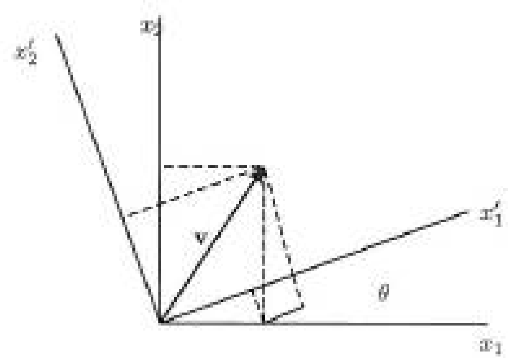
Recitation 2: Stress/Strain Transformations and Mohr's Circle

2.1 General Transformation Rules

2.1.1 2D Vector

Consider a vector $\boldsymbol{v}$ in the (x_1, x_2) coordinate system. A new coordinate system (x_1', x_2') is obtained by rotating the old coordinate system by angle θ. Find the components of $\boldsymbol{v}$ in the new coordinate system.



From geometry, we have:

$$
\begin{aligned}
x_1' &= x_1 \cos\theta + x_2 \sin\theta \\
&= x_1 \cos\theta + x_2 \cos(\frac{\pi}{2} - \theta) \\
&= x_1 L_{11} + x_2 L_{12}
\end{aligned} \tag{2.1}
$$

$$
\begin{aligned}
x_2' &= -x_1 \sin\theta + x_2 \cos\theta \\
&= x_1 \cos(\frac{\pi}{2} + \theta) + x_2 \cos\theta \\
&= x_1 L_{21} + x_2 L_{22}
\end{aligned} \tag{2.2}
$$

where L_{ij} is the direction cosine of the basis vectors:

$$L_{11} = \cos(\boldsymbol{e}_1', \boldsymbol{e}_1),\ L_{12} = \cos(\boldsymbol{e}_1', \boldsymbol{e}_2),\ L_{21} = \cos(\boldsymbol{e}_2', \boldsymbol{e}_1),\ L_{22} = \cos(\boldsymbol{e}_2', \boldsymbol{e}_2) \tag{2.3}$$

2.1.2 General Vector Transformation

A vector is a physical quantity, independent of the coordinate system. But its scalar components DO depend on the coordinate system.

$$\boldsymbol{v} = v_i \boldsymbol{e}_i = v_i' \boldsymbol{e}_i' \tag{2.4}$$

The scalar components v_i' in the new coordinate system can be expressed in terms of the scalar components v_i in the original coordinate system. Multiplying the above equation by $\boldsymbol{e}_j'$ gives:

$$\begin{aligned} v_i \boldsymbol{e}_i \cdot \boldsymbol{e}_j' &= v_i' \boldsymbol{e}_i' \cdot \boldsymbol{e}_j' \\ &= v_i' \delta_{i'j'} \\ &= v_j' \end{aligned} \tag{2.5}$$

The indices i, j are arbitrary and may be reversed to give:

$$v_i' = v_j \boldsymbol{e}_i' \cdot \boldsymbol{e}_j \tag{2.6}$$

Introduce the direction cosine tensor

$$L_{ij} \equiv \boldsymbol{e}_i' \cdot \boldsymbol{e}_j = \cos(\boldsymbol{e}_i', \boldsymbol{e}_j) \quad (i, j = 1, 2, 3) \tag{2.7}$$

then we can write v_i' in index form

$$v_i' = L_{ij} v_j \tag{2.8}$$

and in matrix form

$$\boldsymbol{v}' = [L]\boldsymbol{v} \tag{2.9}$$

where

$$\boldsymbol{v}' = \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix}, \text{ and } \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{2.10}$$

Similarly, the inverse transform

$$v_i = L_{ji} v_j' \tag{2.11}$$

$$\boldsymbol{v} = [L]^T \boldsymbol{v}' \tag{2.12}$$

2.1.3 Tensor Transformations

Using the same definition for L_{ij} defined above, the components of a tensor can be transformed to a new coordinate system by:

$$A_{ij}' = L_{ik} L_{jl} A_{kl} \tag{2.13}$$

or in matrix notation:

$$[\boldsymbol{A}'] = [\boldsymbol{L}][\boldsymbol{A}][\boldsymbol{L}]^T \quad (2.14)$$

The inverse transformation can be found by:

$$A_{kl} = L_{ik}L_{jl}A'_{ij} \quad (2.15)$$

$$[\boldsymbol{A}] = [\boldsymbol{L}]^T[\boldsymbol{A}'][\boldsymbol{L}] \quad (2.16)$$

Note that the transformation matrix $[\boldsymbol{L}]$ is orthogonal, meaning its transpose is equal to its inverse.

$$[\boldsymbol{L}][\boldsymbol{L}]^T = \boldsymbol{I} \quad (2.17)$$

2.2 Eigenvalues and Eigenvecors

If there exists a vector $\boldsymbol{n}_i$ and a scalar λ_i for an arbitrary tensor $[\boldsymbol{A}]$ such that

$$[\boldsymbol{A}]\boldsymbol{n}_i = \lambda_i \boldsymbol{n}_i \quad (2.18)$$

then λ_i and $\boldsymbol{n}_i$ are eigenvalues and eigenvectors of tensor $[\boldsymbol{A}]$, respectively. The three eigenvectors are mutually perpendicular, and form a new coordinate system.

The "eigenvalue problem" for tensor $[\boldsymbol{A}]$ can be written as:

$$(\boldsymbol{A} - \lambda_i \boldsymbol{I})\boldsymbol{n}_i = \boldsymbol{0} \quad (2.19)$$

This equation has solutions $\boldsymbol{n}_i \neq \boldsymbol{0}$ only if

$$\det(\boldsymbol{A} - \lambda_i \boldsymbol{I}) = 0 \quad (2.20)$$

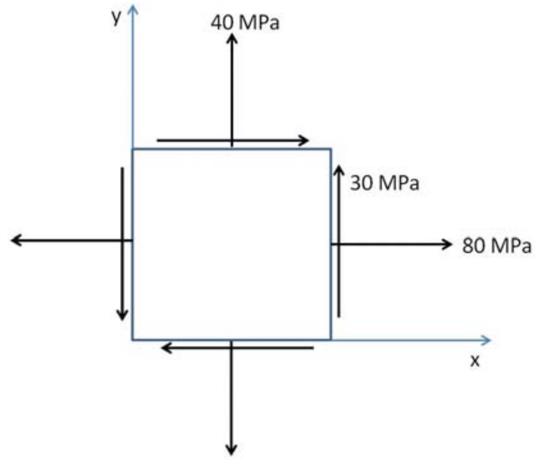
This equation can be solved for λ_i, then λ_i substituted into the eigenvalue problem to solve for $\boldsymbol{n}_i$.

For the stress (strain) tensor, the eigenvalues represent principal stresses (strains), and eigenvectors represent principal axes (i.e., faces with zero shear stress (strain)).

Example

Find the principal stresses and principal axes for $[\sigma] = \begin{bmatrix} 80 & 30 \\ 30 & 40 \end{bmatrix}$.

Solution: The given stress tensor can be represented graphically by



The principal stresses are the eigenvalues (λ_i) of the stress tensor, and are found by solving:

$$\det(\sigma - \lambda \boldsymbol{I}) = 0 \tag{2.21a}$$

$$\det \begin{bmatrix} 80-\lambda & 30 \\ 30 & 40-\lambda \end{bmatrix} = 0 \tag{2.21b}$$

$$(80-\lambda)(40-\lambda) - 30^2 = 0 \tag{2.21c}$$

$$\lambda^2 - 120\lambda + 2300 = 0 \tag{2.21d}$$

$$\lambda_1 = 96.05 \text{ and } \lambda_2 = 23.95 \tag{2.21e}$$

The principal directions are the eigenvectors, found by substituting the eigenvalues into the original eigenvalue problem:

$$(\sigma - \lambda_i \boldsymbol{I})\boldsymbol{n}_i = \boldsymbol{0} \tag{2.22}$$

For $\lambda_1 = 96.05$ MPa,

$$\begin{bmatrix} 80-96.05 & 30 \\ 30 & 40-96.05 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{2.23}$$

$$(80-96.05)x_1 + 30y_1 = 0 \tag{2.24}$$

Any (x_1, y_1) that satisfies this equation is an eigenvector. Therefore, we can choose $y_1 = 1$ and solve for x_1, then normalize the vector as follows:

$$(80-96.05)x_1 + 30 = 0 \tag{2.25}$$

$$x_1 = \frac{30}{16.05} \tag{2.26}$$

$$\boldsymbol{n}_1 = \frac{1}{\sqrt{(\frac{30}{16.05})^2 + 1^2}} \begin{Bmatrix} \frac{30}{16.05} \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.88 \\ 0.47 \end{Bmatrix} \tag{2.27}$$

Similarly, the second eigenvector is found to be:

$$\boldsymbol{n}_2 = \left\{ \begin{array}{c} -0.47 \\ 0.88 \end{array} \right\} \tag{2.28}$$

As a check, it is clear that the two eigenvectors are perpendicular ($\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = 0$).

2.3 Mohr's Circle

Mohr's circle (named after Otto Mohr (1835-1918)) is a graphical technique to transform stress (strain) from one coordinate system to another, and to find maximum normal and shear stresses (strains).

Constructing 2D Mohr's Circle:

1. Establish a rectangular coordinate system with x =normal stress, y =shear stress. Scales must be identical.
2. Plot stresses for 2 orthogonal adjacent faces (values from the original stress (strain) tensor).
3. Connect the 2 points to find center of the circle, C.
4. Draw circle through 2 points with center C.
5. Principal stresses (strains) are values where the circle crosses the x-axis.
6. Max shear stress (strain) is max y-value on the circle.

Sign convention for Mohrs circle: Positive shear stress on a face causes *clockwise* rotation of the unit square.

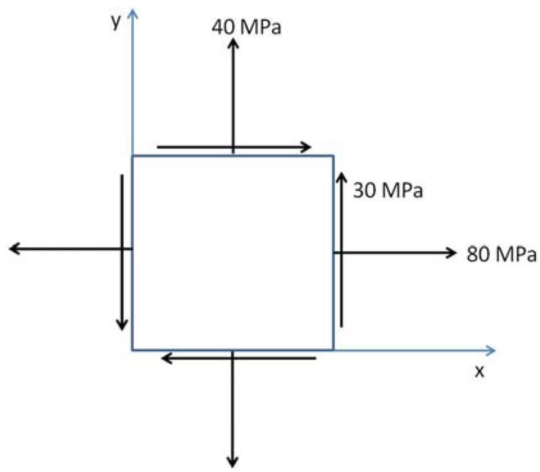
The stress state for a face rotated an angle θ from an original coordinate axis may be found by rotating an angle 2θ on Mohrs circle in the same direction from the original coordinate axis.

Example[1]

Solve the above example using Mohr's circle.

Solution: The given stress tensor is represented graphically by

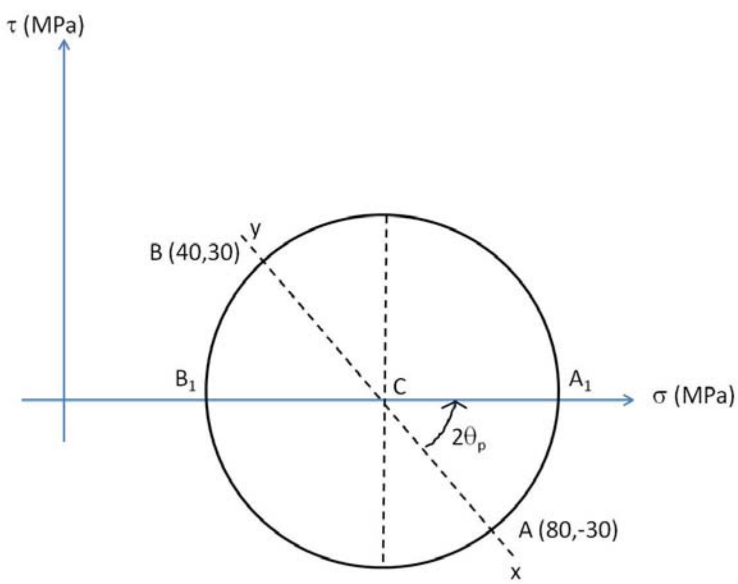
[1]Example 1.3 from Ugural and Fenster, *Advanced Strength and Applied Elasticity*, 2003



The stress state on the positive x-face is $\sigma = +80$ MPa, and $\tau = -30$ MPa (because of the sign convention defined above).

The stress state on the positive y-face is $\sigma = +40$ MPa, and $\tau = +30$ MPa.

The Mohrs circle for the given stress state is as shown:



The center, C, is located at $(40 + 80)/2 = 60$ MPa on the σ axis.

The principal stresses are represented by A_1 and B_1. The coordinates of those points are found from geometry as:

$$\sigma_{1,2} = 60 \pm \sqrt{(80-60)^2 + 30^2} \text{ MPa} \tag{2.29}$$

or

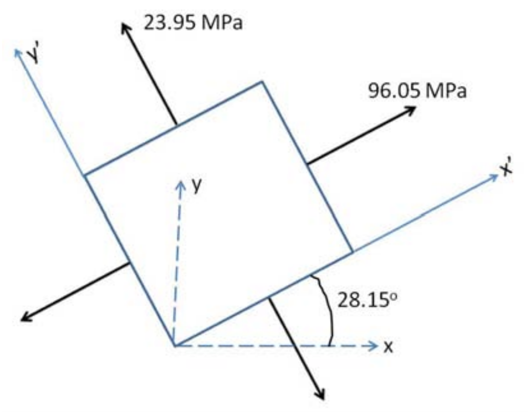
$$\sigma_1 = 96.05 \text{ MPa and } \sigma_2 = 23.95 \text{ MPa} \tag{2.30}$$

The principal directions are found by:

$$2\theta_\text{p} = \arctan \frac{30}{(80-60)} = 56.3^\circ \tag{2.31a}$$

$$\theta_\text{p} = 28.15^\circ \tag{2.31b}$$

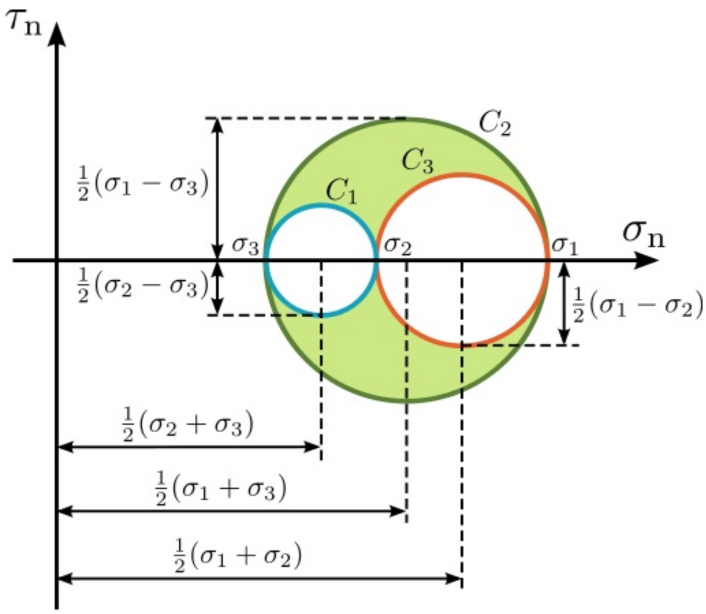
The principal stress state is as shown below:



3D Mohrs Circle

To draw Mohrs circle for a general 3D stress state, the principal stresses and directions must first be evaluated (by solving the eigenvalue problem). Then the Mohrs circle can be constructed as shown below:

The stress state for any rotation will be represented by a point either on one of the 3 circles, or in the shaded green area between the inner and outer circles.



τ_n
σ_n
C_2
C_3
C_1
$\frac{1}{2}(\sigma_1 - \sigma_3)$
σ_3
σ_2
σ_1
$\frac{1}{2}(\sigma_2 - \sigma_3)$
$\frac{1}{2}(\sigma_1 - \sigma_2)$
$\frac{1}{2}(\sigma_2 + \sigma_3)$
$\frac{1}{2}(\sigma_1 + \sigma_3)$
$\frac{1}{2}(\sigma_1 + \sigma_2)$

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