)\\
p_1 &= 10\left(0.6438 \times \frac{1}{2} + j1.189 \times \frac{\sqrt{3}}{2}\right) = 3.219 + j10.30\\
p_2 &= 10\left(0.6438 \times 1 + j1.189 \times 0\right) = 6.438\\
p_3 &= 10\left(0.6438 \times \frac{1}{2} - j1.189 \times \frac{\sqrt{3}}{2}\right) = 3.219 - j10.30\\
p_4 &= 10\left(-0.6438 \times \frac{1}{2} - j1.189 \times \frac{\sqrt{3}}{2}\right) = -3.219 - j10.30\\
p_5 &= 10\left(-0.6438 \times 0 - j1.189 \times 0\right) = -6.438\\
p_6 &= 10\left(-0.6438 \times \frac{1}{2} + j1.189 \times \frac{\sqrt{3}}{2}\right) = -3.219 + j10.30
\end{aligned}$$

and the gain adjusted transfer function of the resulting Type 1 filter is

$$H(s) = \frac{750}{(s^2 + 6.438s + 116.5)(s + 6.438)}$$

The pole-zero plot for the Chebyshev Type 1 filter is shown below.



2.2 The Chebyshev Type 2 Filter

The Chebyshev Type 2 filter has a monotonically decreasing magnitude function in the pass-band, but introduces equi-amplitude ripple in the stop-band by the inclusion of system zeros on the imaginary axis. The Type 2 filter is defined by the power gain function:

$$|H(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 \frac{T_N^2(\Omega_r/\Omega_c)}{T_N^2(\Omega_r/\Omega)}} \tag{12}$$

If we make the substitutions

$$\nu = \frac{\Omega_r \Omega_c}{\Omega} \quad \text{and} \quad \hat{\epsilon} = \frac{1}{\epsilon T_N(\Omega_r/\Omega_c)}$$

Eq. 24 may be written in terms of the modified frequency ν

$$|H(j\nu)|^2 = \frac{\hat{\epsilon}^2 T_N^2(\nu/\Omega_c)}{1 + \hat{\epsilon}^2 T_N^2(\nu/\Omega_c)} \tag{13}$$

which has a denominator similar to the Type 1 filter, but has a numerator that contains a Chebyshev polynomial, and is of order $2N$. We can use a method similar to that used in the Type 1 filter design to find the poles as follows:

1. First define a complex variable, say $\tau = \mu + j\nu$ (analogous to the Laplace variable $s = \sigma + j\Omega$ used in the type 1 design) and write the power transfer function:

$$|H(\tau)|^2 = \frac{\hat{\epsilon}^2 T_N^2(\tau/j\Omega_c)}{1 + \hat{\epsilon}^2 T_N^2(\tau/j\Omega_c)}$$

The poles are found using the method developed for the Type 1 filter, the zeros are found as the roots of the polynomial $T_N(\tau/j\Omega_c)$ on the imaginary axis $\tau = j\nu$. From the definition $T_N(x) = \cos\left(N \cos^{-1}(x)\right)$ it is easy to see that the roots of the Chebyshev polynomial occur at

$$x = \cos\left(\frac{(n-1/2)\pi}{N}\right) \qquad n = 1 \ldots N$$

and from Eq. (25) the system zeros will be at

$$\tau_n = j\Omega_c \cos\left(\frac{(n-1/2)\pi}{N}\right) \qquad n = 1 \ldots N.$$

2. The poles and zeros are mapped back to the s-plane using $s = \Omega_r\Omega_c/\tau$ and the N left half-plane poles are selected as the poles of the filter.
3. The transfer function is formed and the system gain is adjusted to unity at $\Omega = 0$.

■ Example 2

Repeat the previous Chebyshev Type 1 design example using a Chebyshev Type 2 filter.

From the previous example we have $\Omega_c = 10$ rad/s., $\Omega_r = 20$ rad/s., $\epsilon = 1/3$, $\lambda = 4.358$. The procedure to find the required order is the same as before, and we conclude that $N = 3$. Next, define

$$\begin{aligned} \nu &= \frac{\Omega_r\Omega_c}{\Omega} = \frac{200}{\Omega} \\ \hat{\epsilon} &= \frac{1}{\epsilon T_N(\Omega_r/\Omega_c)} = \frac{3}{T_3(2)} = 0.1154 \end{aligned}$$

Determine α:

$$\alpha = \frac{1}{N}\sinh^{-1}\left(\frac{1}{\hat{\epsilon}}\right) = \frac{1}{3}\sinh^{-1}(8.666) = 0.9520$$

and $\sinh\alpha = 1.1024$, and $\cosh\alpha = 1.4884$.

The values of $\gamma_n = (2n-1)\pi/6$ for $n = 1 \ldots 6$ are the same as the design for the

Type 1 filter, so that the poles of $|H(\tau)|^2$ are

$$\begin{aligned}
p_n &= \Omega_c\left(\sinh\alpha\sin\gamma_n + j\cosh\alpha\cos\gamma_n\right)\\
\tau_1 &= 10\left(1.1024\times\frac{1}{2} + j1.4884\times\frac{\sqrt{3}}{2}\right) = 5.512 + j12.890\\
\tau_2 &= 10\left(1.1024\times 1 + j1.4884\times 0\right) = 11.024\\
\tau_3 &= 10\left(1.1024\times\frac{1}{2} - j1.488\times\frac{\sqrt{3}}{2}\right) = 5.512 - j12.890\\
\tau_4 &= 10\left(-1.1024\times\frac{1}{2} - j1.4884\times\frac{\sqrt{3}}{2}\right) = -5.512 - j12.890\\
\tau_5 &= 10\left(-1.1024\times\frac{1}{2} - j1.488\times 0\right) = -11.024\\
\tau_6 &= 10\left(-1.1024\times\frac{1}{2} + j1.4884\times\frac{\sqrt{3}}{2}\right) = -5.512 + j12.890
\end{aligned}$$

The three left half-plane poles (τ_4, τ_5, τ_6) are mapped back to the s-plane using $s = \Omega_r\Omega_c/\tau$ giving three filter poles

$$\begin{aligned}
p_1, p_2 &= -5.609 \pm j13.117\\
p_3 &= -18.14
\end{aligned}$$

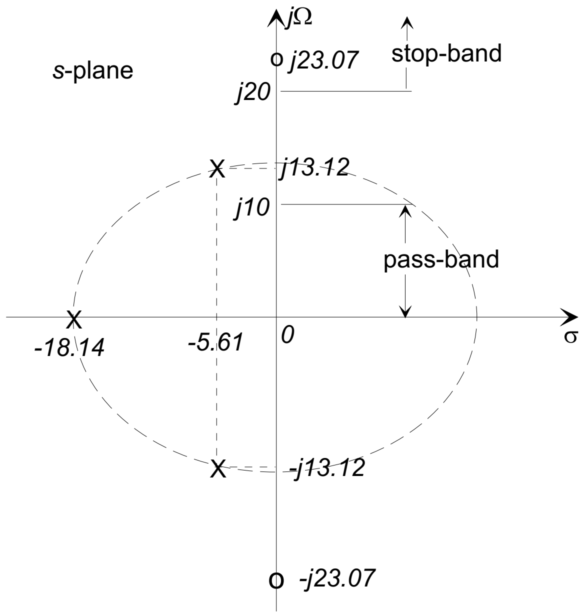
The system zeros are the roots of

$$T_3(\nu/j\Omega_c) = 4(\nu/j\Omega_c)^3 - 3(\nu/j\Omega_c) = 0$$

from the definition of $T_N(x)$, giving $\nu_1 = 0$ and $\nu_2, \nu_3 = \pm j8.666$. Mapping these back to the s-plane gives two finite zeros $z_1, z_2 = \pm j23.07, z_3 = \infty$ (the zero at ∞ does not affect the system response) and the unity gain transfer function is

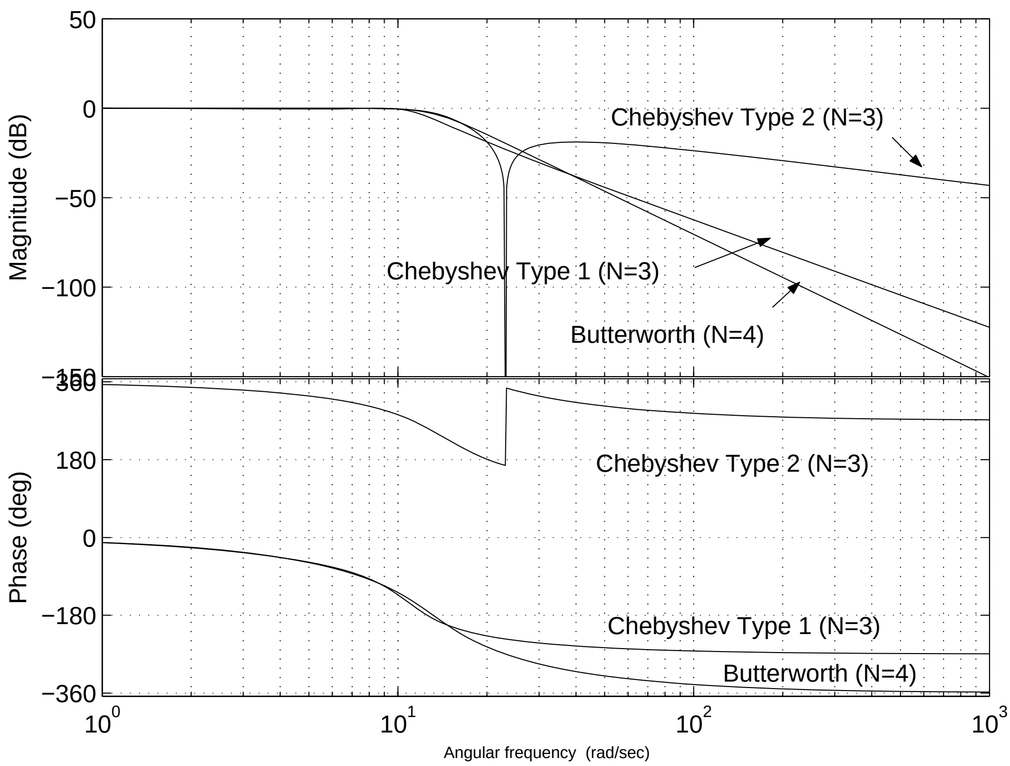
$$\begin{aligned}
H(s) &= \frac{-p_1p_2p_3}{z_1z_2}\frac{(s-z_1)(s-z_2)}{(s-p_1)(s-p_2)(s-p_3)}\\
&= \frac{6.9365(s^2+532.2)}{(s+18.14)(s^2+11.22s+203.5)}
\end{aligned}$$

The pole-zero plot for this filter is shown in below. Note that the poles again lie on ellipse, and the presence of the zeros in the stop-band.



2.3 Comparison of Filter Responses

Bode plot responses for the three previous example filters are shown below:



While all filters meet the design specification, it can be seen that the Butterworth and the Chebyshev Type 1 filters are all-pole designs and have an asymptotic high-frequency magnitude slope of $-20N$ dB/decade, in this case -80 dB/decade for the Butterworth design and -60 dB/decade for the Chebyshev Type 1 design. The Type 2 Chebyshev design has two finite zeros on the imaginary axis at a frequency of 23.07 rad/s, forcing the response to zero at this frequency, but with the result that its asymptotic high frequency response has a slope of only -20 dB/decade. Note also the singularity in the phase response of the Type 2 Chebyshev filter, caused by the two purely imaginary zeros.

The pass-band and stop-band power responses are shown in below. Notice that the design method developed here guarantees that the response will meet the specification at the cut-off frequency (in this case $|H(j\Omega)|^2 = 0.9$ at $\Omega_c = 10$. Other design methods (such as used by MATLAB) may not use this criterion.