2.161 Signal Processing: Continuous and Discrete
Fall 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING
2.161 Signal Processing – Continuous and Discrete
Fall Term 2008

Problem Set 1 Solution: Convolution and Fourier Transforms

Problem 1:

Use the convolution definition $y(t) = f \otimes h = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$

(a) $f(t) = \delta(t+1.5) + \delta(t) + \delta(t-0.75)$

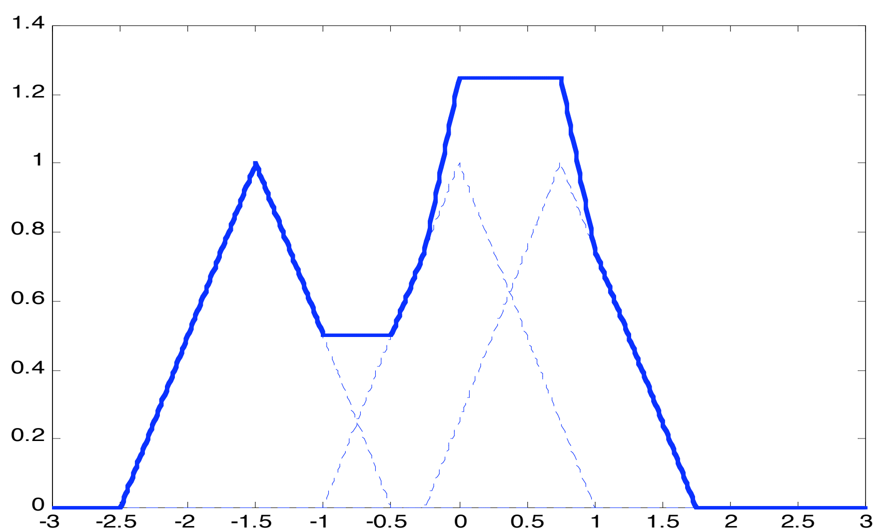
$$h(t) = \begin{cases} 1-|t|, -1 \le t \le 1 \\ 0, \ otherwise \end{cases}$$

$$y(t) = \int_{-\infty}^{\infty} \left(\delta(\tau+1.5) + \delta(\tau) + \delta(\tau-0.75)\right)h(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} \delta(\tau+1.5)h(t-\tau)d\tau + \int_{-\infty}^{\infty} \delta(\tau)h(t-\tau)d\tau + \int_{-\infty}^{\infty} \delta(\tau-0.75)h(t-\tau)d\tau,$$

and using the sifting property of the impulse function,

$$y(t) = h(t+1.5) + h(t) + h(t-0.75)$$

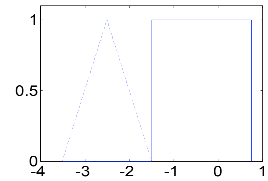
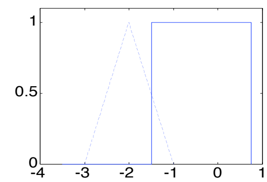
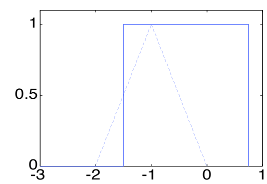
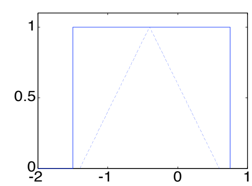


(b) *h(t)* is the as same used in part (a)

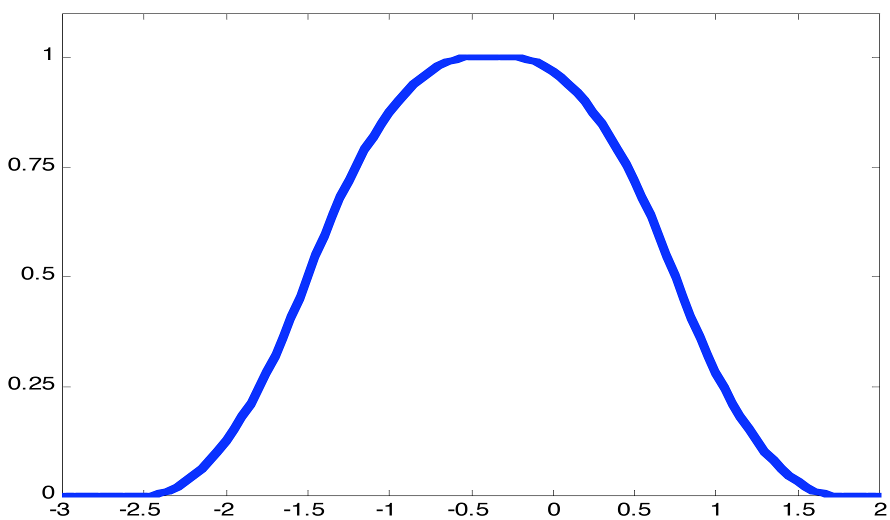
$$f(t) = \begin{cases} 1, -1.5 \le t \le 0.75 \\ 0, \ otherwise \end{cases},$$

then $y(t) = f \otimes h = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau = \int_{-1.5}^{0.75} h(t-\tau)d\tau$

Four basic cases can be observed while varying *t* (sliding the triangle waveform).,

Case	Range	Equation	Picture
Triangle outside	$t \le -2.5$ $1.75 \le t$	**0**	
Less than half triangle inside	$-2.5 \le t \le -1.5$ $0.75 \le t \le 1.75$	$\int_{-1.5}^{t+1}(1-(\tau-t))d\tau = \tau+\tau t-\frac{\tau^2}{2}\Big\vert_{-1.5}^{t+1} = \frac{(t+2.5)^2}{2}$ $\int_{t-1}^{0.75}(1+(\tau-t))d\tau = \tau-\tau t+\frac{\tau^2}{2}\Big\vert_{t-1}^{0.75} = \frac{(t-1.75)^2}{2}$	
More than half triangle inside	$-1.5 \le t \le -0.5$ $-0.25 \le t \le 0.75$	$\int_{-1.5}^{t}(1+(\tau-t))d\tau+0.5 = \tau-\tau t+\frac{\tau^2}{2}\Big\vert_{-1.5}^{t}+0.5 = 1-\frac{(t+0.5)^2}{2}$ $\int_{t}^{0.75}(1-(\tau-t))d\tau+0.5 = \tau+\tau t+\frac{\tau^2}{2}\Big\vert_{t}^{0.75}+0.5 = 1-\frac{(t+0.25)^2}{2}$	
Whole triangle inside	$-0.5 \le t \le -0.25$	**1**	

The result of the convolution, *y(t),* is plotted in the following figure

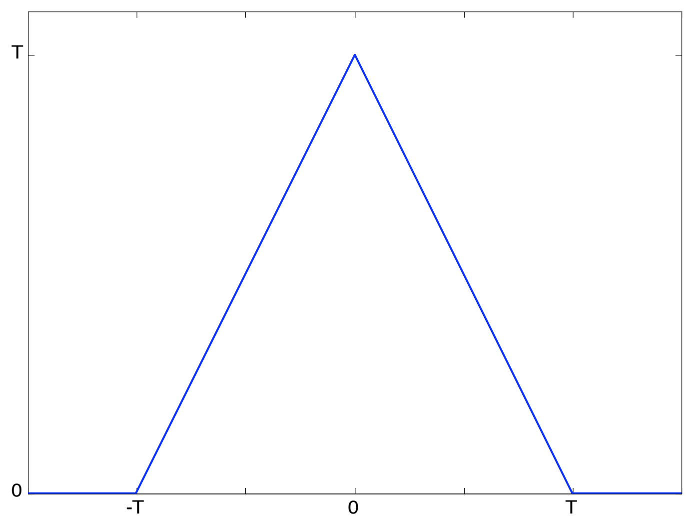


(c)

$$f(t)=\begin{cases}1, |t|\le \dfrac{T}{2}\\ 0,\ otherwise\end{cases}$$

then,

$$y(t)=f\otimes f=\int_{-\infty}^{\infty} f(\tau)f(t-\tau)d\tau=\int_{-T/2}^{T/2} f(t-\tau)d\tau=\begin{cases}\int_{-T/2}^{t+T/2} d\tau=t+T,\ -T\le t\le 0\\ \int_{t-T/2}^{T/2} d\tau=t-T,\ 0\le t\le T\\ 0,\ otherwise\end{cases}$$



$$F(j\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt=\int_{-T/2}^{T/2} e^{-j\omega t}dt=\left.\frac{e^{-j\omega t}}{-j\omega}\right]_{-T/2}^{T/2}=\frac{e^{-j\omega T/2}-e^{j\omega T}}{-j\omega}=\frac{\sin(\omega T/2)}{\omega/2}$$

when *T=1*, $h(t)=f(t)\otimes f(t)$ and from the convolution theorem

$$H(j\omega)=F(j\omega)F(j\omega)=\left(\frac{\sin(\omega/2)}{\omega/2}\right)^2=\left(\frac{\sin(x)}{x}\right)^2$$

Problem 2

$f_1(x) = e^{-ax^2}$, $f_2(x) = e^{-bx^2}$

$$y(x) = f_1 \otimes f_2 = \int_{-\infty}^{\infty} f_1(\tau) f_2(x-\tau) d\tau = \int_{-\infty}^{\infty} e^{-a\tau^2} e^{-b(x-\tau)^2} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-(a+b)\left(\tau^2 - \frac{2bx\tau}{a+b} + \frac{bx^2}{a+b}\right)} d\tau = \int_{-\infty}^{\infty} e^{-(a+b)\left(\tau - \frac{bx}{a+b}\right)^2 - \frac{abx^2}{a+b}} d\tau = e^{-\frac{abx^2}{a+b}} \int_{-\infty}^{\infty} e^{-\left(\sqrt{a+b}\tau - \frac{bx}{\sqrt{a+b}}\right)^2} d\tau$$

using the variable substitution: $\alpha = \sqrt{a+b}\tau - \frac{bx}{\sqrt{a+b}}$, $d\tau = \frac{d\alpha}{\sqrt{a+b}}$

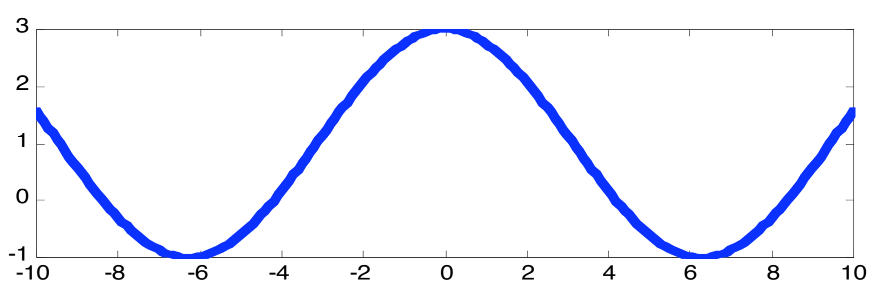
$\therefore\ y(t) = e^{-\frac{ab}{a+b}x^2} \int_{-\infty}^{\infty} e^{-\alpha^2} \frac{d\alpha}{\sqrt{a+b}} = \sqrt{\frac{\pi}{a+b}} e^{-\frac{ab}{a+b}x^2}$, which is a Gaussian function.

Problem 3

$f(t) = \delta(t+0.5) + \delta(t) + \delta(t-0.5)$

$$F(jw) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \delta(t+0.5) e^{-j\omega t} dt + \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt + \int_{-\infty}^{\infty} \delta(t-0.5) e^{-j\omega t} dt$$

$$F(j\omega) = e^{j\frac{\omega}{2}} + 1 + e^{-j\frac{\omega}{2}} = 1 + 2\cos(\frac{\omega}{2})$$



Problem 4

These solutions are all based on the elementary properties of the Fourier transform (see the class handout).

(a) $x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega\,|_{t=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)d\omega = \frac{1}{2\pi}\left(\frac{1}{2}X_0.2W + X_0.W\right) = \frac{1}{\pi}X_0 W$

(b) Using the symmetry properties, we note that $X(j\omega)$ is real, therefore $x(-t) = \overline{x(t)}$, that is they are complex conjugates.

(c) This one is a little tricky! We use the property that

$$\int_{-\infty}^{\infty} x(t)dt = X(j\omega)\,|_{\omega=0}$$

BUT note that there is a singularity at $\omega = 0$. The question is: what is the value of $X(j0)$? The problem statement specifies that $X(j\omega) = X_0$ for $0 \le \omega <$ W, so you can argue that

$$\int_{-\infty}^{\infty} x(t)dt = X_0.$$

On the other hand if you approximate the step discontinuity with a smooth function (say erf()) around $\omega = 0$, you can argue that the value of $X(j0) = 0.75X_0$, or

$$\int_{-\infty}^{\infty} x(t)dt = 0.75X_0.$$

So the answer is dependent on your assumption about the discontinuity!

(d) From Parseval's theorem

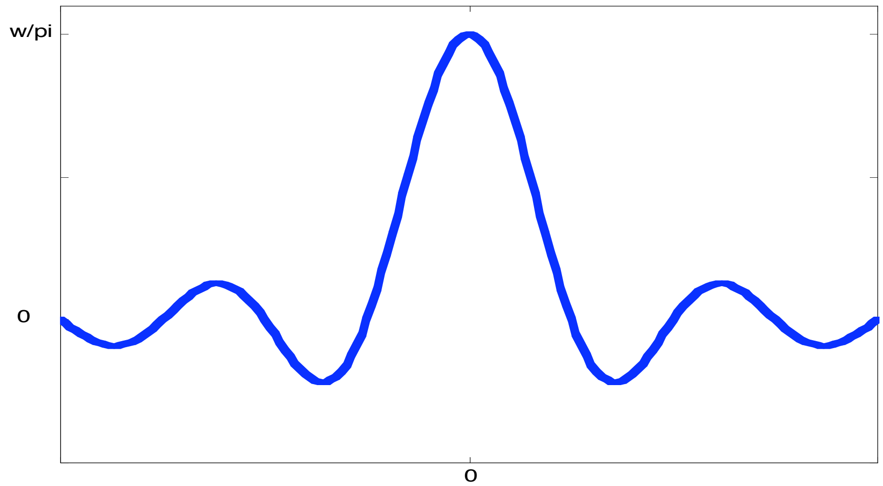
$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega = \frac{1}{2\pi}\left(\frac{1}{4}X_0^2.2W + X_0^2.W\right) = \frac{3}{4\pi}X_0^2 W\,.$$

Problem 5

$$H(j\omega) = \begin{cases} 1, |\omega| < \omega_c \\ 0, otherwise \end{cases}$$

If and impulse is passed through the filter, we obtain the impulse response $h(t) = F^{-1}\{H(j\omega)\}$

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t}d\omega = \frac{1}{2\pi jt}(e^{j\omega_c t} - e^{-j\omega_c t}) = \frac{\omega_c}{\pi}\frac{\sin\omega_c t}{\omega_c t} = \frac{\omega_c}{\pi}\mathrm{sinc}(\omega_c t)$$



The filter is <u>acausal.</u>

Problem 6

$h(t)=5e^{-3t}$ Let's compute the Fourier transform of *h(t)* $H(j\omega)=\int_0^\infty 5e^{-3t}e^{-j\omega t}dt=\frac{5}{j\omega+3}$

Note: lower limit in integral is 0 because a real filter is a causal system.

a) The transfer function can be found by taking the Laplace transform, which can be viewed as a Fourier transform where $j\omega$ is replaced by $s=\sigma+j\omega$.

$$H(s)=\frac{5}{s+3}$$

b) The frequency response is given by *H(jω)* computed previously.

c) We find the cut-off frequency by solving:

$$\frac{|H(j\omega_c)|}{|H(j0)|}=\frac{1}{2}\Rightarrow |H(j\omega_c)=\frac{H(j0)}{2}\Rightarrow\frac{5}{\sqrt{9+\omega_c^2}}=\frac{5}{6}\Rightarrow\omega_c^2=36-9=27\Rightarrow\omega_c=\sqrt{27}\,rad/s$$