Lecture 21 - Solution of the Generalized Eigenvalue Problem

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Reading assignment: Chapters 10 and 11

$$\boldsymbol{M}\ddot{\boldsymbol{U}} + \boldsymbol{K}\boldsymbol{U} = \boldsymbol{R} \tag{1}$$

Aside: $\boldsymbol{M}$ could have zero masses. Then we use Gauss elimination on $\boldsymbol{K}$ to remove zero-mass DOFs, but we denote the final matrix still as $\boldsymbol{K}$. Then, in free vibrations:

$$\boldsymbol{M}\ddot{\boldsymbol{U}} + \boldsymbol{K}\boldsymbol{U} = \boldsymbol{0} \tag{2}$$

where now $\boldsymbol{M}$ and $\boldsymbol{K}$ are assumed to be positive definite matrices, i.e. $\tilde{\boldsymbol{U}}^T\boldsymbol{M}\tilde{\boldsymbol{U}} > 0$, $\tilde{\boldsymbol{U}}^T\boldsymbol{K}\tilde{\boldsymbol{U}} > 0$ for any $\tilde{\boldsymbol{U}} \neq 0$. Then, we obtain the eigenvalue problem

$$\boldsymbol{K}\boldsymbol{\phi} = \lambda\boldsymbol{M}\boldsymbol{\phi} \quad \rightarrow \quad \boldsymbol{K}\boldsymbol{\phi}_i = \lambda_i\boldsymbol{M}\boldsymbol{\phi}_i \tag{A}$$

where $0 < \underbrace{\lambda_1}_{\boldsymbol{\phi}_1} \leq \underbrace{\lambda_2}_{\boldsymbol{\phi}_2} \leq \ldots \leq \underbrace{\lambda_n}_{\boldsymbol{\phi}_n}$.

Recall:

$$\boldsymbol{\phi}_i^T\boldsymbol{M}\boldsymbol{\phi}_j = \delta_{ij}$$
$$\boldsymbol{\phi}_i^T\boldsymbol{K}\boldsymbol{\phi}_j = \omega_i^2\delta_{ij} = \lambda_i\delta_{ij}$$

The Case of Multiple Eigenvalues

Assume $\lambda_1 = \lambda_2 < \lambda_3$, i.e. λ_1 has a multiplicity of 2 ($m = 2$), $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ are two eigenvectors for λ_1 and λ_2, and $\boldsymbol{\phi}_1 \neq \boldsymbol{\phi}_2$. Then, we have

$$\boldsymbol{K}\alpha\boldsymbol{\phi}_1 = \lambda_1\boldsymbol{M}\alpha\boldsymbol{\phi}_1 \quad (\alpha\text{: any constant}) \tag{3}$$
$$\boldsymbol{K}\beta\boldsymbol{\phi}_2 = \lambda_1\boldsymbol{M}\beta\boldsymbol{\phi}_2 \quad (\beta\text{: any constant}) \tag{4}$$

Hence,

$$\boldsymbol{K}\left(\alpha\boldsymbol{\phi}_1 + \beta\boldsymbol{\phi}_2\right) = \lambda_1\boldsymbol{M}\left(\alpha\boldsymbol{\phi}_1 + \beta\boldsymbol{\phi}_2\right) \tag{5}$$

Eq. (5) shows $\alpha\boldsymbol{\phi}_1 + \beta\boldsymbol{\phi}_2 = \tilde{\boldsymbol{\phi}}$ is also an eigenvector corresponding to λ_1! We can change the length of the eigenvector so that for some γ,

$$\left(\gamma\tilde{\boldsymbol{\phi}}\right)^T\boldsymbol{M}\left(\gamma\tilde{\boldsymbol{\phi}}\right) = 1$$

Recall we want $\ddot{x}_i + \omega_i^2x_i = r_i$, having set the mass m to 1 since $\boldsymbol{\phi}_i^T\boldsymbol{M}\boldsymbol{\phi}_j = \delta_{ij}$.

If the eigenvalues for the system (A) are distinct, the eigenvectors are unique. Here, we have a two dimensional eigenspace ($\lambda_1 = \lambda_2$). Any two $\boldsymbol{M}$-orthogonal vectors in this space are eigenvectors and could be used as mode shapes.

Gram-Schmidt (see textbook)

Orthogonalization is used to obtain $\boldsymbol{M}$-orthogonal vectors. For an eigenvalue of multiplicity m, we have an eigenspace of dimension m and can always find m $\boldsymbol{M}$-orthogonal vectors that are in this eigenspace. We need orthogonality to decouple Eq. (2). Next, we will discuss some solution techniques.

Inverse Iteration

Once we have eigenvectors with $\boldsymbol{\phi}_i^T \boldsymbol{M} \boldsymbol{\phi}_j = \delta_{ij}$, we could simply use $\boldsymbol{\phi}_i^T \boldsymbol{K} \boldsymbol{\phi}_j = \lambda_i \delta_{ij}$ to obtain λ_i.

Do we need to iterate on $\boldsymbol{K}\boldsymbol{\phi} = \lambda(\boldsymbol{M}\boldsymbol{\phi})$ to get $\boldsymbol{K}\boldsymbol{\phi}_i = \lambda_i \boldsymbol{M}\boldsymbol{\phi}_i$? Since for the general case there are no explicit formulas available to calculate the roots of $p(\lambda)$ when the order of p is greater than 4, an iterative solution method has to be used.

Iteration

Assume $\lambda_1 > 0$. We pick $\boldsymbol{x}_1$ and use for $k = 1, 2, \ldots$

$$\boldsymbol{K}\bar{\boldsymbol{x}}_{k+1} = \boldsymbol{M}\boldsymbol{x}_k \tag{a}$$

$$\boldsymbol{x}_{k+1} = \frac{\bar{\boldsymbol{x}}_{k+1}}{\left(\bar{\boldsymbol{x}}_{k+1}^T \boldsymbol{M} \bar{\boldsymbol{x}}_{k+1}\right)^{\frac{1}{2}}}$$

Since $\lambda_1 > 0$, $\boldsymbol{K}$ is positive definite and we can solve Eq. (a). We want $\boldsymbol{x}_{k+1}$ to satisfy the mass orthonormality relation $\bar{\boldsymbol{x}}_{k+1}^T \boldsymbol{M} \bar{\boldsymbol{x}}_{k+1} = 1$. If we assume $\boldsymbol{x}_1^T \boldsymbol{M} \boldsymbol{\phi}_1 \neq 0$, then

$$\boldsymbol{x}_{k+1} \rightarrow \boldsymbol{\phi}_1 \text{ as } k \rightarrow \infty$$

$$\lambda_1 = \boldsymbol{\phi}_1^T \boldsymbol{K} \boldsymbol{\phi}_1 \ , \ \boldsymbol{\phi}_1^T \boldsymbol{M} \boldsymbol{\phi}_1 = 1$$

Proof: Consider

$$\boldsymbol{K}\boldsymbol{x}_{k+1} = \boldsymbol{M}\boldsymbol{x}_k \tag{B}$$

We see that (B) is equivalent to working with vectors $\boldsymbol{z}_{k+1}$ and $\boldsymbol{z}_k$.

$$\boldsymbol{\Phi}\boldsymbol{z}_{k+1} = \boldsymbol{x}_{k+1} \quad , \quad \boldsymbol{\Phi}\boldsymbol{z}_k = \boldsymbol{x}_k$$

Substitute into (B):

$$\boldsymbol{\Phi}^T \boldsymbol{K} \boldsymbol{\Phi} \boldsymbol{z}_{k+1} = \boldsymbol{\Phi}^T \boldsymbol{M} \boldsymbol{\Phi} \boldsymbol{z}_k$$

$$\begin{bmatrix} \lambda_1 & & & \text{zeros} \\ & \lambda_2 & & \\ & & \ddots & \\ \text{zeros} & & & \lambda_n \end{bmatrix} \boldsymbol{z}_{k+1} = \boldsymbol{z}_k \tag{C}$$

$$\boxed{\text{Working on (C) is equivalent to working on (B)}}$$

Next, iterate with (C). Assume:

$$\boldsymbol{z}_1^T = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 & & & \text{zeros} \\ & \lambda_2 & & \\ & & \ddots & \\ \text{zeros} & & & \lambda_n \end{bmatrix} \boldsymbol{z}_2 = \boldsymbol{z}_1$$

Then we find

$$\boldsymbol{z}_2^T = \begin{bmatrix} \frac{1}{\lambda_1} & \frac{1}{\lambda_2} & \frac{1}{\lambda_3} & \ldots & \frac{1}{\lambda_n} \end{bmatrix}$$

After l iterations,

$$\boldsymbol{z}_{l+1}^T = \begin{bmatrix} \left(\frac{1}{\lambda_1}\right)^l & \left(\frac{1}{\lambda_2}\right)^l & \left(\frac{1}{\lambda_3}\right)^l & \ldots & \left(\frac{1}{\lambda_n}\right)^l \end{bmatrix}$$

Only the direction of the vector is important.

Assume $\lambda_1 < \lambda_2$. Multiply $\boldsymbol{z}_{l+1}$ by $(\lambda_1)^l$ to obtain a new z_{l+1}:

$$\boldsymbol{z}_{l+1}^T = \begin{bmatrix} 1 & \left(\frac{\lambda_1}{\lambda_2}\right)^l & \left(\frac{\lambda_1}{\lambda_3}\right)^l & \ldots & \left(\frac{\lambda_1}{\lambda_n}\right)^l \end{bmatrix}$$

This $\boldsymbol{z}_{l+1}^T$ converges to $[1 \quad 0 \quad 0 \quad \ldots \quad 0]$ as $l \to \infty$.

Note that if $\boldsymbol{z}_1$ is orthogonal to $\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, we will never reach the eigenvector corresponding to λ_1.

Finally, assume $\lambda_1 = \lambda_2 < \lambda_3$. Then we obtain

$$\boldsymbol{z}_{l+1}^T = \begin{bmatrix} 1 & 1 & 0 & \ldots & 0 \end{bmatrix}$$

To obtain the 2nd eigenvector for $\lambda_1 = \lambda_2$, choose a starting vector $\boldsymbol{x}_1$ that is $\boldsymbol{M}$-orthogonal to $\boldsymbol{\phi}_1$ and enforce this orthogonality in each iteration. To avoid round-off error, see the textbook.

In practice, the inverse iteration method is hardly used by itself, but rather as an ingredient in a more complex scheme. The next lecture introduces the widely used "subspace iteration method" which employs the inverse iteration method to efficiently solve for the first few lowest frequencies/eigenvalues and modeshapes of large systems.

2.092 / 2.093 Finite Element Analysis of Solids and Fluids I
Fall 2009