Bernoulli/Area Estimation Summary

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2.086: Unit II
February 29, 2012

updated 1 October 2012

Estimation of θ (A Coin)

A realization: unknown θ, given a coin

draw $b_1, b_2, \ldots, b_n$ (flip coin n times),

calculate estimate (sample mean)

$$\hat{\theta}_n = \frac{1}{n} \sum_{j=1}^{n} b_j$$ (fraction heads);

calculate confidence interval for θ,

$$[ci]_{\theta;n} = \left[\hat{\theta}_n - z_\gamma \sqrt{\frac{\hat{\theta}_n(1-\hat{\theta}_n)}{n}}, \hat{\theta}_n + z_\gamma \sqrt{\frac{\hat{\theta}_n(1-\hat{\theta}_n)}{n}} \right] .$$

Some details:

$$\gamma \text{ (confidence level)} \rightarrow z_\gamma$$

γ	z_γ
$\mathbf{0.8}$	$\mathbf{1.28}$
$\mathbf{0.95}$	$\mathbf{1.96}$
$\mathbf{1}$	∞

;

check $n\,\hat{\theta}_n > 5$, $n(1-\hat{\theta}_n) > 5$ (normal approximation).

Frequentist interpretation: demo

If perform n_{exp} ($\to \infty$) realizations

$$\underbrace{\underbrace{n \text{ flips} \to \hat{\theta}_n}_{\text{one realization}} \text{ performed } n_{\text{exp}} \text{ times}}_{n_{\text{exp}} \text{ realizations}},$$

then in a fraction γ of these n_{exp} realizations

θ is inside the interval $[ci]_{\theta;n}$.

Error measures: with confidence γ

$$|\theta - \hat{\theta}_n| \le z_\gamma \underbrace{\sqrt{\frac{\hat{\theta}_n(1-\hat{\theta}_n)}{n}}}_{\text{estimate for standard deviation of } \widehat{\Theta}_n} \equiv \text{Half Length}_{\theta;n} ,$$

NOTE: $\dfrac{1}{\sqrt{n}}$ (binomial); z_γ .

$$\frac{|\theta - \hat{\theta}_n|}{\hat{\theta}_n} \le z_\gamma \sqrt{\frac{(1-\hat{\theta}_n)}{n\,\hat{\theta}_n}} \equiv \text{RelErr}_{\theta;n} ,$$

NOTE: $\dfrac{1}{\sqrt{\hat{\theta}_n}}$.

Cumulative sample means (Section 10.3):

$$
\begin{array}{llllllll}
b_1 & \rightarrow & \hat{\theta}_1 & \rightarrow & [ci]_{\theta;1} & \rightarrow & \mathbf{RelErr}_{\theta;1} & \leq \text{ tol ? if no} \\
\text{keep, } b_2 & \rightarrow & \hat{\theta}_2 & \rightarrow & [ci]_{\theta;2} & \rightarrow & \mathbf{RelErr}_{\theta;2} & \leq \text{ tol ? if no} \\
\text{keep, } b_3 & \rightarrow & \hat{\theta}_3 & \rightarrow & [ci]_{\theta;3} & \rightarrow & \mathbf{RelErr}_{\theta;3} & \leq \text{ tol ? if no}
\end{array}
$$

$\vdots$

"n required" $\leq$ tol ? yes STOP

$\vdots$

keep, $b_{n_{\max}} \rightarrow \hat{\theta}_{n_{\max}} \rightarrow [ci]_{\theta;n_{\max}} \rightarrow \mathbf{RelErr}_{\theta;n_{\max}}$.

Estimation of Area (A_D)

A realization: or cumulative

throw darts $(x_1, x_2)_1, \ldots, (x_1, x_2)_n$ uniform over R;

evaluate Bernoulli b_j $\theta = A_D / A_R$

$$b_j = \begin{cases} 0 & (x_1, x_2)_j \text{ not in } D \\ 1 & (x_1, x_2)_j \text{ in } D \end{cases} ;$$

calculate estimate for θ, A_D

$$\hat{\theta}_n = \frac{1}{n} \sum_{j=1}^{n} b_j \quad \Rightarrow \quad (\hat{A}_D)_n = A_R \hat{\theta}_n ;$$

A realization (cont'd): confidence level γ

calculate $[ci]_{A_D;n}$ for A_D

$$\Bigg[\underbrace{A_R \cdot \left(\hat{\theta}_n - z_\gamma \sqrt{\frac{\hat{\theta}_n(1-\hat{\theta}_n)}{n}}\right)}_{\text{lower bound}}, \underbrace{A_R \cdot \left(\hat{\theta}_n + z_\gamma \sqrt{\frac{\hat{\theta}_n(1-\hat{\theta}_n)}{n}}\right)}_{\text{upper bound}}\Bigg] .$$

Note: only require " in D vs. not in D" decision.

Error measures:

$$|A_D - (\hat{A}_D)_n| \leq A_R \cdot z_\gamma \sqrt{\frac{\hat{\theta}_n(1-\hat{\theta}_n)}{n}}$$

$$\equiv \text{Half Length}_{A_D;n} ;$$

$$\frac{|A_D - (\hat{A}_D)_n|}{(\hat{A}_D)_n} \leq z_\gamma \sqrt{\frac{(1-\hat{\theta}_n)}{n\,\hat{\theta}_n}}$$

$$\equiv \text{RelErr}_{A_D;n} .$$

Note: $\dfrac{1}{\sqrt{\hat{\theta}_n}} \approx \sqrt{\dfrac{1}{\theta}} = \sqrt{\dfrac{A_R}{A_D}}$.

2.086 Numerical Computation for Mechanical Engineers
Fall 2012