# Lecture 19 - Modeling for Dynamic Analysis & Solution, cont'd

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In the last lecture, we discussed two methods for solving this general system:

$$\boldsymbol{M}\ddot{\boldsymbol{U}} + \boldsymbol{C}\dot{\boldsymbol{U}} + \boldsymbol{K}\boldsymbol{U} = \boldsymbol{R}(t) \quad ; \quad {}^0\boldsymbol{U},\ {}^0\dot{\boldsymbol{U}}$$

I. Mode Superposition

$$\boldsymbol{U} = \sum_{i=1}^{p} \boldsymbol{\phi}_i x_i(t)$$

So far, we considered the case where $p = n$.

II. Direct Integration

We concluded that implicit methods are only of interest if they are unconditionally stable.

How many modes do we need to include in the mode superposition method? (What should we select for $p$?) We consider three cases:

I. Initial Conditions

$${}^0\boldsymbol{U} = \alpha_1\boldsymbol{\phi}_1,\ {}^0\dot{\boldsymbol{U}} = \boldsymbol{0},\ \boldsymbol{R} = \boldsymbol{0}$$

The response is only in $\boldsymbol{\phi}_1$. Therefore, if the initial conditions "use" certain mode shapes only, the response caused by these initial conditions will *only* be in these certain mode shapes.

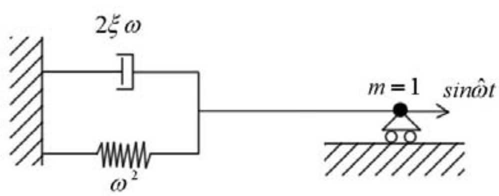
II. Loads, Spatial Distribution

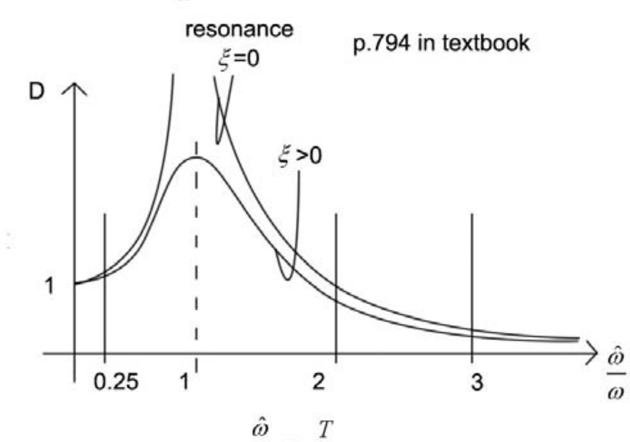
$$\boldsymbol{R} = \alpha\boldsymbol{M}\boldsymbol{\phi}_1 \quad \left({}^0\boldsymbol{U} = {}^0\dot{\boldsymbol{U}} = \boldsymbol{0}\right)$$

Again, the response is only in $\phi_1$ and the conclusion of section I holds here as well.

III. Loads, Frequency Content

$$\ddot{x}_i + 2\xi_i\omega_i\dot{x}_i + \omega_i^2 x_i = \sin\hat{\omega}t$$




$$\frac{\hat{\omega}}{\omega} = \frac{T}{\hat{T}}$$

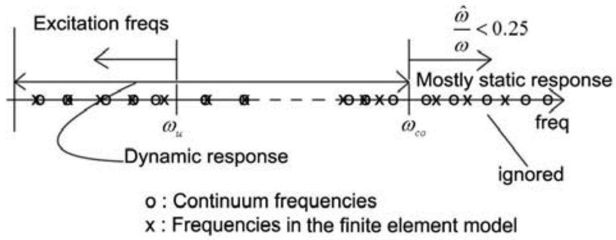
$$\text{Dynamic load factor D} = \frac{\text{dynamic peak response}}{\text{static peak response}}$$

$\hat{\omega}$, the excitation frequency, is given.

When $\hat{\omega}$ is very large, there is no response. When $\hat{\omega}$ is very small, the mass follows the excitation, producing a static response.

In actual analysis, Fourier analysis of loads is performed to see what frequencies are contained in the loading.

Step 1: Identify the highest frequency in the loading. The largest $\hat{\omega} = \omega_u = \omega_{\text{upper}}$. Define $\omega_{\text{co}} = \omega_{\text{cut-off}} = 4\omega_u$.




Step 2: Set up a mesh which represents the continuum frequencies accurately up to the cut-off frequency $\omega_{\text{co}}$.

How do we know whether enough modes are included? Calculate the error defined by the following:

$$\varepsilon_p(t) = \frac{\left\| \boldsymbol{M}\ddot{\boldsymbol{U}}^p + \boldsymbol{C}\dot{\boldsymbol{U}}^p + \boldsymbol{K}\boldsymbol{U}^p - \boldsymbol{R} \right\|_2}{\|\boldsymbol{R}\|_2} \quad (\text{assume } \boldsymbol{R} \neq 0)$$

where

$$\boldsymbol{U}^P = \sum_{i=1}^{p} \boldsymbol{\phi}_i x_i(t)$$

A small $\varepsilon_p(t)$ means that we have a good approximation for the solution. Recall: $\|\boldsymbol{a}\|_2 = \left(\sum_i (a_i)^2\right)^{\frac{1}{2}}$. If any element of $\boldsymbol{a}$ is large, $\|\boldsymbol{a}\|_2$ will not be small. Therefore, using $\varepsilon_p(t)$ is a good way of assessing whether the dynamic solution obtained is accurate enough.

## Static Correction

Consider the load

$$\Delta\boldsymbol{R} = \boldsymbol{R} - \sum_{i=1}^{p} (\boldsymbol{M}\boldsymbol{\phi}_i r_i)$$

in a mode superposition solution.

$$\begin{aligned} \hat{r}_j &= \boldsymbol{\phi}_j^T \Delta\boldsymbol{R} = \boldsymbol{\phi}_j^T \boldsymbol{R} - \sum_{i=1}^{p} \boldsymbol{\phi}_j^T \boldsymbol{M}\boldsymbol{\phi}_i r_i \\ &= \boldsymbol{\phi}_j^T \Delta\boldsymbol{R} = r_j - \sum_{i=1}^{p} \delta_{ij} r_i \end{aligned}$$

By definition, $\hat{\boldsymbol{r}} = \boldsymbol{\Phi}^T \Delta\boldsymbol{R}$. Therefore, for $j = 1, \ldots, p$, we have $\hat{r}_j = 0$.

For a properly modeled problem, the response in $\Delta \boldsymbol{R}$ should be at most a static response. Therefore, a good correction $\Delta \boldsymbol{U}^S$ to the mode superposition solution $\boldsymbol{U}^P$ can be obtained from $\boldsymbol{K}\Delta \boldsymbol{U}^S = \Delta \boldsymbol{R}$. Then, the total solution is

$$\boldsymbol{U} = \boldsymbol{U}^P + \Delta \boldsymbol{U}^S$$

In practice the mode superposition method is used for problems which have low frequency excitations, such as earthquake response, road excitation analysis, etc. It is ineffective for problems which have very high frequency content such as wave propagation. We will explore this topic next lecture.

2.092 / 2.093 Finite Element Analysis of Solids and Fluids I
Fall 2009