Lecture 7: Bending Response of Plates and Optimum Design

7.1 Beam Deflection Equation

The three group of equations for the plate bending problem, formulated in Lecture 2, 3 and 4 are:

$$\text{Geometry} \qquad \kappa_{\alpha\beta} = -w_{,\alpha\beta} \tag{7.1}$$

$$\text{Equilibrium} \qquad M_{\alpha\beta,\alpha\beta} + p = 0 \tag{7.2}$$

$$\text{Elasticity law} \qquad M_{\alpha\beta} = D[(1-\nu)\kappa_{\alpha\beta} + \nu\kappa_{\gamma\gamma}\delta_{\alpha\beta}] \tag{7.3}$$

Eliminating $\kappa_{\alpha\beta}$ between Eqs. (7.1) and (7.2)

$$M_{\alpha\beta} = D[(1-\nu)w_{,\alpha\beta} + \nu w_{,\gamma\gamma}\delta_{\alpha\beta}] \tag{7.4}$$

and substituting the result into Eq. (7.3) gives

$$D[(1-\nu)w_{,\alpha\beta} + \nu w_{,\gamma\gamma}\delta_{\alpha\beta}]_{,\alpha\beta} + p = 0 \tag{7.5}$$

The second term in the brackets is non-zero only when $\alpha = \beta$. Therefore Eq. (7.4) transforms to

$$Dw_{,\alpha\alpha\beta\beta}[-1 + \nu - \nu] + p = 0 \tag{7.6}$$

or finally

$$Dw_{,\alpha\alpha\beta\beta} = p \tag{7.7}$$

Introducing the definition of Laplacian ∇^2 and bi-Laplacian ∇^4 in the rectangular coordinate system,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \nabla^4 = \nabla^2\nabla^2 \tag{7.8}$$

an alternative form of Eq. (7.6) is

$$D\nabla^4 w = p \tag{7.9}$$

This is a linear inhomogeneous differential equation of the fourth order. The boundary conditions in the local coordinate system were given by Eq. (3.84).

A separate set of equations must be stetted for the in-plane response of the plate

$$\text{Geometry} \qquad \epsilon^\circ_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \tag{7.10}$$

$$\text{Equilibrium} \qquad N_{\alpha\beta,\beta} = 0 \tag{7.11}$$

$$\text{Elasticity} \qquad N_{\alpha\beta} = C[(1-\nu)\epsilon^\circ_{\alpha\beta} + \nu\epsilon^\circ_{\gamma\gamma}\delta_{\alpha\beta}] \tag{7.12}$$

Eliminating the strains $\epsilon^{\circ}_{\alpha\beta}$ and membrane force $N_{\alpha\beta}$ between the above system, one gets two coupled partial differential equations of the second order for u_α (u_1, u_2).

$$(1-\nu)u_{\alpha,\beta\beta} + (1+\nu)u_{\beta,\alpha\beta} = 0 \tag{7.13}$$

Such system is seldom solved, because in practical application constant membrane forces are considered.

In either case the in-plane and out-of-plane response of plates is uncoupled in the classical, infinitesimal bending theory of plates. These two system are coupled through the finite rotation term $N_{\alpha\beta}w_{,\alpha\beta}$. The extended governing equation in the theory of moderately large deflection is

$$D\nabla^4 w + N_{\alpha\beta}w_{,\alpha\beta} = 0 \tag{7.14}$$

The above equation will be re-derived and solved for few typical loading cases in Lecture 10. The analysis of the differential equation (7.9) in the classical bending theory of plates along with exemplary solutions can be found in the lecture notes of the course 2.081 plates and shells. In this section we will look into the bending problem of circular plates, which is governed by the linear ordinary differential equation.

7.2 Deflections of Circular Plates

The governing equation (7.9) still holds but the Laplace operator ∇^2 should now be defined in the polar coordinate system (r, θ)

$$\nabla^2 w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} \tag{7.15}$$

In the circular plate subjected to axi-symmetric loading $p = p(r)$, the third term in Eq. (7.15) vanishes and the Laplace operator can be put in the form

$$\nabla^2 w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w}{\mathrm{d}r}\right) \tag{7.16}$$

With the above definition, the plate bending equation becomes

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left\{r\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w}{\mathrm{d}r}\right)\right]\right\} = \frac{p(r)}{D} \tag{7.17}$$

and the solution is obtained by four successive integration

$$w(r) = \int\frac{1}{r}\int r\int\frac{1}{r}\int\frac{rp(r)}{D}\,\mathrm{d}r\,\mathrm{d}r\,\mathrm{d}r\,\mathrm{d}r \tag{7.18}$$

Assuming a uniform loading of the intensity p_o, the above integration can be easily performed to give

$$w(r) = C_1\ln r + C_2 r^2 + C_3 r^2\ln r + C_4 + \frac{p_o r^4}{64D} \tag{7.19}$$

As an illustration, consider clamped boundary conditions:

$$\text{at } r = R \quad w = 0 \text{ and } \frac{\mathrm{d}w}{\mathrm{d}r} = 0 \tag{7.20a}$$

$$\text{at } r = 0 \quad \frac{\mathrm{d}w}{\mathrm{d}r} = 0 \text{ and } \bar{V}_r = 0 \tag{7.20b}$$

where the shear force (per unit length), acting on a plate element at a distance r is

$$V_r = -\frac{1}{2\pi r}\int_0^r p_o 2\pi r \, \mathrm{d}r = -\frac{p_o r}{2} \tag{7.21}$$

The two terms in Eq. (7.19) involving logarithms tend to infinity at $r \to 0$. Therefore, in order for the solution to give finite values of deflections at the center, $C_1 = C_3 = 0$. Now, the expression for the slope is

$$\frac{\mathrm{d}w}{\mathrm{d}r} = 2C_2 r + \frac{p_o r^3}{18D} \tag{7.22}$$

Now, the boundary conditions at $r = 0$ are satisfied identically. From two boundary conditions at $r = R$, one finds the integration constants

$$C_2 = -\frac{p_o R^2}{32D}, \quad C_4 = \frac{p_o R^4}{64D} \tag{7.23}$$

The final form of the solution for the plate deflection is

$$w(r) = \frac{p_o R^4}{64D}\left[1 - \left(\frac{r}{R}\right)^2\right]^2 \tag{7.24}$$

For comparison, the solution for the simply supported plate will be derived. The boundary conditions are mixed so the moment-curvature relation must be used

$$M_r = D\left[\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} + \nu\frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}r}\right] \tag{7.25a}$$

$$M_\theta = D\left[\frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}r} + \nu\frac{\mathrm{d}^2 w}{\mathrm{d}r^2}\right] \tag{7.25b}$$

where the definition of moments in the cylindrical coordinate system was used. At the plate edge

$$w = 0 \text{ and } M_r = 0 \quad \text{at } r = R \tag{7.26}$$

From Eqs. (7.19), (7.25) and (7.26), the system of two algebraic equations for C_2 and C_4 is obtained, where solution is

$$C_2 = -\frac{p_o R^2}{32D}\frac{3+\nu}{1+\nu}, \quad C_4 = \frac{p_o R^4}{64D}\frac{5+\nu}{1+\nu} \tag{7.27}$$

The formula for the plate deflection is

$$w(r) = \frac{pR^4}{64D}\left[\left(\frac{r}{R}\right)^4 - 2\left(\frac{r}{R}\right)^2\frac{3+\nu}{1+\nu} + \frac{5+\nu}{1+\nu}\right] \tag{7.28}$$

The ratio of the maximum deflection of the simply supported and clamped plate at $r = 0$ is

$$\frac{w_{\text{simplysupported}}}{w_{\text{clamped}}} = \frac{5+\nu}{1+\nu} \approx 4 \tag{7.29}$$

It is interesting that a similar ratio for beams is exactly 5.

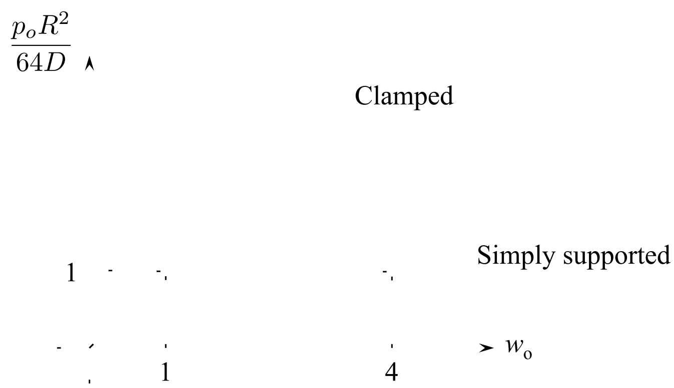

Figure 7.1: Clamped plate is four times stiffer than the simply supported circular plate.

The clamped circular plate can leave at a prototype of the whole family of similar plates. It is therefore of interest to explore the properties of the above solution further. From Eq. (7.24) the radial and circumferential curvatures are:

$$\kappa_r = -\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} = \frac{p_o R^2}{16D}\left(1 - \frac{3r^2}{R^2}\right) \tag{7.30a}$$

$$\kappa_\theta = -\frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}r} = \frac{p_o R^2}{16D}\left(1 - \frac{r^2}{R^2}\right) \tag{7.30b}$$

From the constitutive equations, the radial and circumferential bending moments are

$$M_r = D[\kappa_r + \nu\kappa_\theta] + \frac{p_o R^2}{16}\left[(1+\nu) - (3+\nu)\left(\frac{r}{R}\right)^2\right] \tag{7.31a}$$

$$M_\theta = D[\kappa_\theta + \nu\kappa_r] + \frac{p_o R^2}{16}\left[(1+\nu) - (1+3\nu)\left(\frac{r}{R}\right)^2\right] \tag{7.31b}$$

At the plate center, by symmetry

$$M_r = M_\theta = (1+\nu)\frac{p_o R^2}{16} \tag{7.32}$$

Another extreme value occurs at the clamped edge

$$M_r = \frac{p_o R^2}{8}, \quad M_\theta = -\nu\frac{p_o R^2}{8} \quad \text{at } r = R \tag{7.33}$$

By comparing Eqs. (7.32) and (7.33), it is seen that the maximum bending moment occurs at the edge $r = R$. From the stress formula, Eq. (4.92)

$$|\sigma_{rr}| = \left|\frac{M_r z}{h^3/12}\right|_{z=\frac{h}{2}} = p_o\frac{3}{4}\left(\frac{R}{h}\right)^2 \tag{7.34}$$

At the same time, the circumferential bending moment at $r = R$ is

$$|\sigma_{\theta\theta}| = \frac{M_\theta z}{h^3/12} = p_o \frac{1}{4}\left(\frac{R}{h}\right)^2 \tag{7.35}$$

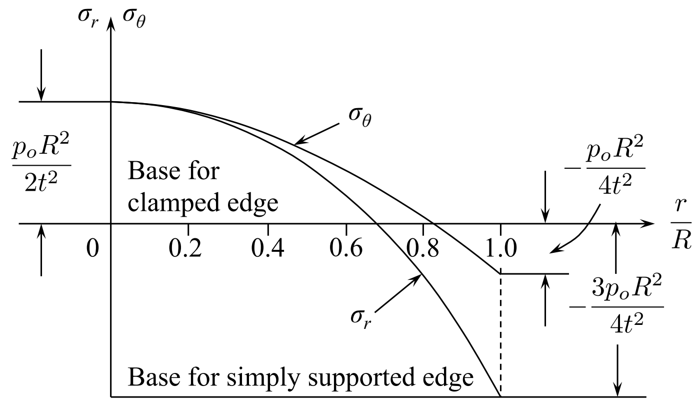

Figure 7.2: Variation of radial and circumferential stresses along the radius of the plate.

7.3 Equivalence of Square and Circular Plates

In the section of Lecture 7 on stiffened plates, the analogy between the response of circular and square plates was exploit to demonstrate the effectiveness of stiffeners. It was stated that stiffness of these two types of plates are similar if the arial surface was identical. We are now in the position to assess accuracy of the the earlier assertion.

Consider a clamped square plate $2a \times 2a$, uniformly loaded by the pressure p_o. The total potential energy of the system Π is

$$\Pi = \frac{D}{2}\int_S \left[(\kappa_x^2 + \kappa_y^2) + 2(1-\nu)\kappa_G\right] \mathrm{d}s - \int_S -q_o w \, \mathrm{d}s \tag{7.36}$$

It can be shown (Shames & Dign 1985) that for the fully clamped boundary conditions, the integral of the Gaussian curvature κ_{G} vanishes. The expression for Π simplifies to

$$\Pi = \frac{D}{2}\int_0^a \int_0^a \left[\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 w}{\mathrm{d}y^2}\right] \mathrm{d}x \, \mathrm{d}y - a^2 q \int_0^a \int_0^a w \, \mathrm{d}x \, \mathrm{d}y \tag{7.37}$$

For simplicity only one quarter of the plate is considered with the origin at the plate center. As a trial deflection shape, we assume

$$w(x, y) = C(x^2 - a^2)^2 (y^2 - a^2)^2 \tag{7.38a}$$

$$\frac{\mathrm{d}w}{\mathrm{d}x} = C2(x^2 - a^2)2x(y^2 - a^2)^2 \tag{7.38b}$$

$$\frac{\mathrm{d}w}{\mathrm{d}y} = C(x^2 - a^2)^2 2(y^2 - a^2)2y \tag{7.38c}$$

It is seen that both the deflections and slopes are zero at the clamped boundary. Furthermore, the slopes at the plate center $x = y = 0$ vanishes, as they should due to symmetry. The maximum amplitude is at center and is equal to $Ca^8 = w_o$. Thus, the kinematic boundary conditions are satisfied identically for any value of the unknown constant C. Substituting the expression (7.38) into Eq. (7.37) and performing integration yields

$$\Pi = 9a^4DC^2 - 0.384q_oC \tag{7.39}$$

According to the Ritz method, equilibrium is maintained if

$$\delta\Pi = \frac{\partial\Pi}{\partial C}\delta C = 0 \tag{7.40}$$

This means that for a given load intensity and the assumed normalized shape function, the true deflection amplitude is chosen by the condition

$$\frac{\partial\Pi}{\partial C} = 0 \quad \text{or} \quad 18a^4DC - 0.383q_o = 0 \tag{7.41}$$

Having found the amplitude C, the load-displacement relation of the square plate becomes

$$w_o = \frac{p_oa^4}{47D} \tag{7.42}$$

The corresponding solution for the clamped circular plate is

$$w_o = \frac{p_oR^4}{64D} \tag{7.43}$$

The stiffnesses of both plates are identical if $\frac{R^4}{64} = \frac{a^4}{47}$ or if $a = 0.92R$. The area equivalence $4a^2 = \pi R^2$ gives a similar result $a = 0.88R$. For simplicity in the qualitative analysis throughout the present lecture notes one can approximately assume $a = R$. The difference between the exact and approximate solution from the area and stiffness equivalence does exist, but it is small. It is interesting that the approximate solution obtained by the Ritz method is very close to the exact series solution where the coefficient 47 in Eq. (7.42) should be replaced by 49.5.

7.4 Design Concept for Plates

The plates loaded in the transverse direction can be design for:

- Stiffness
- Strength(yielding or plastic collapse)
- fracture

Plastic collapse and fracture of ductile materials will be covered in separate lectures. Stiffness is a global property of the plate and is the ratio of force to displacement. For a uniformly loaded plate the stiffness is defined as

$$K = \frac{\pi R^2 p_o}{w_o} \tag{7.44}$$

For the clamped plate with $\nu = 0.3$

$$K = 18.5 \equiv \frac{h^3}{R^2} \left[\frac{\mathrm{N}}{\mathrm{m}}\right] \tag{7.45}$$

Stiffness can be controlled by choosing a suitable material (E), thickness (h) and distance between support (R). The boundary conditions enter through the numerical coefficient. The concept of optimum design includes the weight and cost of a given structure. Leaving the complex issue of cost, the wight can be easily included by calculating stiffness per unit weight. The wight of the circular plate $W = \pi R^2 \rho$, so the stiffness per unit weight is

$$\bar{K} = \frac{K}{W} = \frac{\pi R^2 p_o}{\pi R^2 \rho w_o} = \frac{p_o}{\rho w_o} \tag{7.46}$$

In the case of a clamped plate

$$\bar{K} = 4.8 \frac{E}{\rho} \frac{h^2}{R^4} \left[\frac{\mathrm{N}}{\mathrm{mKg}}\right] \tag{7.47}$$

The dependance of K and $\bar{K}$ on h and R is different. While the stiffness favors thicker plates, the stiffness per unit weight increases faster with a large radius. The effect of the ratio E/ρ can be shown on the example of steel and aluminum plates, see Table (7.1).

Table 7.1: Basic properties of steel and aluminum

	E[GPa]	ρ[g/cm^3]	E/ρ
Steel	2.1	7.8	3.7
Al	0.8	2.8	3.5

Aluminum alloys seem much lighter but they lose elasticity modulus in the same proportion. It is seen that there are not much gain in the stiffness per unit weight by replacing steel by aluminum. So, what else could be done to increase plate stiffness? The answer is:

- Sandwich plates
- Stiffened plates

Each of the above concept is studied separately.

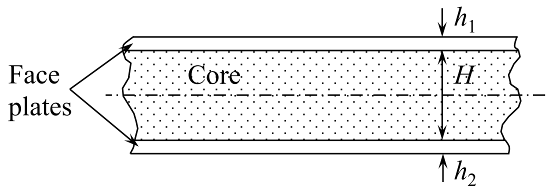

Figure 7.3: There are two materials and three *different thicknesses* in sandwich plates.

7.5 Sandwich Plates

Sandwich plates are composed of face sheets and a lightweight core, Fig. (7.3).

The core is transmitting shear stresses while the face plates are working mostly in tension or compression. Typical materials for a core are polyuritine foam, Aluminum foam, aluminum or nomex honeycombs, polymeric material of various kinds etc. In many steel structures, there is a discrete core made of corrugated sheets welded stiffeners of different topologies or truss structures. Pictures of some typical core materials and sandwich sets are shown in Fig. (7.4).

In order to determine the bending and axial stiffness of the sandwich plate, we must revisit the definition of bending moment. For cylindrical bending,

$$M_{xx} = \int \sigma z \, \mathrm{d}z = \int_{-\frac{H}{2}}^{\frac{H}{2}} \sigma_{\mathrm{c}} z \, \mathrm{d}z + \sigma_{\mathrm{f}} h H \tag{7.48a}$$

$$N_{xx} = \int \sigma \, \mathrm{d}z = \int_{-\frac{H}{2}}^{\frac{H}{2}} \sigma_{\mathrm{c}} \, \mathrm{d}z + 2\sigma_{\mathrm{f}} h \tag{7.48b}$$

The Young's modulus of the core material is usually two orders of magnitude smaller than that of the face plates, so $\sigma_{\mathrm{f}} \gg \sigma_{\mathrm{c}}$. Neglecting the first term in Eq. (7.48) and extending the above definitions to plates, the bending moments and axial forces are

$$M_{\alpha\beta} = \sigma_{\alpha\beta} h H \tag{7.49a}$$

$$N_{\alpha\beta} = 2\sigma_{\alpha\beta} h \tag{7.49b}$$

where $\sigma_{\alpha\beta}$ is related to the face plate strain by the plane stress elasticity law, Eqs. (4.54-4.56).

The Love-Kirchhoff hypothesis is still valid so the strains in the face plates are

$$\epsilon_{\alpha\beta} = \epsilon^{\circ}_{\alpha\beta} \pm \frac{H}{2} \kappa_{\alpha\beta} \tag{7.50}$$

where the "±" sign apply to the tensile and compressive side of the panel. The resulting moment-curvature relations become

$$M_{\alpha\beta} = D_{\mathrm{s}}[(1-\nu)\kappa_{\alpha\beta} + \nu\kappa_{\gamma\gamma}\delta_{\alpha\beta}] \tag{7.51a}$$

$$N_{\alpha\beta} = C_{\mathrm{s}}[(1-\nu)\epsilon^{\circ}_{\alpha\beta} + \nu\epsilon_{\gamma\gamma}\delta_{\alpha\beta}] \tag{7.51b}$$

where the bending and axial rigidities of the sandwich plates are

$$D_{\rm s} = \frac{EhH^2}{(1-\nu^2)}; \quad C_{\rm s} = \frac{EhH}{1-\nu^2} \tag{7.52}$$

Now, there is more room for the optimum design, because instead of a thickness of a monolithic plate, we have two geometrical parameter to play with. Replacing the bending rigidity D of the monolithic plate by Eq. (7.52), the bending stiffness of the circular sandwich plate become

$$K_{\rm s} = 222E\frac{hH^2}{R^2} \tag{7.53}$$

Assuming the mass density of the core to be two orders of magnitude smaller than the face plate, the total wight of the sandwich plate is

$$W_{\rm s} = \pi R^2 2h\rho \tag{7.54}$$

Then, the formula for the bending stiffness per unit weight is

$$\bar{K}_{\rm s} = 35\frac{E}{\rho}\frac{H^2}{R^4}\left[\frac{\rm N}{\rm mKg}\right] \tag{7.55}$$

Two observations can be made. First, $K_{\rm s}$ is independent on the thickness of face-plates. Secondly, the stiffness per unit weight increases parabolically with the core thickness H. Does it mean that one can make $K_{\rm s}$ as large as desired by increasing H? This is too good to be true. With increasing H, the sandwich plate may fail in either of the three failure modes:

(i) Yielding or fracture of face plate on the tensile side;
(ii) Face plate buckling on the compressive side;
(iii) Delamination due to excessive shear.

None of these failure modes are present in monolithic plates. It can be concluded that sandwich plates bring substantial improvements in the bending stiffness but at the same time introduces new unwanted features. Fracture, buckling and shear stresses will be the subject of subsequent lecture. But even at this point we can say that optimization of sandwich plates are possible by determining the maximum core thickness $H_{\rm opt}$ slightly less than that causing one of the above failure modes.

7.6 Stiffened Plates

Another way of light weighting plates is to provide a system of uni-directional or orthogonal stiffeners. As opposed to sandwich structures which are symmetric, stiffened plates are asymmetric with the neutral axis positioned usually outside the profile of the plate. A stiffened plates consists of a system of beams interacting with a uniform thickness plate. Photos of typical stiffened plates used in civil engineering, ship buildings and other segment of the economy are shown in Fig. (7.5).

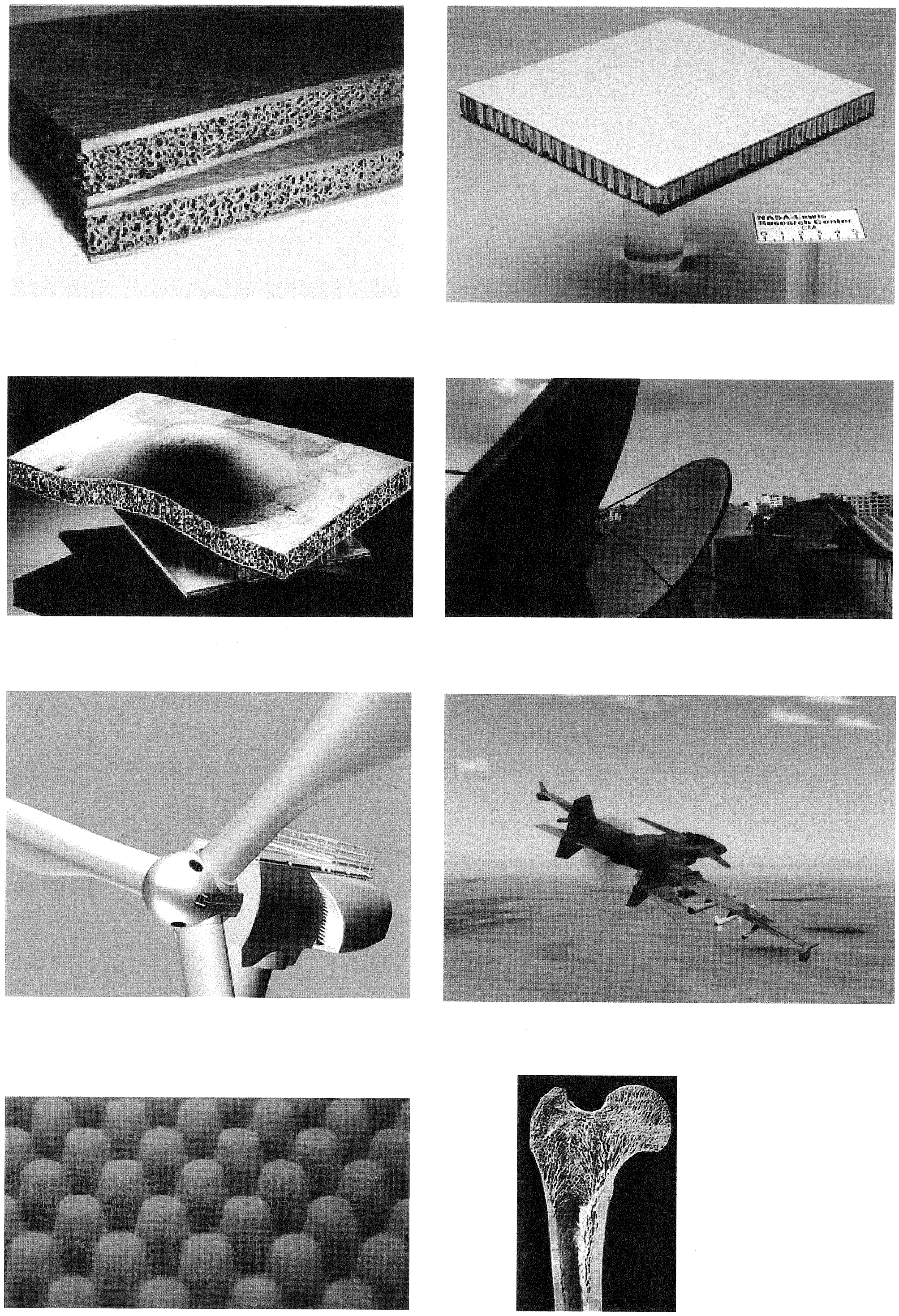

Figure 7.4: Pictures of foam-filled and honeycomb core sandwich plates and panels with some applications.

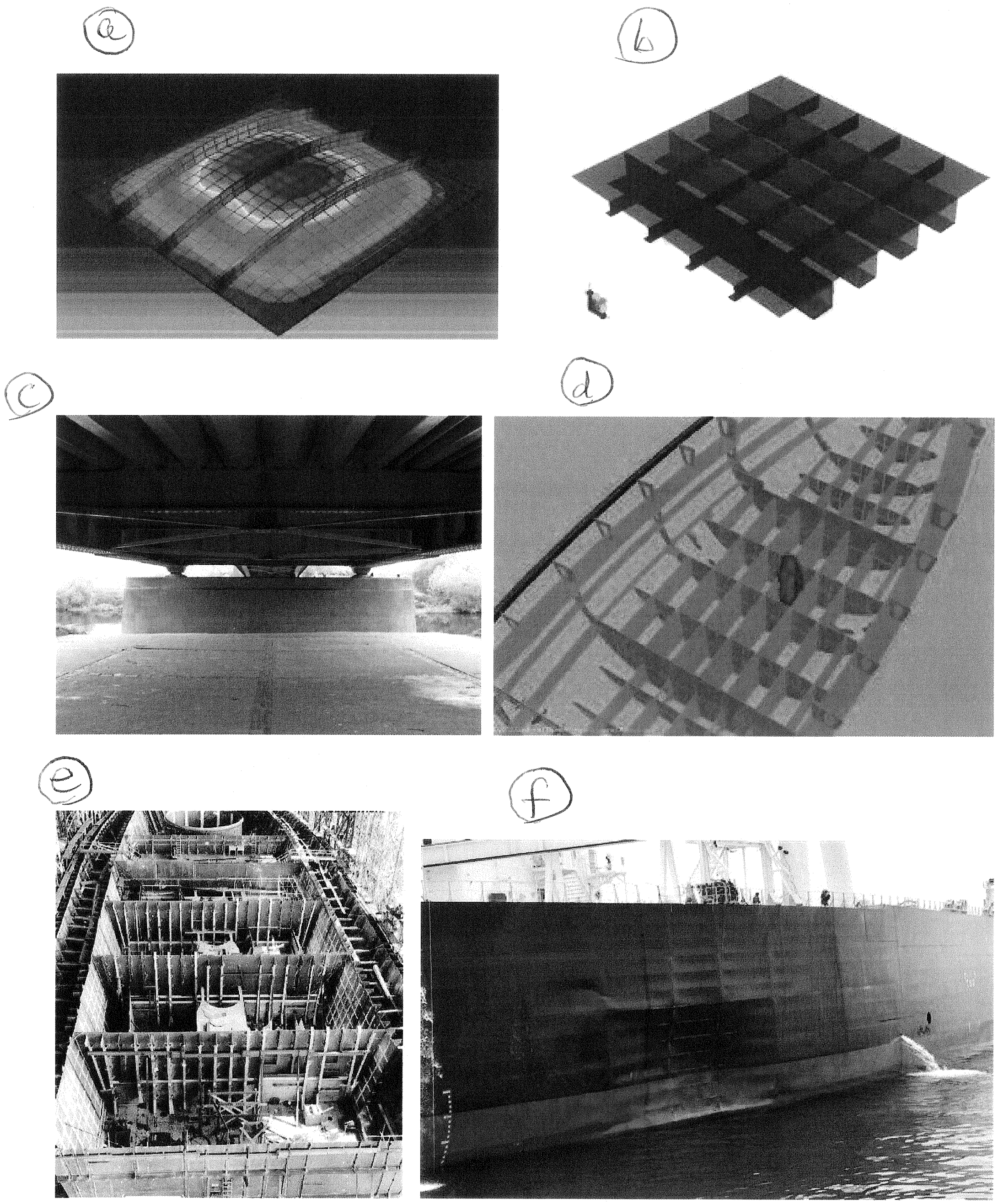

Figure 7.5: Stiffened panels are fundamental building blocks of modern structures.

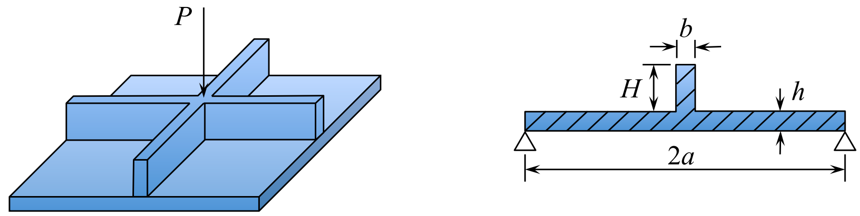

Figure 7.6: Geometry of the stiffened plate.

To illustrate the response of stiffened plates to transverse loads consider an example of a simply supported plate stiffened by two cross-beams, Fig. (7.6).

The structure is loaded by a point force P at the center. For simplicity, shown is the simplest flat bar stiffener but the analysis is valid for any beam defined by the moment of inertia I. Can we determine the stiffness of the system using our existing knowledge of beams and plates? Let's see.

The solution for the beam problem is

$$w(x) = w_o^{\mathrm{b}} \frac{x}{l} \left[3 - 4 \left(\frac{x}{l} \right)^2 \right] \tag{7.56a}$$

$$w_o^{\mathrm{b}} = \frac{P^{\mathrm{b}} l^3}{48EI} \tag{7.56b}$$

where $l = 2a$ is the length of the stiffener. The solution for the circular plate under the concentrated point load is given by Eq. (7.28)

$$\bar{w} = \frac{w(r)}{w_o^{\mathrm{p}}} = \left[2 \left(\frac{r}{a} \right)^2 \ln \frac{r}{a} + \frac{3+\nu}{1+\nu} \left(1 - \left(\frac{r}{a} \right)^2 \right) \right] \tag{7.57a}$$

$$w_o^{\mathrm{p}} = \frac{p^{\mathrm{p}} a^2}{16\pi D} \tag{7.57b}$$

Here the analogy between the response of a circular and square plate was used with $a \approx R$.

The comparison of the deflected shapes of the beam and the plate is shown in Fig. (7.7). This means that in terms of vertical deflections w the beam shape fits on the deform plate. What about the horizontal displacements? This is shown in Fig. (7.8). The beam and the plate deform separately and there is an incompatibility of the displacement u. This corresponds to the situation that both components are not connected, with sliding allowed. Should sliding be prevented, for example by welding, the neutral axis of the plate-beam combination will be shifted. Therefore the actual stiffness of the welded stiffen structure will be greater than simply adding their individual contributions. The present model will give only the lower bound. Let's calculate the lower bound stiffness of the system.

The almost perfect compatibility of the vertical displacement, shown in Fig. (7.7), means that we are dealing with two linear springs in parallel, Fig. (7.9). The total resisting force is the sum of individual components, while the displacements are the same

$$P = P_{\mathrm{p}} + P_{\mathrm{b}}; \quad w_o^{\mathrm{p}} = w_o^{\mathrm{b}} = w_o \tag{7.58}$$

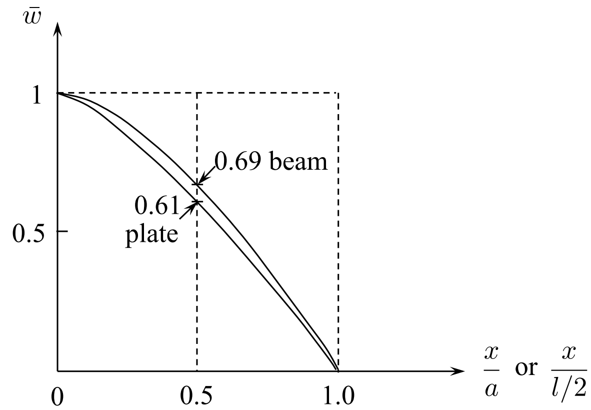

Figure 7.7: Normalized deflected shapes of a beam and a plate is very similar.

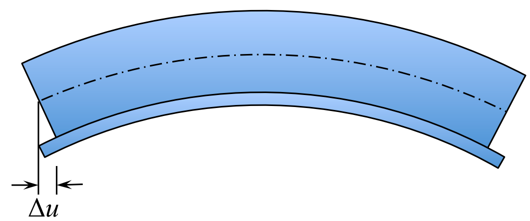

Figure 7.8: The length of the bent plate is the same as the beam axes. Relative displacement at the interface is denoted by Δu.

From Eqs. (7.56) and (7.57) one gets

$$P = \left(\frac{16\pi D}{a^2} + \frac{6EI}{a^3} \right) w_o \tag{7.59}$$

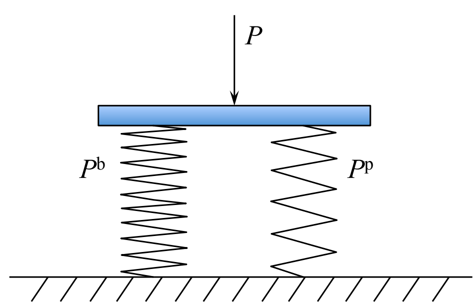

Figure 7.9: Two spring in series.

If we assume for simplicity that the flat bar stiffener is of the same thickness as the plate, $b = h$, then Eq. (7.59) simplifies to

$$P = \frac{1}{2} E h w_o \left[\frac{8\pi}{3(1-\nu^2)} \left(\frac{h}{a} \right)^2 + \left(\frac{H}{a} \right)^3 \right] \tag{7.60}$$

To get a feel, the plate and beam will equally contribute to the stiffen of the system if

$$\left(\frac{3h}{a}\right)^2 = \left(\frac{H}{a}\right)^3 \tag{7.61}$$

For a typical plate with span to thickness ratio $a/h = 30$, the hight of the stiffener is $H \cong 0.2a$. How good the above lower bound solution is? This is a difficult question to which no general and simple solution can be derived.

It is helpful to distinguish three limiting cases shown in Fig. (7.10). A very substantial and sparsely positioned stiffeners acts as an almost rigid clamped support to the plate, case (a). Light and densely distributed stiffeners, case (c), deform together with the plate. there is one deflection line and stiffeners contribute to the bending stiffness of the plate. The case (b) is a combination of the above two extreme cases. Cases (a) and (c) will be studied below.

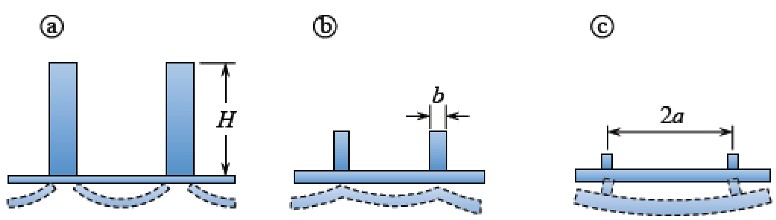

Figure 7.10: Heavy, intermediate and light stiffeners.

7.7 Plates versus Grillages

Case (a). Two heavy stiffeners are subdividing the square plate shown in Fig. (7.6) into four smaller square plates. An example of this type of design is the "hungry horse" deformation pattern of the ship hall, shown in Fig. (7.5b).

The point load is still applied at the intersection of both beams. The solution given by Eq. (7.59) is still valid but now the beam stiffness is much higher than the plate bending stiffness, and the first term in Eq. (7.59) can be neglected. The solution of two intersecting beam, each carrying half of the load is exact. The stiffness of the beam system is

$$K|_{\text{two beams}} = \frac{P}{w_o} = \frac{12EI}{a^3} = \frac{EbH^3}{a^3} \tag{7.62}$$

while the plate stiffness from Eq. (7.57) is

$$K|_{\text{plate}} = \frac{16\pi D}{a^2} = \frac{16\pi E h^3}{12a^2(1-\nu^2)}\frac{1+\nu}{3+\nu} \tag{7.63}$$

Two intersecting beams form the simplest grillage

The question is which of the two types of structures, plates or grillages are more weight efficient? So, let's keep the volume of both types of structures the same and compare their stiffnesses.

$$V_{\text{plate}} = V_{\text{beam}} \;\rightarrow\; ah = bH \tag{7.64}$$

The ratio of stiffnesses, keeping the volume (weight) the same is

$$\frac{K_{\text{beams}}}{K_{\text{plate}}} = 0.6\frac{b}{a}\left(\frac{H}{h}\right)^3 = 0.6\left(\frac{H}{h}\right)^2 \tag{7.65}$$

The stiffness of grillage is the same as that of the plate if $H = 1.25h$. Stiffeners alone or their assemblages into a grillage can thus transmit considerable concentrated loads. They cannot resist distributed pressure. For that purpose plates or stiffened plates must be used.

7.8 The Concept of Equivalent Thickness

Densely spaced and weak stiffeners follow the deflection line of the plate to which they are attached. The main load-resisting mechanism is plate bending with an additional contribution of stiffeners. The solution for the plate is still valid but the plate thickness must be increased to form an *equivalent thickness* h_{eq}. In plate bending problem the equivalence should be based on equal moment of inertia of two structures, Fig. (7.11).

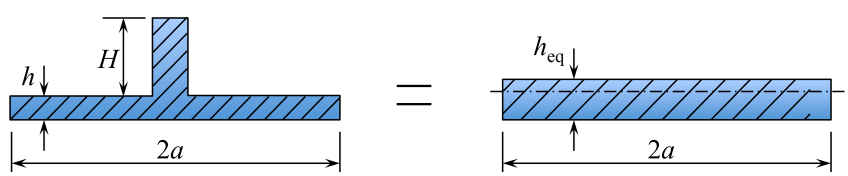

Figure 7.11: Geometry of plate/beam combination and the equivalent thickness plate.

The integrated beam/stiffener system is bending about the common bending axis. The equivalent plate is bending about the middle plane axis. The bending axis of any beam is defined by vanishing the first moment of inertia of the cross-section

$$Q = \int_A z \, \mathrm{d}A = 0 \tag{7.66}$$

For simplicity, the flat bar stiffener is considered. The position of the neutral axis, normalized with respect to the plate thickness, is related to the remaining parameters of the problem by

$$\frac{\eta}{h} = \frac{1}{2}\frac{1 - \dfrac{b}{a}\left(\dfrac{H}{h}\right)^2}{1 + \dfrac{b}{a}\left(\dfrac{H}{h}\right)} \tag{7.67}$$

The plot of the function η/h versus the normalized hight of the stiffener H/h for several values of the plate-to-stiffener aspect ratio a/b is shown in Fig. (7.12). In the limiting case of no stiffener, $H = 0$ and the position of the neutral axis is at the middle axis of the plate.

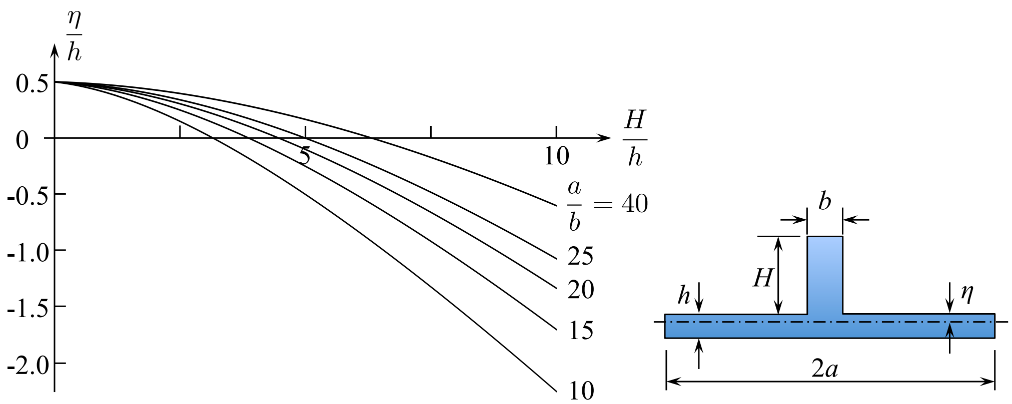

Figure 7.12: Neutral axis of the plate/beam combination moves from the plate center towards the original axis of the beam.

The moment of inertia of the plate/beam combination and the equivalent plate are, respectively

$$I = \frac{2a}{3}\left[(h^3 - 3h^2\eta + 3h\eta^2) + \frac{b}{2a}(H^3 + 3H^2\eta + 3H\eta^2)\right] \tag{7.68a}$$

$$I_{\text{eq}} = \frac{2a}{12}h_{\text{eq}}^3 \tag{7.68b}$$

By equating the respective moments of inertia, the equivalent plate thickness, normalized by the thickness of the un-stiffened plate is

$$\left(\frac{h_{\text{eq}}}{h}\right)^3 = 4\left\{\left[1 - 3\eta + 3\left(\frac{\eta}{h}\right)^2\right] + \frac{b}{2a}\left[\left(\frac{H}{h}\right)^2 + 3\left(\frac{H}{h}\right)^2\frac{\eta}{h} + 3\frac{H}{h}\left(\frac{\eta}{h}\right)^2\right]\right\} \tag{7.69}$$

The plot of h_{eq}/h versus H/h for several values of the a/b ratios is given in Fig. (7.13).

The growth of plate stiffness, according to Eq. (7.69), is parabolic with respect to $\dfrac{H}{h}$. At the same time the increase in weight (volume) of the orthogonally stiffened plate is linear

$$\frac{V_{\text{eq}}}{V} = 1 + \frac{b}{a}\frac{H}{h} \tag{7.70}$$

Therefore the stiffness per unit weight will still be an increasing function of the height of the stiffeners.

The next question is what should be the height H and spacing of stiffeners to fall under the category (c) of light stiffeners. This question can be answered by explaining the concept of the *shear lag*.

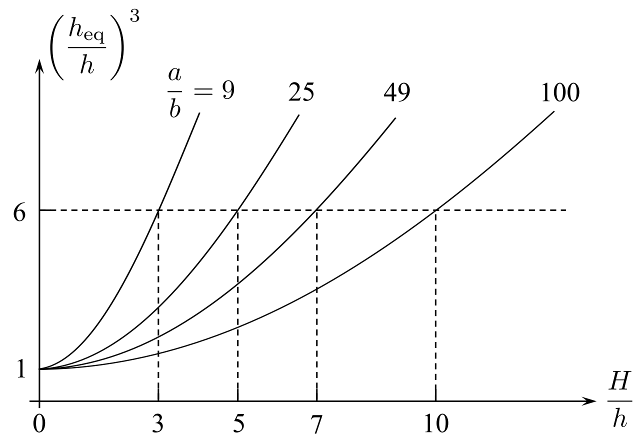

Figure 7.13: The equivalent thickness growth rapidly with the height of the stiffener.

7.9 Shear Lag

The question is how to remove the incompatibility of in-plane displacements between the beam and plate, shown in Fig. (7.8). Let's magnify this figure to see what is happening at the edge.

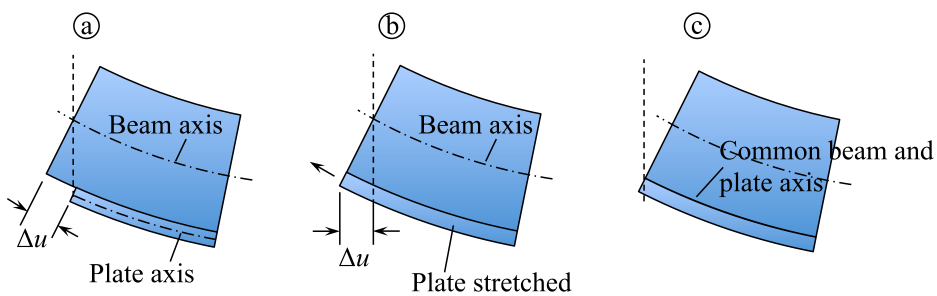

Figure 7.14: Incompatible and compatible interface between beam (stiffener) and plate.

In Fig. (7.14a), the beam and the plate are bent separately about their respective bending axes. One way of making the incompatible edge displacement to vanish, $\Delta u = 0$, would be to stretch the plate to match the tensile side of the beam. This will entail considerable in-plane sheer stresses and strain on both sides of the foot of stiffener.

The finite region of the plate subjected to large in-plane shear is called the "*effective breath*". Most of literature dealing with bending of stiffened plates took the approach called the shear lag. This approach is based on the continuity of shear forces and stresses at the beam/plate interface. The determination of the effective breadth falls behind the scope of the present lecture notes.

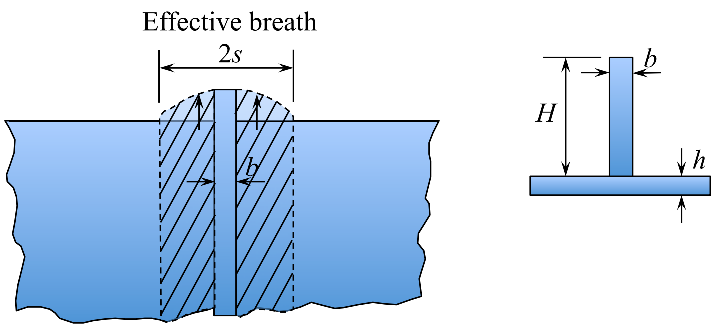

Figure 7.15: In-plane shear induced by the stiffener is restricted to an immediate vicinity of the stiffener.