"Stiff" Equations: An Example

An equation is "stiff" if the solution exhibits *disparate* time scales
- within the dynamics, and/or
- between the dynamics and forcing.

Example: Heat Transfer Problem $\quad \lambda < 0$

$$\frac{du}{dt} = \lambda u + \cos(\omega t), \quad 0 < t \le t_f$$

$$u(0) = u_0 = 0 \qquad 1/\omega \gg 1/|\lambda| \; ; \; t_f \sim 1/\omega \; ; \text{ interest in } t \sim 1/\omega$$

$$\Downarrow$$

$$u(t) = \frac{\omega}{\omega^2 + \lambda^2} \sin(\omega t) - \frac{\lambda}{\omega^2 + \lambda^2}\left(\cos(\omega t) - e^{\lambda t}\right)$$

$$\approx \underbrace{-\frac{1}{\lambda}\left(\cos(\omega t)\right.}_{\text{steady-periodic}}\left. - e^{\lambda t}\right) \quad \text{for } |\lambda| \gg \omega$$

$$1/\omega \gg \frac{1}{|\lambda|}$$

Textbook: Figure 21.3

Reprise: Error Equation, Euler Forward $\quad \lambda < 0$

$$\boxed{|e^j| \le |1 + \lambda \Delta t|\,|e^{j-1}| + \Delta t\,|\tau^j|, \quad 1 \le j \le J}$$

We are interested in solution for $t \sim O(\frac{1}{\omega})$,

but for stability require $\Delta t \sim O(\frac{1}{|\lambda|})$

and hence

$$J \sim \left(\frac{1}{\omega}\right) \Big/ \left(\frac{1}{|\lambda|}\right) \sim |\lambda| / |\omega| \; ;$$

need many timesteps for $|\lambda|/\omega \gg 1$.

Resolve initial $e^{\lambda t}$ transient, BUT pay steep price.

Reprise: Error Equation, Euler Backward $\quad \lambda < 0$

$$\boxed{|1 - \lambda \Delta t|\,|e^j| \le |e^{j-1}| + \Delta t\,|\tau^j|, \quad 1 \le j \le J}$$

$$\Downarrow$$

$$|e^1| \le (1 - \lambda\Delta t)^{-1} \Delta t\,|\tau^1|$$

$$|e^2| \le (1 - \lambda\Delta t)^{-2} \Delta t\,|\tau^1| + (1 - \lambda\Delta t)^{-1} \Delta t\,|\tau^2|$$

$$|e^3| \le (1 - \lambda\Delta t)^{-3} \Delta t\,|\tau^1| + (1 - \lambda\Delta t)^{-2} \Delta t\,|\tau^2| + (1 - \lambda\Delta t)^{-1} \Delta t\,|\tau^3|$$

$$\Downarrow$$

$e(t^j)$ does not "see" early τ^j'

If choose $\underline{\Delta t \sim \varepsilon/\omega}$, then $\qquad |\lambda|/\omega \gg 1$

$$J \sim 1/\omega \, / \, \varepsilon/\omega \sim 1/\varepsilon \qquad \text{independent of } |\lambda|/\omega;$$

furthermore

$$\left(t \sim 1/|\lambda|, \; |\tau^j| \sim \Delta t \, |u_{tt}| \sim \Delta t \, \lambda^2 e^{\lambda t} \;\; \text{large but...} \right)$$

$$t \sim 1/\omega, \; |\tau^j| \sim \Delta t \, |u_{tt}| \sim \Delta t \, \omega^2 \cos^2(\omega t) \;\; \text{small}$$

hence

$$|e^j| \text{ for } t^j \sim 1/\omega \text{ small}$$

(though do not resolve transient). Textbook: Figure 21.6

System of First-Order Equations

with "mass" matrix:

w $\quad n \times 1$ state vector

M $\quad n \times n$ mass matrix $\quad$ (for us: diagonal, invertible)

$\bar{g}(t, w)$ $\quad n \times 1$ "dynamics" vector

$$\boxed{\begin{cases} M \dfrac{dw}{dt} = \bar{g}(t, w), & 0 < t < t_f \\ w(0) = w_0 \end{cases}}$$

$n = 2$: $\qquad w = (w_1 \;\; w_2)^T$

$$M_{11} \frac{dw_1}{dt} = \bar{g}_1\left(t, \begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right), \quad w_1(0) = (w_0)_1$$

$$M_{22} \frac{dw_2}{dt} = \bar{g}_2\left(t, \begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right), \quad w_2(0) = (w_0)_2$$

linear case: $\qquad$ LTI: $\bar{A} \neq \bar{A}(t)$

$$\underset{n \times 1}{\bar{g}(t, w)} = \underset{n \times n}{\bar{A}} \; \underset{n \times 1}{w} + \underset{n \times 1}{\bar{F}(t)}$$

$$\Downarrow$$

$$\begin{cases} M \dfrac{dw}{dt} = \bar{A} w + \bar{F}(t) \\ w(0) = w_0 \end{cases}$$

$n = 2$:

$$M_{11} \frac{dw_1}{dt} = \bar{A}_{11} w_1 + \bar{A}_{12} w_2 + \bar{F}_1(t), \quad w_1(0) = (w_0)_1$$

$$M_{22} \frac{dw_2}{dt} = \bar{A}_{21} w_1 + \bar{A}_{22} w_2 + \bar{F}_2(t), \quad w_2(0) = (w_0)_2$$

without "mass" matrix:

w $n \times 1$ state vector

M $n \times n$ mass matrix (for us: diagonal, invertible)

$g(t, w)$ $n \times 1$ "dynamics" vector $\quad g = M^{-1}\bar{g}$

$$\boxed{\begin{cases} \dfrac{dw}{dt} = g(t, w), \quad 0 < t \le t_f \\ w(0) = w_0 \end{cases}}$$

$n = 2$: $\quad w = (w_1 \; w_2)^T$

$$\frac{dw_1}{dt} = g_1\left(t, \begin{matrix} w_1 \\ w_2 \end{matrix}\right), \quad w_1(0) = (w_0)_1$$

$$\frac{dw_2}{dt} = g_2\left(t, \begin{matrix} w_1 \\ w_2 \end{matrix}\right), \quad w_2(0) = (w_0)_2$$

linear case:

$$\underset{n \times 1}{g(t, w)} = \underset{n \times n}{A} \underset{n \times 1}{w} + \underset{n \times 1}{F(t)} \qquad A = M^{-1}\bar{A}, \; F = M^{-1}\bar{F}$$

$$\Downarrow$$

$$\begin{cases} \dfrac{dw}{dt} = A w + F(t) \\ w(0) = w_0 \end{cases}$$

$n = 2$:

$$\frac{dw_1}{dt} = A_{11} w_1 + A_{12} w_2 + F_1(t), \quad w_1(0) = (w_0)_1$$

$$\frac{dw_2}{dt} = A_{21} w_1 + A_{22} w_2 + F_2(t), \quad w_2(0) = (w_0)_2$$

Temporal Discretization

$I \to M, \; A \to \bar{A}, \; F \to \bar{F}$

$$I \frac{dw}{dt} = A w + F(t)$$

EF $\quad \tilde{w}^j = \tilde{w}^{j-1} + \Delta t \left(A \tilde{w}^{j-1} + F(t^{j-1}) \right)$

evaluation – fast, but if stiff...

EB $\quad \tilde{w}^j = \tilde{w}^{j-1} + \Delta t \left(A \tilde{w}^j + F(t^j) \right)$

$\quad (I - \Delta t A) \tilde{w}^j = \tilde{w}^{j-1} + \Delta t F(t^j)$

solution – slower, but if stiff... Unit V

CN $\quad \tilde{w}^j = \tilde{w}^{j-1} + \frac{\Delta t}{2} \left(A \tilde{w}^j + F(t^j) + A \tilde{w}^{j-1} + F(t^{j-1}) \right)$

$\quad \left(I - \frac{\Delta t}{2} A\right) \tilde{w}^j = \left(I + \frac{\Delta t}{2} A\right) \tilde{w}^{j-1} + \frac{\Delta t}{2} \left(F(t^j) + F(t^{j-1}) \right)$

solution – slower, but if stiff...

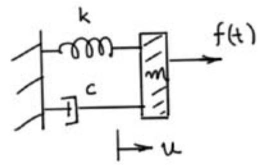
Reduction to a

System of First-Order Equations

general theory;

general numerical methods

Example I : linear oscillator



$$m\ddot{u} = -ku - c\dot{u} + f(t), \text{ or}$$

$$m\ddot{u} + c\dot{u} + ku = f(t), \text{ or}$$

$$\begin{cases} \ddot{u} + \left(\frac{c}{m}\right)\dot{u} + \left(\frac{k}{m}\right)u = f(t)/m \\ u(0) = u_0 \ , \ \dot{u}(0) = \dot{u}_0 \end{cases}$$

$$\begin{cases} \ddot{u} + \left(\frac{c}{m}\right)\dot{u} + \left(\frac{k}{m}\right)u = f(t)/m \\ u(0) = u_0 \ , \ \dot{u}(0) = \dot{u}_0 \end{cases}$$

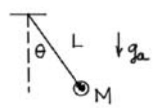
"state" variables : $w_1 \equiv u$, $w_2 = \dot{u}$ $\qquad w = (w_1 \ w_2)^T$ $n = 2$

"state" equations : $g_1(t, {}^{w_1}_{w_2})$ $\quad g_2(t, {}^{w_1}_{w_2})$

$$\underbrace{\begin{pmatrix} \frac{dw_1}{dt} \\ \frac{dw_2}{dt} \end{pmatrix}}_{\frac{dw}{dt}} = \underbrace{\begin{pmatrix} w_2 \\ -\frac{k}{m}w_1 - \frac{c}{m}w_2 - \frac{f(t)}{m} \end{pmatrix}}_{g(t,w)} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}}_{w} + \underbrace{\begin{pmatrix} 0 \\ f(t)/m \end{pmatrix}}_{F(t)}$$

$$\underbrace{\begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix}}_{w(0)} = \underbrace{\begin{pmatrix} u_0 \\ \dot{u}_0 \end{pmatrix}}_{w_0}$$

Example II : (nonlinear) pendulum



$$\begin{cases} \ddot{\theta} + d_1\dot{\theta} + d_2|\dot{\theta}|\dot{\theta} + \frac{g_a}{L}\sin\theta = 0 \\ \theta(0) = \theta_0, \ \dot{\theta}(0) = \dot{\theta}_0 \end{cases}$$

state variables : $w_1 = \theta$, $w_2 = \dot{\theta}$ $\qquad w = (w_1 \ w_2)^T$ $n = 2$

state equations : $g_1(t, {}^{w_1}_{w_2})$ $\quad g_2(t, {}^{w_1}_{w_2})$

$$\underbrace{\begin{pmatrix} \frac{dw_1}{dt} \\ \frac{dw_2}{dt} \end{pmatrix}}_{\frac{dw}{dt}} = \underbrace{\begin{pmatrix} w_2 \\ -\frac{g_a}{L}\sin(w_1) - d_1w_2 - d_2|w_2|w_2 \end{pmatrix}}_{g(t,w)}$$

$$\underbrace{\begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix}}_{w(0)} = \underbrace{\begin{pmatrix} \theta_0 \\ \dot{\theta}_0 \end{pmatrix}}_{w_0}$$

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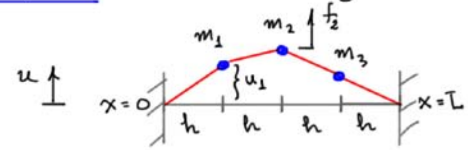
Example III : masses on string $\quad$ (→ string)



tension T

small angle approximation

n_m masses, $h = L/(n_m + 1)$

$m_i = m_i' \cdot h, \quad c_i = c_i' \cdot h, \quad f_i = f_i' \cdot h$

0 for $i = 1$

0 for $i = n_m$

$$m_i \cdot \ddot{u}_i = -c_i\dot{u}_i - T\left(\frac{u_i - u_{i-1}}{h}\right) - T\left(\frac{u_i - u_{i+1}}{h}\right) + f_i$$

$$\text{or} \begin{cases} m_i'\ddot{u}_i + c_i'\dot{u}_i + T\left(\dfrac{-u_{i-1} + 2u_i - u_{i+1}}{h^2}\right) = f_i' \\ u_i(0) = (u_0)_i \ , \ \dot{u}_i(0) = (\dot{u}_0)_i \qquad 1 \le i \le n_m \end{cases}$$

Aside:

as $h \to 0$, our equation approaches

$$m' \frac{\partial^2 u}{\partial t^2} + c' \frac{\partial u}{\partial t} = T \frac{\partial^2 u}{\partial x^2} + f'$$

PDE: wave equation IVP/BVP

u_i, $1 \le i \le n_m$: discretization of BVP in x h

$\tilde{u}_i^j$, $0 \le j \le J$: discretization of IVP in t Δt

state variables:

$w_1 = u_1$, $w_2 = \dot{u}_1$, $w_3 = u_2$, $w_4 = \dot{u}_2$, . . .

$w = (w_1\ w_2\ w_3\ \ldots\ w_n)^T$ $n = 2 \cdot n_m$

state equations:

choice

$$\begin{cases} m_1' \dot{w}_1 = m_1' w_2 \\ m_1' \dot{w}_2 = -\frac{2T}{h^2} w_2 + \frac{T}{h^2} w_4 - c_1' w_1 + f_1'(t) \\ m_2' \dot{w}_2 = \ldots \\ \vdots \end{cases}$$

$$\Rightarrow \underset{n\times n}{M} \underset{n\times 1}{\frac{dw}{dt}} = \underset{n\times n}{\bar{A}} \underset{n\times 1}{w} + \underset{n\times 1}{\bar{F}(t)} , \quad \underset{n\times 1}{w(0)} = \underset{n\times 1}{w_0}$$

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