Direct

Solution of Linear Systems

Square $n \times n$

Motivation

inverse iteration: string in tension

Key "Value-Added": sparsity

mathematical sparsity

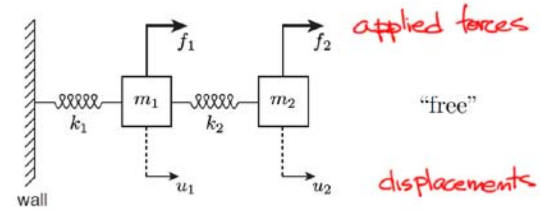
"declared sparsity

but before we look for a solution...

Existence and Uniqueness

$n = 2$ (2×2) → general case

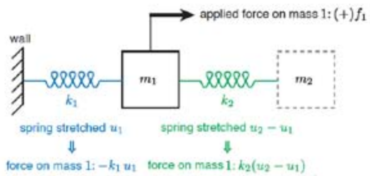
Two Springs



Note: $u_1 = 0$, $u_2 = 0$ ≡ unstretched state.

Equilibrium

mass 1



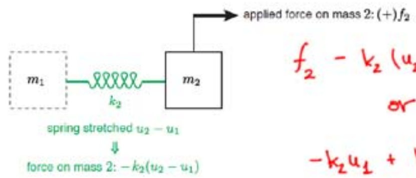
$$f_1 - k_1 u_1 + k_2(u_2 - u_1) = 0$$

or

$$(k_1 + k_2)u_1 - k_2 u_2 = f_1$$

eqn 1

mass 2



$$f_2 - k_2(u_2 - u_1) = 0$$

or

$$-k_2 u_1 + k_2 u_2 = f_2$$

eqn 2

Matrix Form

$$\begin{aligned} (k_1 + k_2)\, u_1 - k_2\, u_2 &= f_1 \qquad \text{eqn 1} \\ -k_2\, u_1 + k_2\, u_2 &= f_2 \qquad \text{eqn 2} \end{aligned}$$

$$\Downarrow$$

$$\underbrace{\begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}}_{\substack{K \\ 2\times 2}} \overset{\text{unknown}}{\underbrace{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}}_{\substack{u \\ 2\times 1}}} = \overset{\text{known}}{\underbrace{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}}_{\substack{f \\ 2\times 1}}} \quad \cdots \begin{array}{l} \leftarrow \text{eqn 1} \\ \leftarrow \text{eqn 2} \end{array}$$

1st component of Ku = 1st component of f ⇒ eqn 1

2nd component of Ku = 2nd component of f ⇒ eqn 2

Existence and Uniqueness

Given

$$2 \times 2 \text{ matrix } \; A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix};$$

$$2 \times 1 \text{ vector } \; f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

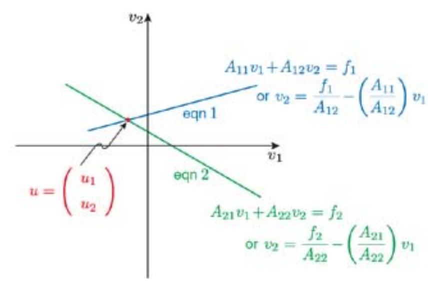
look for 2×1 vector u which satisfies

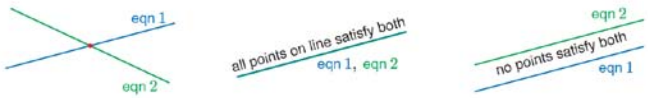
$$Au = f\,, \quad \text{or} \quad \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad \text{or} \quad \left.\begin{aligned} A_{11}u_1 + A_{12}u_2 &= f_1 \\ A_{21}u_1 + A_{22}u_2 &= f_2 \end{aligned}\right\}.$$

$$\boxed{\begin{array}{l}\text{Does } u \text{ exist? if so,} \\ \text{Is } u \text{ unique?}\end{array}}$$

2

E & U: Row View



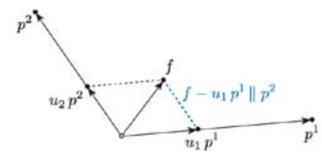


(i)	*(ii)*	*(iii)*
exists ✓	exists ✓	exists ✗
unique ✓	unique ✗	~~unique~~
	redundant information,	*inconsistent* information
	infinity of solutions	*no* solution

E & U: COLUMN View

$$Au = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \underbrace{\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix}}_{p^1} u_1 + \underbrace{\begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix}}_{p^2} u_2 \ ,$$

$$\boxed{Au = f \quad \Leftrightarrow \quad p^1 u_1 + p^2 u_2 = f}$$



(i)	*(ii)*	*(iii)*
exists ✓	exists ✓	exists ✗
unique ✓	unique ✗	~~unique~~
	(only p^1, *or* more p^1 and some p^2, *or* ...)	

$p^2 \neq \gamma p^1$	$p^2 = \gamma p^1$ <u>and</u> $f = \beta p^1$	$p^2 = \gamma p^1$ <u>and</u> $f \neq \beta p^1$

Familiar determinant condition:

$$\det \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = A_{11}A_{22} - A_{21}A_{12} \ ;$$

if

$\underline{p^2 = \gamma p^1}$ then $A_{12} = \gamma A_{11}$, $A_{22} = \gamma A_{21}$,

and $A_{11}A_{22} - A_{21}A_{12} = \gamma A_{11}A_{21} - \gamma A_{21}A_{11} = 0$;

det(A) = 0

similarly, if

$A_{11}A_{22} - A_{21}A_{12} = 0$, then $A_{12}/A_{11} = A_{22}/A_{12} \ (= \gamma)$,

det(A) = 0

so $\underline{p^2 = \gamma p^1}$.

case (ii) : $p^2 = \gamma p^1 , f = \beta p^1$

$f = \beta p^1$

$$
\begin{aligned}
f &= p^1 \cdot \beta + p^2 \cdot 0 \\
&= \begin{pmatrix} p^1 & p^2 \end{pmatrix} \begin{pmatrix} \beta \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \underbrace{\begin{pmatrix} \beta \\ 0 \end{pmatrix}}_{u^*} \\
&= Au^*
\end{aligned}
$$

Thus u^* is a particular solution to $Au = f$.

$p^2 = \gamma p^1 , f = \beta p^1$

$p^2 = \gamma p^1$

$$
\begin{aligned}
0 &= p^1 \cdot (-\gamma) + p^2 \cdot (1) \\
&= \begin{pmatrix} p^1 & p^2 \end{pmatrix} \begin{pmatrix} -\gamma \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} -\gamma \\ 1 \end{pmatrix} \\
&= A \begin{pmatrix} -\gamma \\ 1 \end{pmatrix} .
\end{aligned}
$$

Note $(A - 0 \cdot I) \begin{pmatrix} -\gamma \\ 1 \end{pmatrix} = 0$

$\Downarrow$

eigenvalue 0

eigenvector $\begin{pmatrix} -\gamma \\ 1 \end{pmatrix}$.

Thus $\begin{pmatrix} -\gamma \\ 1 \end{pmatrix}$ is the homogeneous solution to $Au = f$.

Finally, $p^2 = \gamma p^1 , f = \beta p^1$

$$u = \underbrace{u^* + \alpha \begin{pmatrix} -\gamma \\ 1 \end{pmatrix}}_{\text{infinity of solutions}} \qquad \text{any } \alpha$$

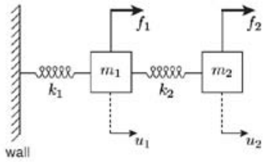
is the general solution to $Au = f$:

$$
\begin{aligned}
A \left(u^* + \alpha \begin{pmatrix} -\gamma \\ 1 \end{pmatrix} \right) &= Au^* + A \left(\alpha \begin{pmatrix} -\gamma \\ 1 \end{pmatrix} \right) \\
&= Au^* + \alpha A \begin{pmatrix} -\gamma \\ 1 \end{pmatrix} \\
&= f + \alpha \cdot 0 \\
&= f .
\end{aligned}
$$

A Tale of

Two Spring Scenarios

System

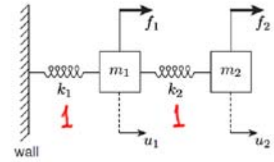


$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \qquad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

given   to find

$$Au = f \quad \text{for} \quad A = K \equiv \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}$$

Scenario (I) $k_1 = k_2 = 1$; any f $\bar{k}$



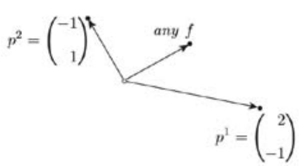
$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \qquad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

given   to find

$$Au = f \quad \text{for} \quad A = K \equiv \begin{array}{c} \begin{matrix} 2 & \quad -1 \end{matrix} \\ \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \\ \begin{matrix} -1 & \quad 1 \end{matrix} \end{array}$$

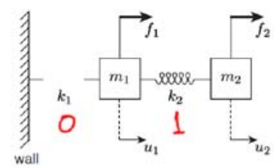
COLUMN VIEW :

$$A = \begin{array}{c} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \\ \begin{matrix} p^1 & p^2 \end{matrix} \end{array}$$



case (i): exists ✓, unique ✓

Scenario (III): $k_1 = 0$, $k_2 = 1$ $f_1 = 1$, $f_2 = 1$ $\bar{k}, \bar{f}$



$$f = \begin{array}{c} 1 \\ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \\ 1 \end{array}, \qquad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

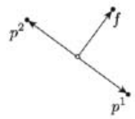
given   to find

$$Au = f \quad \text{for} \quad A = K \equiv \begin{array}{c} \begin{matrix} 1 & \quad -1 \end{matrix} \\ \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \\ \begin{matrix} -1 & \quad 1 \end{matrix} \end{array}$$

COLUMN View:

$$f = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

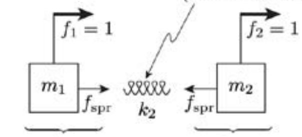
$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$
$\quad p^1 \quad p^2$



case (*iii*): exists ✗, ~~unique~~

PHYSICAL View:

(assume stretched)



$\underbrace{\quad}_{1 + f_{spr} = 0}$ $\quad$ $\underbrace{\quad}_{1 - f_{spr} = 0}$ *not* possible

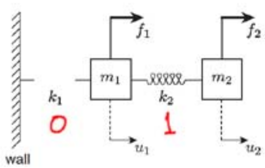
$\uparrow \qquad \uparrow$

$m_1 \ddot{u}_1 \neq 0 \qquad m_2 \ddot{u}_2 \neq 0$ faulty *model* assumption

f_{spr} = tension in spring
$= k_2 (u_2 - u_1)$

Net force on system; no ground/wall (reaction force)

Scenario (II) $k_1 = 0$, $k_2 = 1$ $\quad f_1 = -1$, $f_2 = 1$ $\qquad \bar{k}, \bar{f}$



$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$
$f_1 = -1$, $f_2 = 1$

given $\qquad$ to find

$$Au = f \quad \text{for} \quad A = K \equiv \begin{array}{c} \begin{array}{cc} 1 & -1 \end{array} \\ \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \\ \begin{array}{cc} -1 & 1 \end{array} \end{array}$$

COLUMN View:

$$f = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$
$\quad p^1 \quad p^2$

case (*ii*): exists ✓, unique ✗

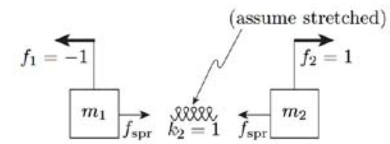
$p^2 = \underbrace{\gamma}_{-1} p^1 \qquad f = \underbrace{\beta}_{-1} p^1 \Rightarrow u^* = \begin{pmatrix} \beta \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$$\Rightarrow \quad u = u^* + \alpha\begin{pmatrix}-\gamma\\ 1\end{pmatrix}$$

$$= \underbrace{\begin{pmatrix}-1\\ 0\end{pmatrix} + \alpha\begin{pmatrix}1\\ 1\end{pmatrix}}_{\text{general solution}} \quad \text{for any } \alpha$$

$$\boxed{\begin{aligned} A\left(\begin{pmatrix}-1\\ 0\end{pmatrix} + \begin{pmatrix}\alpha\\ \alpha\end{pmatrix}\right) &= \begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}-1+\alpha\\ \alpha\end{pmatrix} \\ &= \begin{pmatrix}(-1+\alpha)-\alpha\\ (1-\alpha)+\alpha\end{pmatrix} \\ &= \begin{pmatrix}-1\\ 1\end{pmatrix} \\ &= f \end{aligned}}$$

PHYSICAL View:



$$-1 + f_{\text{spr}} = 0 \qquad -f_{\text{spr}} + 1 = 0$$

$$\Downarrow$$

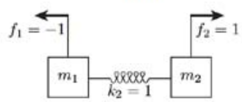
$$f_{\text{spr}} = 1$$ a solution

no net force on system

$$\Downarrow$$

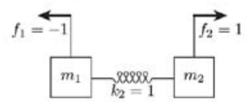
$$\underbrace{1}_{k_2}(u_2 - u_1) = \underbrace{1}_{f_{\text{spr}}}$$ only *difference* in displacement matters

a solution



$$u = u^* = \begin{pmatrix}-1\\ 0\end{pmatrix}$$

another solution



$$u = \underbrace{u^*}_{\begin{pmatrix}-1\\ 0\end{pmatrix}} + \underbrace{\alpha\begin{pmatrix}1\\ 1\end{pmatrix}}_{\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}}$$

shift by α ⟶

Existence and Uniqueness: General Case

$$\underset{n\times n}{A}\,\underset{n\times 1}{u} = \underset{n\times 1}{f}$$

A unique solution exists IF:

A non-singular
A^{-1} exists

- A has independent columns (IFF);
- A has independent rows (IFF);
- A has non-zero determinant (IFF);
- A has no zero eigenvalues (IFF);
- A (row-permuted) has no zero pivots (IFF);
- A is SPD.

Otherwise,

either non-existence or non-uniqueness.