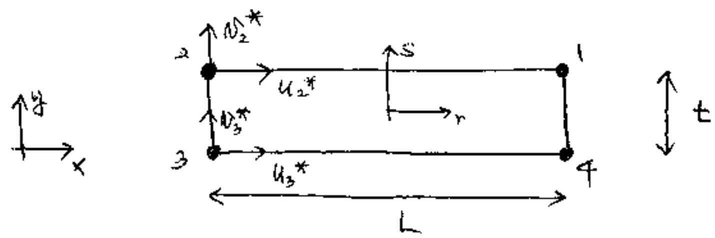
Lecture 19 - Slender structures

Prof. K.J. Bathe MIT OpenCourseWare

Beam analysis, $\frac{t}{L} \ll 1$ (e.g. $\frac{t}{L} = \frac{1}{100}, \frac{1}{1000}, \cdots$)

Reading: Sec. 5.4, 6.5

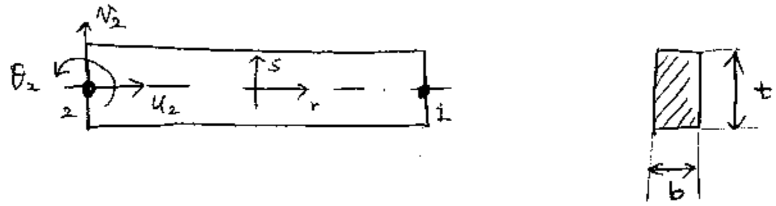


(plane stress)

$$\boldsymbol{J} = \begin{bmatrix} \frac{L}{2} & 0 \\ 0 & \frac{t}{2} \end{bmatrix} \tag{19.1}$$

$$h_2 = \frac{1}{4}(1-r)(1+s) \tag{19.2}$$

$$h_3 = \frac{1}{4}(1-r)(1-s) \tag{19.3}$$



Beam theory assumptions (Timoshenko beam theory):

$$v_2^* = v_3^* = v_2 \tag{19.4}$$

$$u_3^* = u_2 + \frac{t}{2}\theta_2 \tag{19.5}$$

$$u_2^* = u_2 - \frac{t}{2}\theta_2 \tag{19.6}$$

$$\begin{matrix} & u_2^* & v_2^* & u_3^* & v_3^* \end{matrix}$$

$$\boldsymbol{B}^* = \begin{bmatrix} -\frac{1}{4}(1+s)\frac{2}{L} & 0 & -\frac{1}{4}(1-s)\frac{2}{L} & 0 \\ 0 & \frac{1}{4}(1-r)\frac{2}{t} & 0 & -\frac{1}{4}(1-r)\frac{2}{t} \\ \frac{1}{4}(1-r)\frac{2}{t} & -\frac{1}{4}(1+s)\frac{2}{L} & -\frac{1}{4}(1-r)\frac{2}{t} & -\frac{1}{4}(1-s)\frac{2}{L} \end{bmatrix} \text{ etc} \tag{19.7}$$

$$\boldsymbol{B}_{\text{beam}} = \begin{array}{c} \\ \\ \\ \end{array} \begin{bmatrix} & u_2 & v_2 & \theta_2 \\ & -\frac{1}{L} & 0 & \frac{t}{2L}s \\ \cdots & \emptyset & \emptyset & \emptyset \\ & 0 & -\frac{1}{L} & -\frac{1}{2}(1-r) \end{bmatrix} \sim \begin{pmatrix} \frac{\partial u}{\partial x} \\ \cancelto{0}{\frac{\partial v}{\partial y}} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix} \tag{19.8}$$

$$v(r) = \frac{1}{2}(1-r)v_2 \tag{19.9}$$

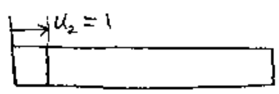
$$u(r) = \frac{1}{2}(1-r)u_2 - \frac{st}{4}(1-r)\theta_2 \tag{19.10}$$

at $r = -1$,

$$v(-1) = v_2 \tag{19.11}$$

$$u(-1) = -\frac{st}{2}\theta_2 + u_2 \tag{19.12}$$

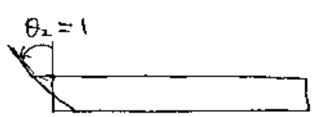
Kinematics is



$$u(r) = \frac{1}{2}(1-r)u_2 \tag{19.13}$$

results into ϵ_{xx}

$$\rightarrow \epsilon_{xx} = \frac{\partial u}{\partial r} \cdot \frac{2}{L} = -\frac{1}{L} \tag{19.14}$$

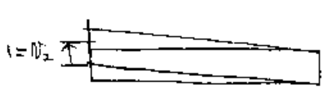


$$u(r,s) = -\frac{st}{4}(1-r)\theta_2 \tag{19.15}$$

results into ϵ_{xx}, γ_{xy}

$$\rightarrow \epsilon_{xx} = \frac{st}{2L} \tag{19.16}$$

$$\gamma_{xy} = \frac{\partial u}{\partial s} \cdot \frac{2}{t} = -\frac{1}{2}(1-r) \tag{19.17}$$



$$v(r) = \frac{1}{2}(1-r)v_2 \tag{19.18}$$

results into γ_{xy}

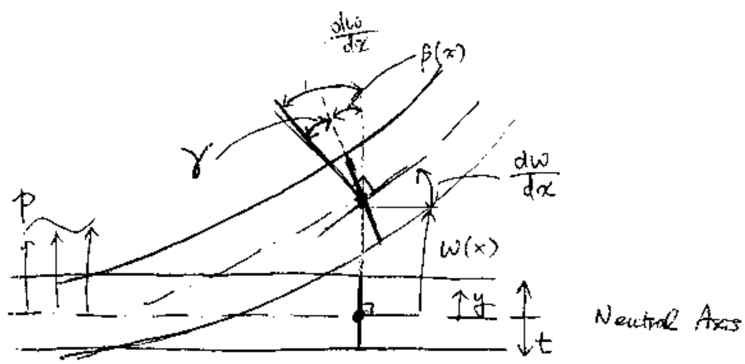
$$\rightarrow \gamma_{xy} = -\frac{1}{L} \tag{19.19}$$

For a pure bending moment, we want

$$-\frac{1}{L}v_2 - \frac{1}{2}(1-r)\theta_2 = 0 \tag{19.20}$$

for all r! $\Rightarrow$ $\boxed{\text{Impossible}}$ (except for $v_2 = \theta_2 = 0$) $\Rightarrow$ So, the element has a spurious shear strain!

Beam kinematics (Timoshenko, Reissner-Mindlin)



$$\gamma = \frac{dw}{dx} - \beta \tag{19.21}$$

$$\left(I = \frac{1}{12}bt^3\right) \tag{19.22}$$

Principle of virtual work

$$EI\int_0^L \frac{d\overline{\beta}}{dx}\frac{d\beta}{dx}dx + A_S G\int_0^L \left(\frac{d\overline{w}}{dx} - \overline{\beta}\right)\left(\frac{dw}{dx} - \beta\right)dx = \int_0^L p\overline{w}dx \tag{19.23}$$

$$A_s = kA = kbt \tag{19.24}$$

To calculate k Reading: p. 400

$$\int_A \frac{1}{2G}(\tau_a)^2 dA = \int_{A_S}\frac{1}{2G}\left(\frac{V}{A_s}\right)^2 dA_s \tag{19.25}$$

where τ_a is the actual shear stress:

$$\tau_a = \frac{3}{2}\cdot\frac{V}{A}\left[\frac{\left(\frac{t}{2}\right)^2 - y^2}{\left(\frac{t}{2}\right)^2}\right] \tag{19.26}$$

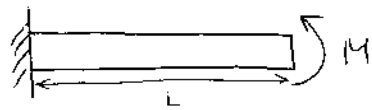
and V is the shear force. Reading: Ex. 5.23

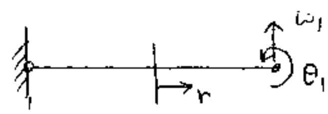
$$\boxed{\Rightarrow k = \frac{5}{6}} \tag{19.27}$$

Now interpolate

$$w(r) = h_1 w_1 + h_2 w_2 \tag{19.28}$$
$$\beta(r) = h_1 \theta_1 + h_2 \theta_2 \tag{19.29}$$

Revisit the simple case:





$$w = \frac{1+r}{2} w_1 \tag{19.30}$$
$$\beta = \frac{1+r}{2} \theta_1 \tag{19.31}$$

Shearing strain

$$\gamma = \frac{w_1}{L} - \frac{1+r}{2} \theta_1 \tag{19.32}$$

Shear strain is not zero all along the beam. But, at $r = 0$, we can have the shear strain = 0.

$$\frac{w_1}{L} - \frac{\theta_1}{2} \text{ can be zero} \tag{19.33}$$

Namely,

$$\frac{w_1}{L} - \frac{\theta_1}{2} = 0 \quad \text{for } \boxed{\theta_1 = \frac{2}{L} w_1} \tag{19.34}$$