23 APPENDIX 2: ADDED MASS VIA LAGRANGIAN DYNAMICS

The development of rigid body inertial dynamics presented in a previous section depends on the rates of change of vectors expressed in a moving frame, specifically that of the vehicle. An alternative approach is to use the *lagrangian*, wherein the dynamic behavior follows directly from consideration of the kinetic co-energy of the vehicle; the end result is exactly the same. Since the body dynamics were already developed, we here develop the lagrangian technique, using the analogous example of added mass terms. Among other effects, the equations elicit the origins of the Munk moment.

23.1 Kinetic Energy of the Fluid

The added mass matrix for a body in six degrees of freedom is expressed as the matrix M_a, whose negative is equal to:

$$-M_a = \begin{bmatrix} X_{\dot{u}} & X_{\dot{v}} & X_{\dot{w}} & X_{\dot{p}} & X_{\dot{q}} & X_{\dot{r}} \\ Y_{\dot{u}} & Y_{\dot{v}} & Y_{\dot{w}} & Y_{\dot{p}} & Y_{\dot{q}} & Y_{\dot{r}} \\ Z_{\dot{u}} & Z_{\dot{v}} & Z_{\dot{w}} & Z_{\dot{p}} & Z_{\dot{q}} & Z_{\dot{r}} \\ K_{\dot{u}} & K_{\dot{v}} & K_{\dot{w}} & K_{\dot{p}} & K_{\dot{q}} & K_{\dot{r}} \\ M_{\dot{u}} & M_{\dot{v}} & M_{\dot{w}} & M_{\dot{p}} & M_{\dot{q}} & M_{\dot{r}} \\ N_{\dot{u}} & N_{\dot{v}} & N_{\dot{w}} & N_{\dot{p}} & N_{\dot{q}} & N_{\dot{r}} \end{bmatrix}, \quad (260)$$

(Continued on next page)

where (X, Y, Z) denotes the force, (K, M, N) the moment, (u, v, w) denotes the velocity and (p, q, r) the angular velocity. The sense of M_a is that the fluid forces due to added mass are given by

$$\left\{ \begin{array}{c} X_{am} \\ Y_{am} \\ Z_{am} \\ K_{am} \\ M_{am} \\ N_{am} \end{array} \right\} = -M_a \frac{d}{dt} \left\{ \begin{array}{c} u \\ v \\ w \\ p \\ q \\ r \end{array} \right\}. \tag{261}$$

The added mass matrix M_a is completely analagous to the actual mass matrix of the vehicle,

$$M = \begin{bmatrix} m & 0 & 0 & 0 & z_G & -y_G \\ 0 & m & 0 & -z_G & 0 & x_G \\ 0 & 0 & m & y_G & -x_G & 0 \\ 0 & -z_G & y_G & I_{xx} & I_{xy} & I_{xz} \\ z_G & 0 & -x_G & I_{xy} & I_{yy} & I_{yz} \\ -y_G & x_G & 0 & I_{xz} & I_{yz} & I_{zz} \end{bmatrix}, \tag{262}$$

where $[x_G, y_G, z_G]$ are the (vessel frame) coordinates of the center of gravity. The mass matrix is symmetric, nonsingular, and positive definite. These properties are also true for the added mass matrix M_a, although symmetry fails when there is a constant forward speed.
The kinetic co-energy of the fluid E_k is found as:

$$E_k = -\frac{1}{2}\, q^T M_a \, q \tag{263}$$

where $q^T = (u, v, w, p, q, r)$. We expand to find in the non-symmetric case:

$$\begin{aligned} -2E_k \; = \; & X_{\dot{u}}u^2 + X_{\dot{v}}uv + X_{\dot{w}}uw + X_{\dot{p}}up + X_{\dot{q}}uq + X_{\dot{r}}ur + \\ & Y_{\dot{u}}uv + Y_{\dot{v}}v^2 + Y_{\dot{w}}vw + Y_{\dot{p}}vp + Y_{\dot{q}}vq + Y_{\dot{r}}vr + \\ & Z_{\dot{u}}uw + Z_{\dot{v}}vw + Z_{\dot{w}}w^2 + Z_{\dot{p}}wp + Z_{\dot{q}}wq + Z_{\dot{r}}wr + \\ & K_{\dot{u}}up + K_{\dot{v}}vp + K_{\dot{w}}wp + K_{\dot{p}}p^2 + K_{\dot{q}}pq + K_{\dot{r}}pr + \\ & M_{\dot{u}}uq + M_{\dot{v}}vq + M_{\dot{w}}wq + M_{\dot{p}}pq + M_{\dot{q}}q^2 + M_{\dot{r}}qr + \\ & N_{\dot{u}}ur + N_{\dot{v}}vr + N_{\dot{w}}wr + N_{\dot{p}}pr + N_{\dot{q}}qr + N_{\dot{r}}r^2. \end{aligned} \tag{264}$$

For the purposes of this discussion, we will assume from here on that M_a is symmetric, that is $M_a = M_a^T$, or, for example, $Y_{\dot{u}} = X_{\dot{v}}$. In general, this implies that the i'th force due to the j'th acceleration is equal to the j'th force due to the i'th acceleration, for i and j = [1,2,3,4,5,6]. Refering to the above equation, these terms occur as pairs, so the asymmetric case could be reconstructed easily from what follows. By restricting the added mass to be symmetric, then, we find:

$$
\begin{aligned}
-2E_k \;=\;& X_{\dot u}u^2 + Y_{\dot v}v^2 + Z_{\dot w}w^2 + K_{\dot p}p^2 + M_{\dot q}q^2 + N_{\dot r}r^2 + \\
& 2X_{\dot v}uv + 2X_{\dot w}uw + 2X_{\dot p}up + 2X_{\dot q}uq + 2X_{\dot r}ur + \\
& 2Y_{\dot w}vw + 2Y_{\dot p}vp + 2Y_{\dot q}vq + 2Y_{\dot r}vr + \\
& 2Z_{\dot p}wp + 2Z_{\dot q}wq + 2Z_{\dot r}wr + \\
& 2K_{\dot q}pq + 2K_{\dot r}pr + \\
& 2M_{\dot r}qr. \qquad (265)
\end{aligned}
$$

23.2 Kirchhoff's Relations

To derive the fluid inertia terms in the body-referenced equations of motion, we use Kirchhoff's relations with the co-energy E_k; see the derivation below, or Milne-Thomson (1960). These relations state that if $\vec{v} = [u, v, w]$ denotes the velocity vector and $\vec{\omega} = [p, q, r]$ the angular velocity vector, then the inertia terms expressed in axes affixed to a moving vehicle are

$$\vec{F} = -\frac{\partial}{\partial t}\left(\frac{\partial E_k}{\partial \vec{v}}\right) - \vec{\omega} \times \frac{\partial E_k}{\partial \vec{v}}, \qquad (266)$$

$$\vec{Q} = -\frac{\partial}{\partial t}\left(\frac{\partial E_k}{\partial \vec{\omega}}\right) - \vec{\omega} \times \frac{\partial E_k}{\partial \vec{\omega}} - \vec{v} \times \frac{\partial E_k}{\partial \vec{v}}. \qquad (267)$$

where $\vec{F} = [X, Y, Z]$ express the force vector, $\vec{Q} = [K, M, N]$ the moment vector, and $\times$ denotes the cross product.

23.3 Fluid Inertia Terms

Applying Kirchhoff's relations to the expression for the kinetic co-energy with a symmetric added mass matrix, we derive the following terms containing the fluid inertia forces:

$$
\begin{aligned}
X \;=\;& +X_{\dot u}\dot u + X_{\dot v}(\dot v - ur) + X_{\dot w}(\dot w + uq) + X_{\dot p}\dot p + X_{\dot q}\dot q + X_{\dot r}\dot r \\
& -Y_{\dot v}vr + Y_{\dot w}(vq - wr) - Y_{\dot p}pr - Y_{\dot q}qr - Y_{\dot r}r^2 \\
& +Z_{\dot w}wq + Z_{\dot p}pq + Z_{\dot q}q^2 + Z_{\dot r}qr. \qquad (268)
\end{aligned}
$$

The Y and Z forces can be obtained, through rotational symmetry, in the form:

$$
\begin{aligned}
Y \;=\;& +X_{\dot v}(\dot u + ur) - X_{\dot w})up \\
& +Y_{\dot v}(\dot v + vr) + Y_{\dot w}(\dot w - vp + wr) + Y_{\dot p}(\dot p + pr) + Y_{\dot q}(\dot q + qr) + Y_{\dot r}(\dot r + r^2) \\
& -Z_{\dot w}wp - Z_{\dot p}p^2 - Z_{\dot q}pq - Z_{\dot r}pr \qquad (269)
\end{aligned}
$$

$$\begin{aligned} Z \;=\; & -X_{\dot{u}}uq + X_{\dot{v}}(up - vq) + X_{\dot{w}}(\dot{u} - wq) - X_{\dot{p}}pq - X_{\dot{q}}q^2 - X_{\dot{r}}qr \\ & +Y_{\dot{v}}vp + Y_{\dot{w}}(\dot{v} + wp) + Y_{\dot{p}}p^2 + Y_{\dot{q}}pq + Y_{\dot{r}}pr \\ & +Z_{\dot{w}}\dot{w} + Z_{\dot{p}}\dot{p} + Z_{\dot{q}}\dot{q} + Z_{\dot{r}}\dot{r} \end{aligned} \qquad (270)$$

The apparent imbalance of coefficients comes from symmetry, which allows us to use only the 21 upper-right elements of the added mass matrix in Equation 260, e.g., $M_{\dot{v}} = Y_{\dot{q}}$. The moments K, M, and N are obtained in a similar manner as:

$$\begin{aligned} K \;=\; & -X_{\dot{v}}wu + X_{\dot{w}}uv + X_{\dot{r}}uq + X_{\dot{p}}\dot{u} - X_{\dot{q}}ur \\ & -Y_{\dot{v}}vw + Y_{\dot{w}}(v^2 - w^2) + Y_{\dot{p}}(\dot{v} - wp) - Y_{\dot{q}}(vr + wq) + Y_{\dot{r}}(vq - wr) \\ & +Z_{\dot{w}}vw + Z_{\dot{p}}(\dot{w} + vp) + Z_{\dot{q}}(vq - wr) + Z_{\dot{r}}(wq + vr) \\ & +K_{\dot{p}}\dot{p} + K_{\dot{q}}(\dot{q} - rp) + K_{\dot{r}}(\dot{r} + pq) \\ & +M_{\dot{r}}(q^2 - r^2) - M_{\dot{q}}qr \\ & +N_{\dot{r}}qr \end{aligned} \qquad (271)$$

$$\begin{aligned} M \;=\; & +X_{\dot{u}}uw + X_{\dot{v}}vw + X_{\dot{w}}(w^2 - u^2) + X_{\dot{p}}(ur + wp) + X_{\dot{q}}(\dot{u} + wq) + X_{\dot{r}}(wr - up) \\ & -Y_{\dot{w}}uv + Y_{\dot{p}}vr + Y_{\dot{q}}\dot{v} - Y_{\dot{r}}vp \\ & -Z_{\dot{w}}uw + Z_{\dot{p}}(wr - up) + Z_{\dot{q}}(\dot{w} - uq) - Z_{\dot{r}}(ur + wp) \\ & +K_{\dot{p}}pr + K_{\dot{q}}(\dot{p} + qr) + K_{\dot{r}}(r^2 - p^2) \\ & +M_{\dot{q}}\dot{q} - M_{\dot{r}}pq + M_{\dot{r}}\dot{r} \\ & -N_{\dot{r}}pr \end{aligned} \qquad (272)$$

$$\begin{aligned} N \;=\; & -X_{\dot{u}}uv + X_{\dot{v}}(u^2 - v^2) - X_{\dot{w}}vw - X_{\dot{p}}(uq + vp) + X_{\dot{q}}(up - vq) + X_{\dot{r}}(\dot{u} - vr) \\ & +Y_{\dot{v}}uv + Y_{\dot{w}}uw + Y_{\dot{p}}(up - vq) + Y_{\dot{q}}(uq + vp) + Y_{\dot{r}}(\dot{v} + ur) \\ & -Z_{\dot{p}}wq + Z_{\dot{q}}wp + Z_{\dot{r}}\dot{w} \\ & -K_{\dot{p}}pq + K_{\dot{q}}(p^2 - q^2) + K_{\dot{r}}(\dot{p} - qr) \\ & +M_{\dot{q}}pq + M_{\dot{r}}(\dot{q} + pr) \\ & +N_{\dot{r}}\dot{r} \end{aligned} \qquad (273)$$

23.4 Derivation of Kirchhoff's Relations

We can derive Kirchhoff's relation for a lagrangian $L(\vec{v}, \vec{\omega}, t)$, involving the velocity $\vec{v}$ and angular velocity $\vec{\omega}$, whose components will be expressed in a local coordinate system rotating with the angular velocity $\vec{\omega}$, i.e. in a reference system fixed on the body. The principle to satisfy is that of least action, i.e. to minimize the integral I (Crandall *et al.*, 1968):

$$I = \int_{t_1}^{t_2} L(\vec{v}, \vec{\omega}, t)dt \tag{274}$$

At the minimum value of I - the admissible condition - the variation of I, δI, and hence of L with the velocity $\vec{v}$ and angular rate $\vec{\omega}$ vanishes.
Our condition $\delta I = 0$, as written, involves only the lagrangian, which in the more general case is the kinetic co-energy minus the potential energy of the system. Since we are considering the motion of a body in an unbounded, homogeneous fluid, there is no potential energy, so the lagrangian is exactly the kinetic energy:

$$L = E_k. \tag{275}$$

Hamilton's principle in its general form also accounts for applied forces and moments $\vec{\Xi}$. They are defined to align with the generalized, infinitesimal linear and angular displacements $\delta\vec{\eta}$ and $\delta\vec{\phi}$, leading to

$$\delta I = \int_{t_1}^{t_2} \left[\delta L(\vec{v}, \vec{\omega}, t) + \left\langle \vec{\Xi}_{u,v,w}, \delta\vec{\eta} \right\rangle + \left\langle \vec{\Xi}_{p,q,r}, \delta\vec{\phi} \right\rangle \right] dt. \tag{276}$$

Now, the lagrangian is invariant under coordinate transformation, so it is a function of the *free* vectors of velocity and angular velocity. Using the notation detailed in the Nomenclature section below, $\delta\vec{\eta}$ and $\delta\vec{\phi}$ are interpreted as free vectors, while $\delta\underline{\eta}$ and $\delta\underline{\phi}$ are the projections of the free vectors onto a given reference frame. The following relationships link the displacements with the body-referenced rates:

$$\begin{aligned} \underline{v} &= \frac{\partial \underline{\eta}}{\partial t} + \underline{\omega} \otimes \underline{\eta}, \text{ and} \qquad (277) \\ \underline{\omega} &= \frac{\partial \underline{\phi}}{\partial t}. \qquad (278) \end{aligned}$$

A variation of the lagrangian at a given time t is, to first order,

$$\delta L(\vec{v}, \vec{\omega}, t) = \left\langle \frac{\partial L}{\partial \vec{v}}, \delta\vec{v} \right\rangle + \left\langle \frac{\partial L}{\partial \vec{\omega}}, \delta\vec{\omega} \right\rangle \tag{279}$$

The variations $\delta\vec{v}$ and $\delta\vec{\omega}$, in view of equations 277 and 278, can be written as

$$\begin{aligned} \delta\vec{v} &= \delta\underline{v}^T \hat{x} + \{\underline{v}^T \delta\hat{x}\} \\ &= \left(\frac{\partial \delta\underline{\eta}}{\partial t} + \underline{\omega} \otimes \delta\underline{\eta} \right)^T \hat{x} + \vec{v} \times \delta\vec{\phi}, \qquad (280) \end{aligned}$$

$$\begin{aligned} \delta\vec{\omega} &= \delta\underline{\omega}^T\hat{x} + \{\underline{\omega}^T\delta\hat{x}\} \\ &= \left(\frac{\partial\delta\underline{\phi}}{\partial t}\right)^T \hat{x} + \vec{\omega}\times\delta\vec{\phi}. \end{aligned} \tag{281}$$

The terms $\{\underline{v}^T\delta\hat{x}\}$ and $\{\underline{w}^T\delta\hat{x}\}$ above represent the effects of the variation of body orientation, and lead to one of the more subtle points of the derivation. It can be shown that $\delta\hat{x}$, the displacement of the unit triad $\hat{x}$ is $\delta\vec{\phi}\times\hat{x}$, leading for example to $\underline{v}^T(\delta\vec{\phi}\times\hat{x})$. This form, however, fails to capture what is meant by $\{\underline{v}^T\delta\hat{x}\}$ and $\{\underline{w}^T\delta\hat{x}\}$, specifically: how the free vectors $\delta\vec{v}$ and $\delta\vec{w}$ change as the triad rotates, but the projections $\underline{v}$ and $\underline{w}$ *remain constant*. With this in mind, one can easily derive the proper interpretations, as written above.

We now return to the evaluation of the lagrangian L. Combining terms, we see that there will be five inner products to consider. Writing the terms involving the time derivatives as δL_1 and δI_1, we have from an integration by parts

$$\begin{aligned} \delta I_1 &= \int_{t_1}^{t_2}\left[\left\langle\frac{\partial L}{\partial\vec{v}},\left(\frac{\partial\delta\underline{\eta}}{\partial t}\right)^T\hat{x}\right\rangle + \left\langle\frac{\partial L}{\partial\vec{\omega}},\left(\frac{\partial\delta\underline{\phi}}{\partial t}\right)^T\hat{x}\right\rangle\right]dt \\ &= -\int_{t_1}^{t_2}\left[\left\langle\frac{\partial}{\partial t}\frac{\partial L}{\partial\vec{v}},\delta\vec{\eta}\right\rangle + \left\langle\frac{\partial}{\partial t}\frac{\partial L}{\partial\vec{\omega}},\delta\vec{\phi}\right\rangle\right]dt. \end{aligned} \tag{282}$$

There are no terms remaining at the time boundaries because the lagrangian is zero at these points.

In evaluating the remaining terms, in δI_2, one needs to use the following property of triple vector products:

$$< \vec{a},\vec{b}\times\vec{c} > = < \vec{c},\vec{b}\times\vec{a} > = < \vec{b},\vec{c}\times\vec{a} > . \tag{283}$$

Hence, the variation of the action I_2, expressed in terms of vectors, is

$$\begin{aligned} \delta I_2 &= \int_{t_1}^{t_2}\left[\left\langle\frac{\partial L}{\partial\vec{v}},\vec{\omega}\times\delta\vec{\eta}+\vec{v}\times\delta\vec{\phi}\right\rangle + \left\langle\frac{\partial L}{\partial\vec{\omega}},\vec{\omega}\times\delta\vec{\phi}\right\rangle\right]dt \\ &= -\int_{t_1}^{t_2}\left[\left\langle\vec{\omega}\times\frac{\partial L}{\partial\vec{v}},\delta\vec{\eta}\right\rangle + \left\langle\vec{v}\times\frac{\partial L}{\partial\vec{v}}+\vec{\omega}\times\frac{\partial L}{\partial\vec{\omega}},\delta\vec{\phi}\right\rangle\right]dt. \end{aligned} \tag{284}$$

Finally, we write the part of δI due to the generalized forces:

$$\delta I_3 = \int_{t_1}^{t_2}\left[\left\langle\vec{\Xi}_{u,v,w},\delta\vec{\eta}\right\rangle + \left\langle\vec{\Xi}_{p,q,r},\delta\vec{\phi}\,\right\rangle\right]dt. \tag{285}$$

The Kirchoff relations follow directly from combining the three action variations. The kinetic co-energy we developed is that of the fluid, and so the generalized forces indicate what forces the body would exert on the fluid; with a sign change, these are the forces that the fluid exerts on the body.

23.5 Nomenclature

23.5.1 Free versus Column Vector

We make the distinction between a **free vector** $\vec{f}$, which is an element of a linear vector field, and a **column vector** $\underline{f}$ which denotes the components of the vector $\vec{f}$ in a given coordinate system. The connection between the two concepts is given via the free triad $\hat{x}$, containing as elements the unit vectors of the chosen Cartesian system, $\hat{i}, \hat{j}, \hat{k}$, i.e.:

$$\hat{x} = (\hat{i}, \hat{j}, \hat{k})^T \tag{286}$$

The notation is hybrid, but convenient since it allows us to write:

$$\vec{f} = \underline{f}^T \hat{x} \tag{287}$$

where $\underline{f}^T$ denotes the transpose of $\underline{f}$, i.e. a row vector. The product between the row vector $\underline{f}^T$ and the column vector $\hat{x}$ is in the usual matrix multiplication sense.

23.5.2 Derivative of a Scalar with Respect to a Vector

The derivative of a scalar L with respect to a vector $\vec{x}$ is denoted as:

$$\frac{\partial L}{\partial \vec{x}} \tag{288}$$

and is a vector with the same dimension as $\vec{x}$, whose element i is the derivative of L with respect to the ith element of $\vec{x}$, and in the same direction:

$$\left(\frac{\partial L}{\partial \vec{x}}\right)_i = \frac{\partial L}{\partial \vec{x}_i} \tag{289}$$

23.5.3 Dot and Cross Product

The dot product of two free vectors $\vec{f}$ and $\vec{g}$ is denoted as $< \vec{f}, \vec{g} >$, and we use the following notation for column vectors: $< \underline{f}^T \hat{x}, \underline{g}^T \hat{x} >$. The cross product of two vectors $\vec{f}$ and $\vec{g}$ is $\vec{z} = \vec{f} \times \vec{g}$, denoted in terms of column vectors as $\underline{z} = \underline{f} \otimes \underline{g}$.