1.138J/2.062J, WAVE PROPAGATION

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CHAPTER THREE

TWO DIMENSIONAL WAVES

1 Reflection and tranmission of sound at an interface

Reference : Brekhovskikh and Godin §.2.2.

The governing equation for sound in a homogeneous fluid is given by (7.31) and (7.32) in Chapter One. In term of the velocity potential defined by $\mathbf{u} = \nabla\phi$ it is

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \nabla^2\phi \tag{1.1}$$

where $c$edenotes the sound speed. Recall that the fluid pressure $pe= -\rho_o\partial\phi/\partial te$also satisfies the same equation.

We first generalize the plane sinusoidal wave in three dimensional space

$$\phi(\mathbf{x}, t) = \phi_o\ ^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = \phi_o\ ^{i(k\mathbf{n}\cdot\mathbf{x}-\omega t)} \tag{1.2}$$

where $\mathbf{n}$ is the unit vector in the direction of $\mathbf{k}$. Here the phase function is

$$\theta(\mathbf{x}, t) = \mathbf{k}\cdot\mathbf{x} - \omega te \tag{1.3}$$

The equation of constant phase $\theta(\mathbf{x}, t) = \theta_o$ describes a moving surface. The wave number vector $\mathbf{k} = k\mathbf{n}$ is defined to be

$$\mathbf{k} = k\mathbf{n} = \nabla\theta \tag{1.4}$$

hence is orthogonal to the surface of constant phase, and represents the direction of wave propagation. The frequency is defined to be

$$\omega = -\frac{\partial\theta}{\partial te} \tag{1.5}$$

Is (1.2) a solution? Let us check (1.7).

$$\nabla\phi = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi = i\mathbf{k}\phi$$

$$\nabla^2\phi = \nabla\cdot\nabla\phi = i\mathbf{k}\cdot i\mathbf{k}\phi = -k^2\phi$$

$$\frac{\partial^2\phi}{\partial t^2} = -\omega^2\phi$$

Hence (1.1) is satisfied if

$$\omega = kc \tag{1.6}$$

Consider two semi-infinite fluids separated by the plane interface along $z = 0$. The sound speeds in the upper and lower fluids are c and c_1 respectively. Let a plane incident wave Arrive from $z > 0$ at the incident angle of θ with respect to the z axis,

$$p_i = \exp[ik(x\sin\theta - z\cos\theta)] \tag{1.7}$$

implying that

$$\mathbf{k}^i = (k_x^i, k_z^i) = k(\sin\theta, -\cos\theta) \tag{1.8}$$

The motion is confined in the x, z plane.

On the same (incidence) side of the interface we have the reflected wave

$$p_r = R\exp[ik(x\sin\theta + z\cos\theta)] \tag{1.9}$$

where R denotes the reflection coefficient. The wavenumber vector is

$$\mathbf{k}^r = (k_x^r, k_z^r) = k(\sin\theta, \cos\theta) \tag{1.10}$$

In the lower medium $z < 0$ the transmitted wave has the pressure

$$p_t = T\exp[ik_1(x\sin\theta_1 - z\cos\theta_1)] \tag{1.11}$$

where T is the transmission coefficient. Along the interface $z = 0$ we require the continuity of pressure and normal velocity, i.e.,

$$[p] = 0, \quad z = 0 \tag{1.12}$$

and

$$[w] = 0 \quad z = 0, \tag{1.13}$$

where the square brackets signify the jump across the interface:

$$[f] \equiv f(z = 0+) - f(z = 0-) \tag{1.14}$$

We define the impedance of a simple harmonic waves by

$$Z = -\frac{pe}{we} \tag{1.15}$$

where w is the vertical component of the fluid velocity. Because

$$\rho \frac{\partial we}{\partial te} = -i\omega \rho w = -\frac{\partial pe}{\partial ze}, e \tag{1.16}$$

$$\frac{pe}{we} = -\frac{-i\omega \rho pe}{\frac{\partial p}{\partial z}} \tag{1.17}$$

It follows from the two continuity requirements that the impedance must be continuous

$$[Z] = 0 \quad z = 0 \tag{1.18}$$

Note first that to satisfy the conditions of continuity for all x it is necessary that the ye factors match, so that

$$k \sin\theta = k_1 \sin\theta_1 \tag{1.19}$$

or

$$\frac{\sin\theta}{ce} = \frac{\sin\theta_1}{c_1} \tag{1.20}$$

Eq. (1.19) or (1.20) is the famous Snell's law of refraction. If $c_1 < c$, waves are incident from the faster medium, the direction of the refracted (or transmitted) wave is closer to the normal to the interface. Now (1.12) requires that

$$1 + R = Te \tag{1.21}$$

The impedance of the incident wave, the reflected wave, and the transmitted waves are respectively

$$Z_i = \frac{\rho ce}{\cos\theta}, \quad Z_r = -\frac{\rho ce}{\cos\theta}, \quad Z_1 = \frac{\rho_1 c_1}{\cos\theta_1} \tag{1.22}$$

which are all constants, and the total impedance on the incidence/reflection side is

$$Z = \frac{\rho ce}{\cos\theta} \frac{\exp(-2ikz\cos\theta) + Re}{\exp(-2ikz\cos\theta) - Re} \tag{1.23}$$

which is in general a complex function of z. Next we impose (1.12) and get

$$Z_1 = \frac{\rho c}{\cos\theta}\frac{1+R}{1-R} \tag{1.24}$$

hence

$$R = \frac{Z_1\cos\theta - \rho c}{Z_1\cos\theta + \rho c} \tag{1.25}$$

This formula is written in a general form where the impedance of the lower medium can be anything . For the present example it is given by (1.22) and

$$R = \frac{\rho_1 c_1\cos\theta - \rho c\cos\theta_1}{\rho_1 c_1\cos\theta + \rho c\cos\theta_1} \tag{1.26}$$

Let

$$m = \frac{\rho_1}{\rho}, \quad n = \frac{c}{c_1} \tag{1.27}$$

where the ratio of sound speeds n is called the index of refraction. We get after using Snell's law that

$$R = \frac{m\cos\theta - n\cos\theta_1}{m\cos\theta + n\cos\theta_1} = \frac{m\cos\theta - n\sqrt{1-\frac{\sin^2\theta}{n^2}}}{m\cos\theta + n\sqrt{1-\frac{\sin^2\theta}{n^2}}} \tag{1.28}$$

The transmission coefficient follows readily from (1.21),

$$T = 1 + R = \frac{2m\cos\theta}{m\cos\theta + n\sqrt{1-\frac{\sin^2\theta}{n^2}}} \tag{1.29}$$

We now examine the physics.

If $n = c/c_1 > 1$, the incidence is from a faster to a slower medium, then R is always real. If however $n < 1$ then $\theta_1 > \theta$. There is a critical incidence angle θ_c, defined by

$$\sin\theta_c = n \tag{1.30}$$

beyond which ($\theta > \theta_c$) the square roots above become imaginary. We must then take

$$\cos\theta_1 = \sqrt{1-\frac{\sin^2\theta}{n^2}} = i\sqrt{\frac{\sin^2\theta}{n^2} - 1} \tag{1.31}$$

This means that the reflection coefficient is now complex

$$R = \frac{m\cos\theta - in\sqrt{\frac{\sin^2\theta}{n^2}-1}}{m\cos\theta + in\sqrt{\frac{\sin^2\theta}{n^2}-1}} \tag{1.32}$$

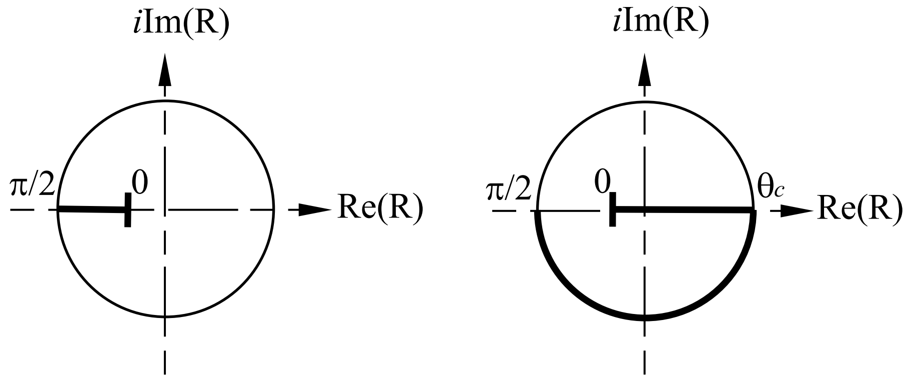

Figure 1: Complex reflection coefficient

with $|R| = 1$, implying complete reflection. As a check the transmitted wave is now given by

$$p_t = T \exp\left[k_1\left(ix\sin\theta_1 + z\sqrt{\sin^2\theta/n^2 - 1}\right)\right] \tag{1.33}$$

so the amplitude attenuates exponentially in z as $z \to -\infty$. Thus the wave train cannot penetrate much below the interface.

The dependence of R on various parameters is best displayed in the complex plane $R = \Re R + i\Im R$.

Case 1: $n > 1$. Here R is always real.

For normal incidence $\theta = 0$,

$$R = \frac{m - ne}{m + ne} \tag{1.34}$$

$R > 0$ if $n < m$ and $R < 0$ if $n > m$. In either case $|R| < 1$ For glancing incidence $\theta = \pi/2$, $R = -1$. For any intermediate incidence angles, R falls in the segment of the real axis as shown in figure 1.a. and 1.b.

Case 2. $n < 1$ then R is real only if $\theta < \theta_c$, otherwise R becomes complex and has the unit amplitude. It is clear from (1.32) that $\Im R < 0$ so that R falls on the half circle in the lower half of the complex plane as shown in figure 1.c and 1.d.

Refs:

Graff: *Wave Motion in Elastic Solids*

Aki & Richards *Quantitative Seismology,* V. 1.

Achenbach. *Wave Propagation in Elastic Solids*

2 Equations for elastic waves

Let the displacement vector at a point x_j and time $t$ebe denoted by $u_i(x_j, t)$, then Newton's law applied to an material element of unit volume reads

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \tau_{ij}}{\partial x_j} \tag{2.1}$$

where τ_{ij} is the stress tensor. We have neglected body force such as gravity. For a homogeneous and isotropic elastic solid, we have the following relation between stress and strain

$$\tau_{ij} = \lambda \ _{kk}\delta_{ij} + 2\mu e_{ij} \tag{2.2}$$

where λ and μ are Lamé constants and

$$_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.3}$$

is the strain tensor. Eq. (2.2) can be inverted to give

$$_{ij} = \frac{1+\nu}{E}\tau_{ij} - \frac{\nu}{Ee}\tau_{kk}\delta_{ij} \tag{2.4}$$

where

$$E = \frac{\mu(3\lambda + \mu)}{\lambda + \mu e} \tag{2.5}$$

is Young's modulus and

$$\nu = \frac{\lambda}{2(\lambda + \mu)}.e \tag{2.6}$$

Substituting (2.2) and (2.3) into (2.1) we get

$$\begin{aligned}
\frac{\partial \tau_{ij}}{\partial x_j} &= \lambda \frac{\partial \ _{kk}}{\partial x_j}\delta_{ij} + \mu e \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \\
&= \lambda \frac{\partial \ _{kk}}{\partial x_i} + \mu e \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \mu e \frac{\partial^2 u_j}{\partial x_i \partial x_j} \\
&= (\lambda + \mu)\frac{\partial^2 u_i}{\partial x_i x_j} + \mu \nabla^2 u_i
\end{aligned}$$

In vector form (2.1) becomes

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u} \tag{2.7}$$

Taking the divergence of (2.1) and denoting the dilatation by

$$\Delta \equiv \ _{kk} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \tag{2.8}$$

we get the equation governing the dilatation alone

$$\rho \frac{\partial^2 \Delta}{\partial t^2} = (\lambda + \mu)\nabla \cdot \nabla \Delta + \mu \nabla^2 \Delta = (\lambda + 2\mu)\nabla^2 \Delta \tag{2.9}$$

or,

$$\frac{\partial^2 \Delta}{\partial t^2} = c_L^2 \nabla^2 \Delta \tag{2.10}$$

where

$$c_L = \sqrt{\frac{\lambda + 2\mu e}{\rho}} \tag{2.11}$$

Thus the dilatation propagates as a wave at the speed c_L. To be explained shortly, this is a longitudinal waves, hence the subscript L. On the other hand, taking the curl of (2.7) and denoting by $\vec{\omega}$ the rotation vector:

$$\vec{\omega} = \nabla \times \mathbf{u} \tag{2.12}$$

we then get the governing equation for the rotation alone

$$\frac{\partial^2 \vec{\omega}}{\partial t^2} = c_T^2 \nabla^2 \vec{\omega} \tag{2.13}$$

where

$$c_T = \sqrt{\frac{\mu e}{\rho}} \tag{2.14}$$

Thus the rotation propagates as a wave at the slower speed c_T. The subscript Teindicates that this is a transverse wave, to be shown later.

The ratio of two wave speeds is

$$\frac{c_L}{c_T} = \sqrt{\frac{\lambda + \mu e}{\lambda}} > e1.e \tag{2.15}$$

Since

$$\frac{\mu e}{\lambda} = \frac{1}{2\nu} - 1 \tag{2.16}$$

it follows that the speed ratio depends only on Poisson's ratio

$$\frac{c_L}{c_T} = \sqrt{\frac{2 - 2\nu}{1 - 2\nu}} \tag{2.17}$$

There is a general theorem due to Helmholtz that any vector can be expressed as the sum of an irrotational vector and a solenoidal vector i.e.,

$$\mathbf{u} = \nabla\phi + \nabla \times \mathbf{H} \tag{2.18}$$

subject to the constraint that

$$\nabla \cdot \mathbf{H} = 0 \tag{2.19}$$

The scalar ϕ and the vector $\mathbf{H}$ are called the displacement potentials. Substituting this into (2.7), we get

$$\rho\frac{\partial^2}{\partial t^2}[\nabla\phi + \nabla \times \mathbf{H}] = \mu\nabla^2[\nabla\phi + \nabla \times \mathbf{H}] + (\lambda+\mu)\nabla\nabla\cdot[\nabla\phi + \nabla \times \mathbf{H}]$$

Since $\nabla \cdot \nabla\phi = \nabla^2\phi$, and $\nabla \cdot \nabla \times \mathbf{H} = 0$ we get

$$\nabla\left[(\lambda+2\mu)\nabla^2\phi - \frac{\partial^2\phi}{\partial t^2}\right] + \nabla \times \left[\mu\nabla^2\mathbf{H} - \rho\frac{\partial^2\mathbf{H}}{\partial t^2}\right] = 0 \tag{2.20}$$

Clearly the above equation is satisfied if

$$(\lambda+2\mu)\nabla^2\phi - \frac{\partial^2\phi}{\partial t^2} = 0 \tag{2.21}$$

and

$$\mu\nabla^2\mathbf{H} - \rho\frac{\partial^2\mathbf{H}}{\partial t^2} = 0 \tag{2.22}$$

Although the governing equations are simplified, the two potentials are usually coupled by boundary conditions, unless the physical domain is infinite.

3 Free waves in infinite space

The dilatational wave equation admits a plane sinusoidal wave solution:

$$\phi(\mathbf{x},t) = \phi_o\ ^{ik(\mathbf{n}\cdot\mathbf{x}-c_L t)} \tag{3.1}$$

Here the phase function is

$$\theta(\mathbf{x},t) = k(\mathbf{n}\cdot\mathbf{x} - c_L t) \tag{3.2}$$

which describes a moving surface. The wave number vector $\mathbf{k} = k\mathbf{n}$ is defined to be

$$\mathbf{k} = k\mathbf{n} = \nabla\theta \tag{3.3}$$

hence is orthogonal to the surface of constant phase, and represents the direction of wave propagation. The frequency is

$$\omega = kc_T = -\frac{\partial \theta}{\partial te} \tag{3.4}$$

A general solution is

$$\phi = f(\mathbf{n} \cdot \mathbf{x} - c_L t) \tag{3.5}$$

Similarly the following sinusoidal wave is a solution to the shear wave equation;

$$\mathbf{H} = \mathbf{H}_o \ ^{ik(\mathbf{n}\cdot\mathbf{x}-c_T t)} \tag{3.6}$$

A general solution is

$$\mathbf{H} = \mathbf{F}(\mathbf{n} \cdot \mathbf{x} - c_T t) \tag{3.7}$$

We can also write (3.5) and (3.9) as

$$\phi = f(t - \frac{\mathbf{n} \cdot \mathbf{x}}{c_L}) \tag{3.8}$$

and

$$\mathbf{H} = \mathbf{F}(t - \frac{\mathbf{n} \cdot \mathbf{x}}{c_T}) \tag{3.9}$$

where

$$\mathbf{s}_L = \frac{\mathbf{n}}{c_L}, e \ \mathbf{s}_T = \frac{\mathbf{n}}{c_T} \tag{3.10}$$

are called the slowness vectors of longitudinal and transverse waves respectively.

In a dilatational wave the displacement vector is parallel to the wave number vector:

$$\mathbf{u}_L = \nabla \phi = f' \mathbf{n} \tag{3.11}$$

where f' is the ordinary derivative of $f$ewith respect to its argument. Hence the dilatational wave is a *longitudinal* (compression) wave. On the other hand in a rotational wave the displacement vector is perpendicular to the wave number vector,

$$\begin{aligned} \mathbf{u}_T &= \nabla \times \mathbf{H} = \mathbf{e}_x \left(\frac{\partial F_z}{\partial ye} - \frac{\partial F_y}{\partial ze} \right) + \mathbf{e}_y \left(\frac{\partial F_x}{\partial ze} - \frac{\partial F_z}{\partial xe} \right) + \mathbf{e}_z \left(\frac{\partial F_y}{\partial xe} - \frac{\partial F_x}{\partial ye} \right) \\ &= \mathbf{e}_x \left(F_z' e n_y - F_y' e n_z \right) + \mathbf{e}_y \left(F_x' e n_z - F_z' e n_x \right) + \mathbf{e}_z \left(F_y' e n_x - F_x' e n_y \right) \\ &= \mathbf{n} \times \mathbf{F}' \end{aligned} \tag{3.12}$$

Hence a rotational wave is a *transverse* (shear) wave.

4 Elastic waves in a plane

Refs. Graff, Achenbach,

Aki and Richards : *Quantitative Seismology, v.1*

Let us examine waves propagating in the vertical plane of x, y. All physical quantities are assumed to be uniform in the direction of z, hence $\partial/\partial z = 0$, then

$$u_x = \frac{\partial \phi}{\partial x e} - \frac{\partial H_z}{\partial y}, \quad u_y = \frac{\partial \phi}{\partial y e} - \frac{\partial H_z}{\partial x}, \quad u_z = -\frac{\partial H_z}{\partial y e} + \frac{\partial H_y}{\partial x e} \tag{4.13}$$

and

$$\frac{\partial H_x}{\partial x e} + \frac{\partial H_y}{\partial y e} = 0 \tag{4.14}$$

where

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c_L^2}\frac{\partial^2 \phi}{\partial t^2}, e \tag{4.15}$$

$$\frac{\partial^2 H_p}{\partial x^2} + \frac{\partial^2 H_p}{\partial x^2} = \frac{1}{c_T^2}\frac{\partial^2 H_p}{\partial t^2}, \quad p = x, y, ze \tag{4.16}$$

Note that u_z is also governed by (4.16).

From Hooke's law the stress components can be written

$$\begin{aligned} \tau_{xx} &= \lambda\left(\frac{\partial u_x}{\partial x e} + \frac{\partial u_y}{\partial y e}\right) + 2\mu e\frac{\partial u_x}{\partial x e} = (\lambda + 2\mu)\left(\frac{\partial u_x}{\partial x e} + \frac{\partial u_y}{\partial y e}\right) - 2\mu e\frac{\partial u_y}{\partial y e} \\ &= (\lambda + 2\mu)\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) - 2\mu e\left(\frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 H_z}{\partial y \partial x e}\right) \end{aligned} \tag{4.17}$$

$$\begin{aligned} \tau_{yy} &= \lambda\left(\frac{\partial u_x}{\partial x e} + \frac{\partial u_y}{\partial y e}\right) + 2\mu e\frac{\partial u_y}{\partial y e} = (\lambda + 2\mu)\left(\frac{\partial u_x}{\partial x e} + \frac{\partial u_y}{\partial y e}\right) - 2\mu e\frac{\partial u_x}{\partial x e} \\ &= (\lambda + 2\mu)\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) - 2\mu e\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 H_z}{\partial x \partial y e}\right) \end{aligned} \tag{4.18}$$

$$\tau_{zz} = \frac{\lambda}{2(\lambda + \mu)}(\tau_{xx} + \tau_{yy}) = \nu(\tau_{xx} + \tau_{yy}) = \lambda\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) \tag{4.19}$$

$$\tau_{xy} = \mu e\left(\frac{\partial u_y}{\partial x e} + \frac{\partial u_x}{\partial y e}\right) = \mu\left(2\frac{\partial^2 \phi}{\partial x \partial y e} - \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2}\right) \tag{4.20}$$

$$\tau_{yz} = \mu e\frac{\partial u_z}{\partial y e} = \mu\left(-\frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_y}{\partial y \partial x e}\right) \tag{4.21}$$

$$\tau_{xz} = 0 \tag{4.22}$$

Note that the governing equations for ϕ and H_z are governed by waves uncoupled from those for H_x and H_y, hence the in-plane displacement components u_x, u_y are independent of the out-of-plane component u_z. The in-plane displacements (u_x, u_y) are associated with dilatation and in-plane shear, represented respectively by ϕ and H_z, which will be referred to as the P wave and the SV wave. The out-of-plane displacement u_z is associated with H_x and H_y, and will be referred to as the SH wave.

Different physical situations arise for different boundary conditions. We shall consider first the half plane problem bounded by the plane $y = 0$.

5 Reflection of elastic waves from a plane boundary

Several types of boundary conditions can be prescribed on the plane boundary : (i) dynamic: the stress components only (the traction condition); (ii) kinematic: the displacement components only, or (iii). a combination of stress components and displacement components. Most difficult are (iv) the mixed conditions in which stresses are given over part of the boundary and displacements over the other.

We consider the simplest case where the plane ye= 0is completely free of external stresses,

$$\tau_{yy} = \tau_{xy} = 0,e \tag{5.23}$$

and

$$\tau_{yz} = 0 \tag{5.24}$$

It is clear that (5.23) affects the P and SV waves only, while (5.24) affects the SH wave only. Therefore we have two uncoupled problems each of which can be treated separately.

5.1 P and SV waves

Consider the case where only $Pe$and SVewaves are present, then $H_x = H_z = 0$ Let all waves have wavenumber vectors inclined in the positive x direction:

$$\phi = f(y)\ ^{i\xi x - i\omega t}, \quad H_z = h_z(y)\ ^{i\zeta x - i\omega t} \tag{5.25}$$

It follows from (4.15) and (4.16) that

$$\frac{d^2 f}{dy^2} + \alpha^2 f e= 0,e \quad \frac{d^2 h_z}{dy^2} + \beta^2 h_z = 0,e \tag{5.26}$$

with

$$\alpha = \sqrt{\frac{\omega^2}{c_L^2} - \xi^2} = \sqrt{k_L^2 - \xi^2},e \quad \beta = \sqrt{\frac{\omega^2}{c_T^2} - \zeta^2} = \sqrt{k_T^2 - \xi^2} \tag{5.27}$$

We first take the square roots to be real; the general solution to (5.26) are sinusoids, hence,

$$\phi = A_P \ {}^{i(\xi x - \alpha y - \omega t)} + B_P \ {}^{i(\xi x + \alpha y - \omega t)}, \quad H_z = A_S \ {}^{i(\zeta x - \beta y - \omega t)} + B_S \ {}^{i(\zeta x + \beta y - \omega t)} \tag{5.28}$$

On the right-hand sides the first terms are the incident waves and the second are the reflected waves. If the incident amplitudes A_P, A_S and are given, what are the properties of the reflected waves B_P, B_S? The wave number components can be written in the polar form:

$$(\xi, \alpha) = k_L(\sin\theta_L, \cos\theta_L),e \ (\zeta, \beta) = k_T(\sin\theta_T, \cos\theta_T) \tag{5.29}$$

where (k_L, k_T) are the wavenumbers, the (θ_L, θ_T) the directions of the P wave and SV wave, respectively. In terms of these we rewrite (5.28)

$$\phi = A_P \ {}^{ik_L(\sin\theta_L x - \cos\theta_L y - \omega t)} + B_P \ {}^{i(\sin\theta_L x + \cos\theta_L y - \omega t)} \tag{5.30}$$

$$H_z = A_S \ {}^{i(\sin\theta_T x - \cos\theta_L y - \omega t)} + B_S \ {}^{i(\sin\theta_T x + \cos\theta_T y - \omega t)} \tag{5.31}$$

In order to satisfy (5.23) on $y = 0$ for all x, we must insist:

$$k_L \sin\theta_L = k_T \sin\theta_T,e \quad (\xi = \zeta) \tag{5.32}$$

This is in the form of Snell's law:

$$\frac{\sin\theta_L}{c_L} = \frac{\sin\theta_T}{c_T} \tag{5.33}$$

implying

$$\frac{\sin\theta_L}{\sin\theta_T} = \frac{c_L}{c_L} = \frac{k_T}{k_L} \equiv \kappa \tag{5.34}$$

When (5.23) are applied on $y e= 0$ the exponential factors cancel, and we get two algebraic conditions for the two unknown amplitudes of the reflected waves (B_P, B_S) :

$$k_L^2(2\sin^2\theta_L - \kappa^2)(A_P + B_P) - k_T^2 \sin 2\theta_T(A_S - B_S) = 0 \tag{5.35}$$

$$k_L^2 \sin 2\theta_L (A_P - B_P) - k_T^2 \cos\theta_T (A_S + B_S) = 0.e \tag{5.36}$$

Using (5.34), we get

$$2\sin^2\theta_L - \kappa^2 = \kappa^2(2\sin^2\theta_T - 1) = -\kappa^2\cos 2\theta_T$$

The two equations can be solved and the solution expressed in matrix form:

$$\begin{Bmatrix} B_P \\ B_S \end{Bmatrix} = \begin{bmatrix} S_{PP} & S_{SP} \\ S_{PS} & S_{SS} \end{bmatrix} \begin{Bmatrix} A_P \\ A_S \end{Bmatrix} \tag{5.37}$$

where

$$\mathbf{S} = \begin{bmatrix} S_{PP} & S_{SP} \\ S_{PS} & S_{SS} \end{bmatrix} \tag{5.38}$$

denotes the scattering matrix. Thus S_{PS} represents the reflected S-wave due to incident P wave of unit amplitude, etc. It is straightforward to verify that

$$S_{PP} = \frac{\sin 2\theta_L \sin 2\theta_T - \kappa^2\cos^2 2\theta_T}{\sin 2\theta_L \sin 2\theta_T + \kappa^2\cos^2 2\theta_T} \tag{5.39}$$

$$S_{SP} = \frac{-2\kappa^2 \sin 2\theta_T \cos 2\theta_T}{\sin 2\theta_L \sin 2\theta_T + \kappa^2\cos^2 2\theta_T} \tag{5.40}$$

$$S_{PS} = \frac{2\sin 2\theta_T \cos 2\theta_T}{\sin 2\theta_L \sin 2\theta_T + \kappa^2\cos^2 2\theta_T} \tag{5.41}$$

$$S_{SS} = \frac{\sin 2\theta_L \sin 2\theta_T - \kappa^2\cos^2 2\theta_T}{\sin 2\theta_L \sin 2\theta_T + \kappa^2\cos^2 2\theta_T} \tag{5.42}$$

In view of (5.33) and

$$\kappa = \frac{c_L}{c_T} = \sqrt{\frac{2-2\nu}{1-2\nu}} \tag{5.43}$$

The scattering matrix is a function of Poisson's ratio and the angle of incidence.

(i) P- wave Incidence : Consider the special case when the only incident wave is a P wave. Then $A_S = 0$ and only S_{PP} and S_{SP} are relevant. . Note first that $\theta_L > \theta_T$ in general . For normal incidence, $\theta_L = 0$. We find

$$S_{PP} = -1, \quad S_{PS} = 0 \tag{5.44}$$

there is no SV wave. On the other hand if

$$\sin 2\theta_L \sin 2\theta_T - \kappa^2\cos^2 2\theta_T = 0 \tag{5.45}$$

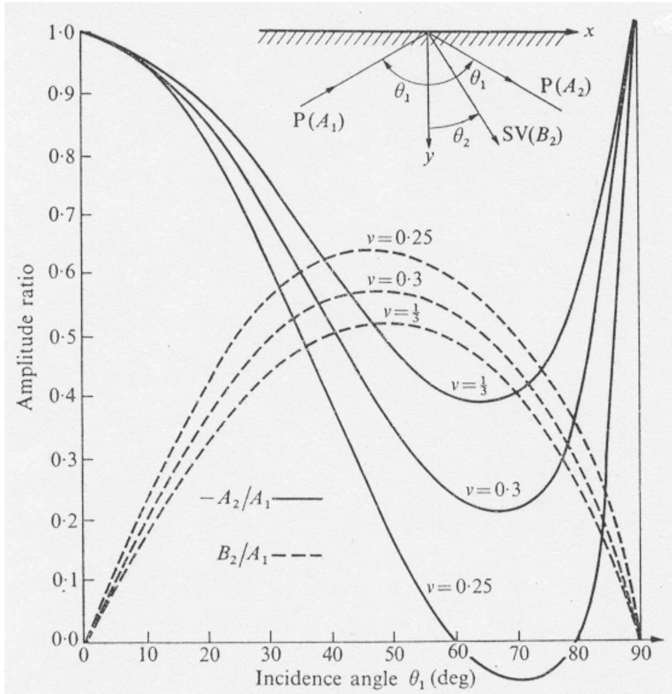

Figure 2: Amplitude ratios for incident P waves

then only *SVe*wave is reflected. This is the case of mode conversion, whereby an incident P waves changes to a SV wave after reflection. The amplitude of the reflected SV wave is

$$\frac{B_S}{A_P} = S_{PS} = \frac{\tan 2\theta_T}{\kappa^2} \tag{5.46}$$

(ii) SV wave Incidence : Let $A_P = 0$. Then only S_{SP} and S_{SS} are relevant. For normal incidence, $S_{SS} = -1$, and $S_{SP} = 0$. Mode conversion also happens when (5.45) is satisfied. Since $\theta_L > e\theta_T$, there is a critical incidence angle θ_T beyond which the P wave cannot propagate into the solid. At the critical angle

$$\sin\theta_L = 1, e \text{ or } \quad \sin\theta_T = 1/\kappa \tag{5.47}$$

Thus for $\nu = 1/3$, $\kappa = 2$ and the critical incidence angle is $\theta_T = 30°$. The P wave propagates along the x axis.

Beyond the critical angle of incidence, the *Pe*waves decay exponentially away from the free surface. The amplitude of the SV wave is linear in *ye*which is unphysical,

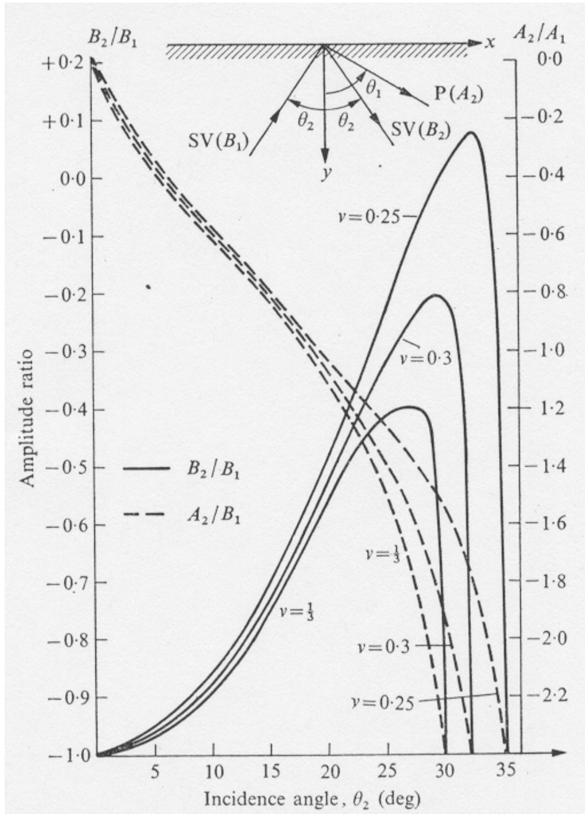

Figure 3: Reßected wave amplitude ratios for incident SV waves

suggesting the limitation of unbounded space assumption.

5.2 SH wave

Because of (4.14) we can introduce a stream function ψ so that

$$H_x = -\frac{\partial \psi}{\partial y}, \quad H_y = \frac{\partial \psi}{\partial x} \tag{5.48}$$

Clearly

$$\nabla^2 \psi = \frac{1}{c_T^2} \frac{\partial^2 \psi}{\partial t^2} \tag{5.49}$$

$$u_z = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi \tag{5.50}$$

and

$$\tau_{yz} = \mu \frac{\partial}{\partial y} \nabla^2 \psi = \frac{\mu}{c_T^2} \frac{\partial}{\partial y} \frac{\partial^2 \psi}{\partial t^2} \tag{5.51}$$

The zero-stress boundary condition implies

$$\frac{\partial \psi}{\partial ye} = 0 \tag{5.52}$$

Thus the problem for ψ is analogous to one for sound waves reflected by a solid plane. Again for monochromatic incident waves, the solution is easily shown to be

$$\psi = \left(Ae^{-i\beta y} - Ae^{i\beta y} \right) \quad ^{i\alpha x - i\omega t} \tag{5.53}$$

where

$$\alpha^2 + \beta^2 = k_T^2 \tag{5.54}$$

We remark that when the boundary is any cylindrical surface with axis parallel to the zeaxis, the stress-free condition reads

$$\tau_{zn} = 0, e \quad \text{on} \quad B.e \tag{5.55}$$

where $ne$is the unit outward normal to B. Since in the pure SH wave problem

$$\tau_{zn} = \mu e \frac{\partial u_z}{\partial ne} = \frac{\partial}{\partial n} \nabla^2 \psi = \frac{\mu e}{c_T^2} \frac{\partial}{\partial ne} \frac{\partial^2 \psi}{\partial t^2}$$

the e condition (5.55) implies

$$\frac{\partial \psi}{\partial ne} = 0, e \quad \text{on} \quad B.e \tag{5.56}$$

Thus the analogy to acoustic scattering by a hard object is true irrespective of the geometry of the scatterer.

6 Scattering of monochromatic SH waves

6.1 Solution in polar coordinates

We consider the scattering of two-dimensional SH waves of single frequency. The time-dependent potential can be written as

$$\psi(x, y, t) = \Re \left[\phi(x, y) \quad ^{-i\omega t} \right] \tag{6.1}$$

where the potential ϕ is governed by the Helmholtz equation

$$\nabla^2 \phi + k^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi^2}{\partial y^2} + k^2 \phi = 0, \quad k = e \frac{\omega}{c_T} \tag{6.2}$$

To be specific consider the scatterer to be a finite cavity of some general geometry. On the stress-free boundary B the shear stress vanishes,

$$\tau_{zn} = -\frac{\mu\omega^2}{c_T^2}\Re\left(\frac{\partial\phi}{\partial ne}{}^{-i\omega t}\right) = 0 \tag{6.3}$$

hence

$$\frac{\partial\phi}{\partial ne} = 0 \tag{6.4}$$

Let the incident waves be a plane wave

$$\phi_I = Ae^{i\mathbf{k}\cdot\mathbf{x}} \tag{6.5}$$

and the angle of incidence is θ_o with respect to the positive x axis. In polar coordinates we write

$$\mathbf{k} = k(\cos\theta_o, \sin\theta_o),e \quad \mathbf{x} = r(\cos\theta, \sin\theta) \tag{6.6}$$

$$\phi_I = A\exp\left[ikr(\cos\theta_o\cos\theta + \sin\theta_o\sin\theta)\right] = Ae^{ikr\cos(\theta-\theta_o)} \tag{6.7}$$

It can be shown (see Appendix A) that the plane wave can be expanded in Fourier-Bessel series :

$$^{ikr\cos(\theta-\theta_o)} = \sum_{n=0}^{\infty}\epsilon_n i^n J_n(kr)\cos n(\theta-\theta_o) \tag{6.8}$$

where ϵ_n is the Jacobi symbol:

$$\epsilon_0 = 0, \quad \epsilon_n = 2,\ n = 1, 2, 3, \ldots e \tag{6.9}$$

Each term in the series (6.8) is called a partial wave.

Let the total wave be the sum of the incident and scattered waves

$$\phi = \phi_I + \phi_S \tag{6.10}$$

then the scattered waves must satisfy the *radiation condition* at infinity, i.e., it can only radiate energy outward from the scatterer.

The boundary condition on the cavity surface is

$$\frac{\partial\phi}{\partial re} = 0, \quad r = ae \tag{6.11}$$

In polar coordinates the governing equation reads

$$\frac{1}{r}\frac{\partial}{\partial re}\left(re\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + k^2\phi = 0 \tag{6.12}$$

Since ϕ_I satisfies the preceding equation, so does ϕ_S.

By the method of separation of variables,

$$\phi_S(r,\theta) = R(r)\Theta(\theta)$$

we find

$$r^2R'' + rR' + (k^2r^2 - n^2)R = 0, \quad \text{and} \quad \Theta'' + n^2\Theta = 0$$

where $n = 0, 1, 2, \ldots$ are eigenvalues in order that Θ is periodic in θ with period 2π. For each eigenvalue n the possible solutions are

$$\Theta_n = (\sin n\theta, \cos n\theta),$$

$$R_n = \left(H_n^{(1)}(kr), H_n^{(2)}(kr)\right),$$

where $H_n^{(1)}(kr), H_n^{(2)}(kr)$ are Hankel functions of the first and second kind, related to the Bessel and Weber functions by

$$H_n^{(1)}(kr) = J_n(kr) + iY_n(kr), \quad H_n^{(2)}(kr) = J_n(kr) - iY_n(kr) \tag{6.13}$$

The most general solution to the Helmholtz equation is

$$\phi_S = A\sum_{n=0}^{\infty} (A_n \sin n\theta + B_n \cos n\theta)\left[C_n H_n^{(1)}(kr) + D_n H_n^{(2)}(kr)\right], \tag{6.14}$$

For large radius the asymptotic form of the Hankel functions are

$$H_n^{(1)} \sim \sqrt{\frac{2}{\pi kr}}\, e^{i(kr - \frac{\pi}{4} - \frac{n\pi}{2})}, \quad H_n^{(2)} \sim \sqrt{\frac{2}{\pi kr}}\, e^{-i(kr - \frac{\pi}{4} - \frac{n\pi}{2})} \tag{6.15}$$

In conjunction with the time factor $\exp(-i\omega t)$, $H_n^{(1)}$ gives an outgoing wave while $H_n^{(2)}$ gives an incoming wave. To satisfy the radiation condition, we must discard all terms involving $H_n^{(2)}$. From here on we shall abbreviate $H_n^{(1)}$ simply by H_n. The scattered wave is now

$$\phi_S = A\sum_{n=0}^{\infty} (A_n \sin n\theta + B_n \cos n\theta) H_n(kr) \tag{6.16}$$

The expansion coefficients (A_n, B_n) must be chosen to satisfy the boundary condition on the cavity surface[1] Once they are determined, the wave is found everywhere. In

[1] In one of the numerical solution techniques, one divides the physical region by a circle enclosing the cavity. Between the cavity and the circle, finite elements are used. Outside the circle, (6.16) is used. By constructing a suitable variational principle, finite element computation yields the nodal coefficients as well as the expansion coefficients. See (Chen & Mei , 1974).

particular in the far field, we can use the asymptotic formula to get

$$\phi_S \sim Ae\sum_{n=0}^{\infty} (A_n \sin n\theta + B_n \cos n\theta) \;\; {}^{-in\pi/2}\sqrt{\frac{2}{\pi kr}} \;\; {}^{ikr-i\pi/4} \tag{6.17}$$

Let us define the dimensionless directivity factor

$$\mathcal{A}(\theta) = \sum_{n=0}^{\infty} (A_n \sin n\theta + B_n \cos n\theta) \;\; {}^{-in\pi/2} \tag{6.18}$$

which indicates the angular variation of the far-field amplitude, then

$$\phi_S \sim A\mathcal{A}(\theta)\sqrt{\frac{2}{\pi kre}} \;\; {}^{ikr-i\pi/4} \tag{6.19}$$

This expression exhibits clearly the asymptotic behavior of ϕ_S as an outgoing wave. By differentiation, we readily see that

$$\lim_{kr\to\infty} \sqrt{r}e\left(\frac{\partial\phi_S}{\partial re} - \phi_S\right) = 0 \tag{6.20}$$

which is one way of stating the radiation condition for two dimensional SH waves.

At any radius $re$the total rate of energy outflux by the scattered wave is

$$\begin{aligned} r\int_0^{2\pi} d\theta \overline{\tau_{rz}\frac{\partial u_z}{\partial te}} &= \mu re\int_0^{2\pi} d\theta \overline{\Re\left[-\mu k^2\frac{\partial\phi}{\partial re} \;\; {}^{-i\omega t}\right]\Re\left[i\omega k^2\phi \;\; {}^{-i\omega t}\right]} \\ &= -\frac{\mu\omega k^4 re}{2}\int_0^{2\pi} d\theta \Re\left[i\phi^*\frac{\partial\phi}{\partial re}\right] = -\frac{\mu\omega k^4 re}{2}\Im\int_0^{2\pi} d\theta \left[\phi^*\frac{\partial\phi}{\partial re}\right] \end{aligned} \tag{6.21}$$

where overline indicates time averaging over a wave period $2\pi/\omega$.

We remark that in the analogous case of plane acoustics where the sound pressure and radial fluid velocity are respectively,

$$pe = -\rho_o\frac{\partial\phi}{\partial te},e \quad \text{and} \quad u_r = \frac{\partial\phi}{\partial re} \tag{6.22}$$

the energy scattering rate is

$$r\int_0^{\infty} d\theta \overline{pu_r} = \frac{\omega\rho_o re}{2}\Re\int_C d\theta \left(-i\phi^*\frac{\partial\phi}{\partial re}\right) = -\frac{\omega\rho_o re}{2}\Im\int_C d\theta \left(\phi^*\frac{\partial\phi}{\partial re}\right) \tag{6.23}$$

6.2 A general theorem on scattering

For the same scatterer and the same frequency ω, different angles of incidence θ_j define different scattering problems ϕ_j. In particular at infinity, we have

$$\phi_j \sim A_j \left\{ {}^{ikr\cos(\theta-\theta_j)} + \mathcal{A}_j(\theta)\sqrt{\frac{2}{\pi kre}}\,{}^{ikr-i\pi/4} \right\} \tag{6.24}$$

Let us apply Green's formula to ϕ_1 and ϕ_2 over a closed area bounded by a closed contour C,

$$\iint_S \left(\phi_2\nabla^2\phi_1 - \phi_1\nabla^2\phi_2\right) dAe= \int_B \left(\phi_2\frac{\partial\phi_1}{\partial ne} - \phi_1\frac{\partial\phi_2}{\partial ne}\right) dse+ \int_C dse\left(\phi_2\frac{\partial\phi_1}{\partial ne} - \phi_1\frac{\partial\phi_1}{\partial ne}\right) dse$$

where $\mathbf{n}$ refers to the unit normal vector pointing out of S. The surface integral vanishes on account of the Helmholtz equation, while the line integral along the cavity surface vanishes by virtue of the boundary condition, hence

$$\int_C dse\left(\phi_2\frac{\partial\phi_1}{\partial ne} - \phi_1\frac{\partial\phi_2}{\partial ne}\right) dse= 0 \tag{6.25}$$

By similar reasoning, we get

$$\int_C dse\left(\phi_2\frac{\partial\phi_1^*}{\partial ne} - \phi_1^*\frac{\partial\phi_2}{\partial ne}\right) dse= 0 \tag{6.26}$$

where ϕ_1^* denotes the complex conjugate of ϕ_1.

Let us choose $\phi_1 = \phi_2 = \phi_o$ in (6.26), and get

$$\int_C dse\left(\phi\frac{\partial\phi^*}{\partial ne} - \phi^*\frac{\partial\phi}{\partial ne}\right) dse= 2\Im\left(\int_C ds\,\phi\frac{\partial\phi^*}{\partial ne}\right) = 0 \tag{6.27}$$

Physically, across any circle the net rate of energy flux vanishes, i.e., the scattered power must be balanced by the incident power.

Making use of (6.24) we get

$$\begin{aligned} 0 &= \Im\int_0^{2\pi} rd\theta \left[{}^{ikr\cos(\theta-\theta_o)} + \sqrt{\frac{2}{\pi kre}}\mathcal{A}_o(\theta)\,{}^{ikr-i\pi/4} \right] \\ &\cdot \left[-ikecos(\theta-\theta_o)\,{}^{-ikr\cos(\theta-\theta_o)} - ike\sqrt{\frac{2}{\pi kre}}\mathcal{A}_o^*(\theta)\,{}^{-ikr+i\pi/4} \right] \\ &= \Im\int_0^{2\pi} rd\theta \left\{ -ikecos(\theta-\theta_o) + \frac{2}{\pi kre}(-ik)|\mathcal{A}_o|^2 \right. \end{aligned}$$

$$+ \quad {}^{ikr[\cos\theta-\theta_o)-1]+i\pi/4}(-ik)\sqrt{\frac{2}{\pi kr}}\,\mathrm{e}\,\mathcal{A}_o^*$$

$$+ \quad {}^{-ikr[\cos\theta-\theta_o)-1]-i\pi/4}(-ik)\cos(\theta-\theta_o)\sqrt{\frac{2}{\pi kre}}\mathcal{A}_o \Bigg\}$$

The first term in the integrand gives no contribution to the integral above because of periodicity. Since $\Im(if) = \Im(if^*)$, we get

$$\begin{aligned} 0 &= -\frac{2}{\pi}\int_0^{2\pi} |\mathcal{A}_o(\theta)|^2 d\theta \\ &+\Im\int_0^{2\pi} r d\theta \left\{\mathcal{A}_o(-ik)\sqrt{\frac{2}{\pi kr}}[1+\cos(\theta-\theta_o)] \;\; {}^{i\pi/4} \;\; {}^{ikr(1-\cos(\theta-\theta_o))}\right\} \\ &= -\frac{2}{\pi}\int_0^{2\pi} |\mathcal{A}_o(\theta)|^2 d\theta \\ &-\Re\left\{ \;{}^{-i\pi/4}\left[\mathcal{A}_o(k)r\sqrt{\frac{2}{\pi kre}}\int_0^{2\pi} d\theta[1+\cos(\theta-\theta_o)] \;\; {}^{ikr(1-\cos(\theta-\theta_o))}\right]\right\} \end{aligned}$$

For large $kre$the remaining integral can be found approximately by the method of stationary phase (see Appendix B), with the result

$$\int_0^{2\pi} d\theta[1+\cos(\theta-\theta_o)] \;\; {}^{ikr(1-\cos(\theta-\theta_o))} \sim \sqrt{\frac{2\pi}{kre}} \;\; {}^{i\pi/4} \tag{6.28}$$

We get finally

$$\int_0^{2\pi} |\mathcal{A}|^2 d\theta = -2\Re\mathcal{A}(\theta_o) \tag{6.29}$$

Thus the total scattered energy in all directions is related to the amplitude of the scattered wave in the forward direction. In atomic physics, where this theorem was originated (by Niels Bohr), measurement of the scattering amplitude in all directions is not easy. This theorem suggests an economical alternative.

Homework For the same scatterer, consider two scattering problems ϕ_1 and ϕ_2. Show that

$$\mathcal{A}_1(\theta_2) = \mathcal{A}_2(\theta_1) \tag{6.30}$$

For general elastic waves, see Mei (1978) : Extensions of some identities in elastodynamics with rigid inclusions. *J . Acoust. Soc. Am.* 64(5), 1514-1522.

6.3 Scattering by a circular cavity

Without loss of generality we can take $\theta_o = 0$. On the surface of the cylindrical cavity $re = a$, we impose

$$\frac{\partial \phi_I}{\partial re} + \frac{\partial \phi_S}{\partial re} = 0, \quad r = e a e$$

It follows that $A_n = 0$ and

$$\epsilon^n i^n A J_n'(ka) + B_n k H_n'(ka) = 0, \quad n = e 0, 1, 2, 3, \ldots n\, e$$

where primes denote differentiation with respect to the argument. Hence

$$B_n = -A\epsilon_n i^n \frac{J_n'(ka)}{H_n'(ka)}$$

The sum of incident and scattered waves is

$$\phi = Ae \sum_{n=0}^{\infty} {}_n i^n \left[J_n(kr) - \frac{J_n'(ka)}{H_n'(ka)} H_n(kr) \right] \cos n\theta \tag{6.31}$$

and

$$\psi = Ae^{-i\omega t} \sum_{n=0}^{\infty} {}_n i^n \left[J_n(kr) - \frac{J_n'(ka)}{H_n'(ka)} H_n(kr) \right] \cos n\theta \tag{6.32}$$

The limit of long waves can be approximately analyzed by using the expansions for Bessel functions for small argument

$$J_n(x) \sim \frac{x^n}{2^n n!}, \quad Y_n(x) \sim \frac{2}{\pi} \log x, e\ Y_n(x) \sim \frac{2^n (n-1)!}{\pi x^n} \tag{6.33}$$

Then the scattered wave has the potential

$$\begin{aligned} \frac{\phi_S}{A} &\sim -H_0(kr) \frac{J_0'(ka)}{H_0'(ka)} - 2i H_1(kr) \frac{J_1'(ka)}{H_1'(ka)} \cos\theta + O(ka)^3 \\ &= \frac{\pi}{2}(ka)^2 \left(-\frac{ie}{2} H_0(kr) - H_1(kr)\cos\theta \right) + O(ka)^3 \end{aligned} \tag{6.34}$$

The term $H_0(kr)$ corresponds to a oscillating source which sends isotropic waves in all directions. The second term is a dipole sending scattered waves mostly in forward and backward directions. For large kr, the angular variation is a lot more complex. The far field pattern for various kae is shown in fig 4.

On the cavity surface, the displacement is proportional to $\psi(a, \theta)$ or $\phi(a, \theta)$. The angular variation is plotted for several kae in figure 5.

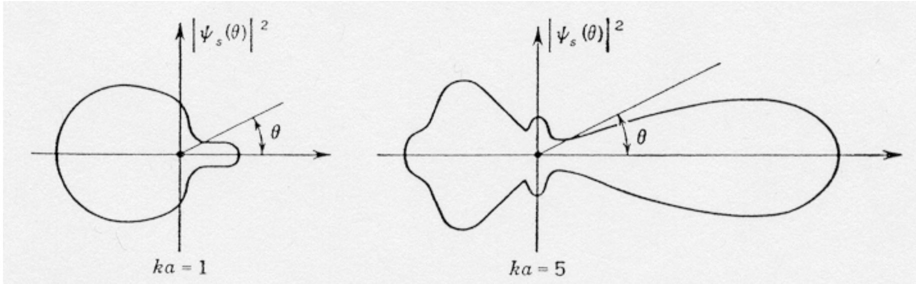

Figure 4: Angular distribution in cylindrical scattering

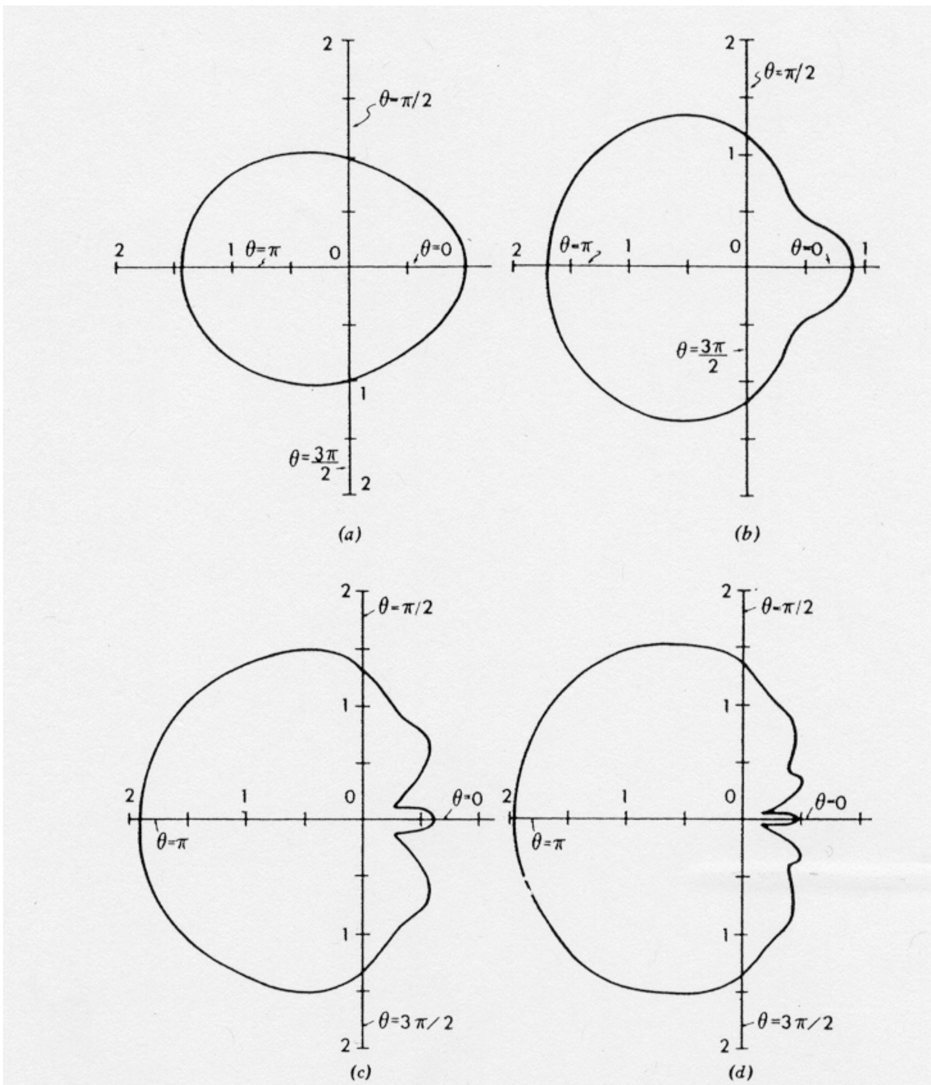

Figure 5: Polar distribution of run-up on a circular cylinder

7 Diffraction of SH wave by a long crack

References

Morse & Ingard, *Theoretical Acoustics* Series expansions.

Born & Wolf, *Principle of Optics* Fourier Transform and the method of steepest descent.

B. Noble. *The Wiener-Hopf Technique.*

If the obstacle is large, there is always a shadow behind where the incident wave cannot penetrate deeply. The phenomenon of scattering by large obstacles is usually referred to as diffraction.

Diffraction of plane incident SH waves by a long crack is identical to that of a hard screen in acoustics. The exact solution was due to A. Sommerfeld. We shall apply the boundary layer idea and give the approximate solution valid far away from the tip $kre\gg 1$ by the *parabolic approximation*, due to V. Fock..

Referring to figure () let us make a crude division of the entire field into the illuminated zone I , dominated by the incident wave alone, the reflection zone II dominated the sum of the incident and the reflected wave, and the shadow zone III where there is no wave. The boundaries of these zones are the rays touching the crack tip. According to this crude picture the solution is

$$\phi = \begin{cases} A_o \exp(ik\cos\theta x + ik\sin\theta y), & Ie \\ A_o[\exp(ik\cos\theta xe+ ik\sin\theta y) + \exp(ik\cos\theta x - ik\sin\theta y)], & II \quad e \\ 0,e & III\,\mathrm{e} \end{cases} \tag{7.1}$$

Clearly (7.1) is inadequate because the potential cannot be discontinuous across the boundaries. A remedy to provide smooth transitions is needed.

Consider the shadow boundary Ox'. Let us introduce a new Cartesian coordinate system so that x' axis is along, while the y' axis is normal to, the shadow boundary. The relations between (x, y) and (x', y') are

$$x' = x\cos\theta + y\sin\theta, \quad y' = y\cos\theta - x\sin\theta \tag{7.2}$$

Thus the incident wave is simply

$$\phi_I = A_o \quad ^{ikx'} \tag{7.3}$$

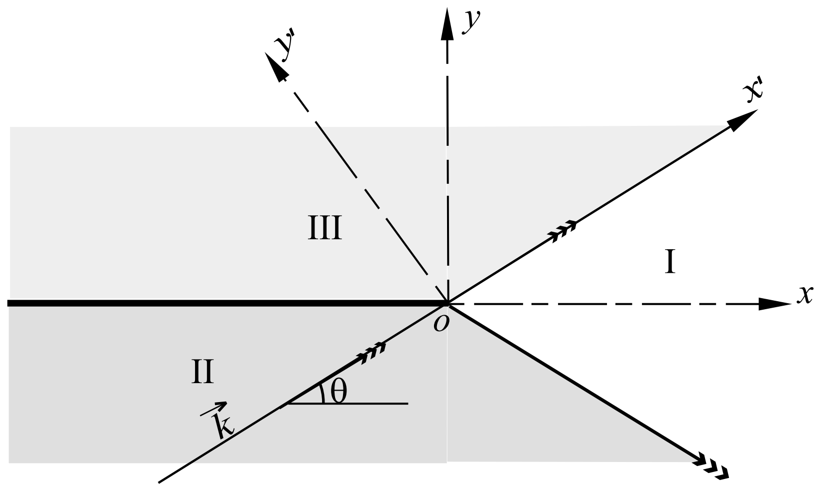

Figure 6: Wave zones near a long crack

Following the chain rule of differentiation,

$$\frac{\partial\phi}{\partial xe} = \frac{\partial\phi}{\partial x'}\frac{\partial x'}{\partial xe} + \frac{\partial\phi}{\partial y'}\frac{\partial y'}{\partial xe} = \cos\theta\frac{\partial\phi}{\partial x'} - \sin\theta\frac{\partial\phi}{\partial y'}$$

$$\frac{\partial\phi}{\partial ye} = \frac{\partial\phi}{\partial x'}\frac{\partial x'}{\partial ye} + \frac{\partial\phi}{\partial y'}\frac{\partial y'}{\partial ye} = \sin\theta\frac{\partial\phi}{\partial x'} + \cos\theta\frac{\partial\phi}{\partial y'}$$

we can show straightforwardly that

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{\partial^2\phi}{\partial x'^2} + \frac{\partial^2\phi}{\partial y'^2}$$

so that the Helmholtz equation is unchanged in form in the x', y' system.

We try to fit a boundary layer along the x' axis and expect the potential to be almost like a plane wave

$$\phi(x,', y') = A(x', y')\ ^{ikx'} \tag{7.4}$$

, but the amplitude is slowly modulated in both x' and y' directions. Substituting (7.4 into the Helmholtz equation, we get

$$^{ikx'}\left\{\frac{\partial^2 Ae}{\partial x'^2} + 2ike\frac{\partial Ae}{\partial x'} - k^2 A + \frac{\partial^2 A}{\partial y'^2} + k^2 A\right\} = 0 \tag{7.5}$$

Expecting that the characteristic scale L_x of A along x' is much longer than a wavelength, $kL_x \gg 1$, we have

$$2ike\frac{\partial Ae}{\partial x'} \gg \frac{\partial^2 A}{\partial x'^2}$$

Hence we get as the first approximation the Schródinger equation[2]

$$2ike\frac{\partial Ae}{\partial x'} + \frac{\partial^2 A}{\partial y'^2} \approx 0 \tag{7.7}$$

In this transition zone where the remaining terms are of comparable importance, hence the length scales must be related by

$$\frac{ke}{x'} \sim \frac{1}{y'^2},e \quad \text{implying} \quad ky' \sim \sqrt{kx'}$$

Thus the transition zone is the interior of a parabola.

Equation (7.7) is of the parabolic type. The boundary conditions are

$$A(x, \infty) = 0 \tag{7.8}$$

$$A(x, -\infty) = A_o \tag{7.9}$$

The initial condition is

$$A(0, y') = \begin{cases} 0, & y' > e0,e \\ A_0, & y' < e0 \end{cases} \tag{7.10}$$

he initial-boundary value for Aehas no intrinsic length scales except x', y' themselves. Therefore the condition $kL_x \gg 1$ means $kx' \gg 1$ i.e., far away from the tip. This problem is somewhat analogous to the problem of one-dimensional heat diffusion across a boundary. A convenient way of solution is the method of similarity.

Assume the solution

$$Ae = A_o f(\gamma) \tag{7.11}$$

where

$$\gamma = \frac{-ky'}{\sqrt{\pi k x'}} \tag{7.12}$$

is the similarity variable. We find upon substitution that fesatisfies the ordinary differential equation

$$f'' - i\pi\gamma f' = 0 \tag{7.13}$$

[2]In one-dimensional quantum mechanics the wave function in a potential-free field is governed by the Schrödinger equation

$$ih\frac{\partial \psi}{\partial t} + \frac{1}{2M}\frac{\partial^2 \psi}{\partial x^2} = 0 \tag{7.6}$$

subject to the boundary conditions that

$$fe\to 0,e\ \gamma\to-\infty;\quad fe\to 1,e\ \gamma\to\infty.e \tag{7.14}$$

Rewriting (7.13) as

$$\frac{f''}{f'}=i\pi\gamma$$

we get

$$\log f' = i\pi\gamma/2+\text{constant}.e$$

One more integration gives

$$fe= Ce\int_{-\infty}^{\gamma}\exp\left(\frac{i\pi u^2}{2}\right)due$$

Since

$$\int_0^{\infty}\exp\left(\frac{i\pi u^2}{2}\right)due=\frac{{}^{i\pi/4}}{\sqrt{2}}$$

we get

$$Ce=\frac{{}^{-i\pi/4}}{\sqrt{2}}$$

and

$$fe=\frac{A}{A_o}=\frac{{}^{-i\pi/4}}{\sqrt{2}}\int_{-\infty}^{\gamma}\exp\left(\frac{i\pi u^2}{2}\right)due=\frac{{}^{-i\pi/4}}{\sqrt{2}}\left\{\frac{{}^{i\pi/4}}{\sqrt{2}}+\int_0^{\gamma}\exp\left(\frac{i\pi u^2}{2}\right)du\right\} \tag{7.15}$$

Defining the cosine and sine Fresnel integrals by

$$C(\gamma)=\int_0^{\gamma}\cos\left(\frac{\pi v^2}{2}\right)dv,e\quad S(\gamma)=\int_0^{\gamma}\sin\left(\frac{\pi v^2}{2}\right)dve \tag{7.16}$$

we can then write

$$\frac{{}^{-i\pi/4}}{\sqrt{2}}\left\{\left[\frac{1}{2}+C(\gamma)\right]+ie\left[\frac{1}{2}+S(\gamma)\right]\right\} \tag{7.17}$$

In the complex plane the plot of $C(\gamma)+iS(\gamma)$ vs. γ is the famous Cornu's spiral, shown in figure (**??**).

The wave intensity is given by

$$\frac{|A|^2}{A_o^2}=\frac{1}{2}\left\{\left[\frac{1}{2}+C(\gamma)\right]^2+\left[\frac{1}{2}+S(\gamma)\right]^2\right\} \tag{7.18}$$

Since $C, Se\to 0$ as $\gamma\to\infty$, the wave intensity diminishes to zero gradually into the shadow. However, $C, Se\to 1/2$ as $\gamma\to\infty$ in an oscillatory manner. The wave intensity oscillates while approaching to unity asymptotically. In optics this shows up as alternately light and dark diffraction bands.

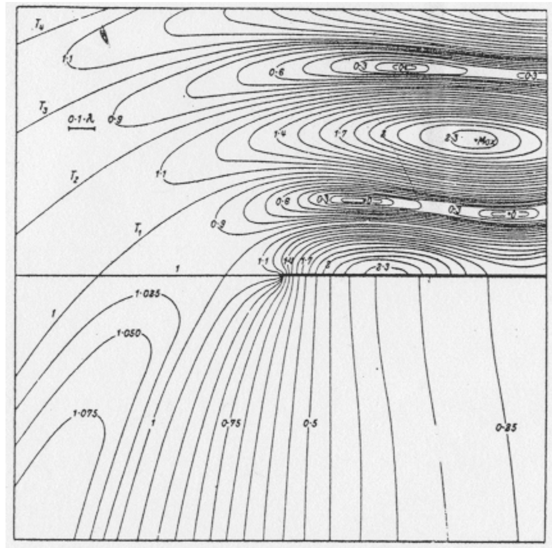

Figure 7: Amplitude contours of H_z

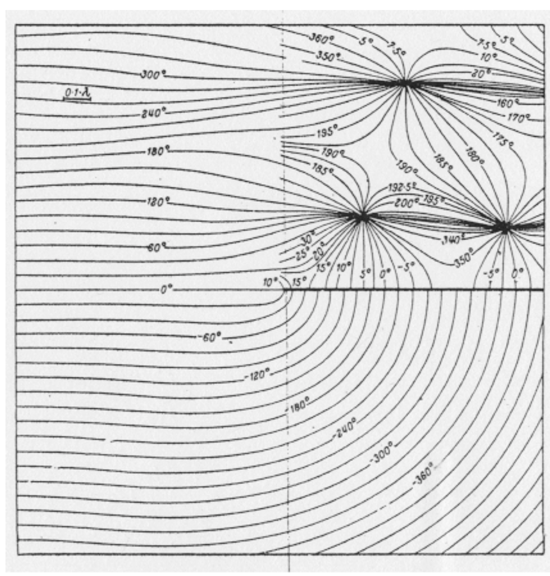

Figure 8: Phase contours of H_z

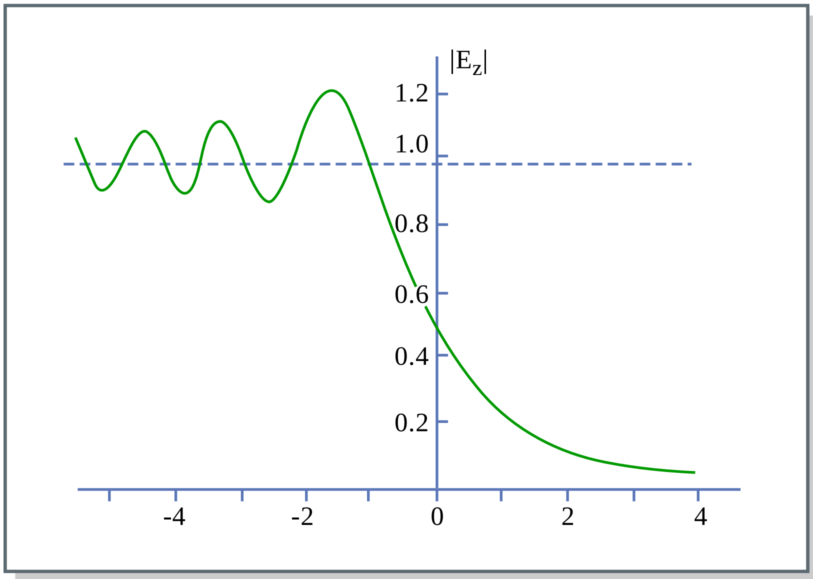

Figure 9: Diffraction of a normally incident E-polarized plane wave

Figure by MIT OpenCourseWare.

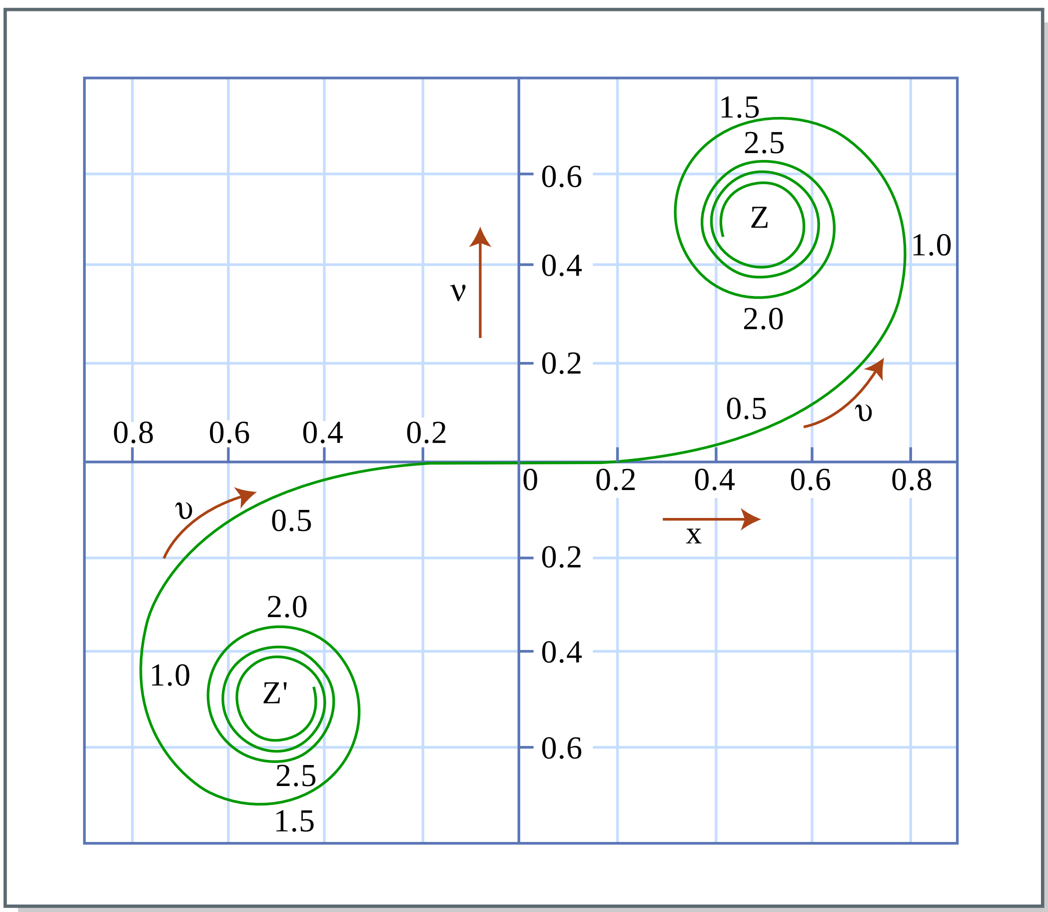

Figure 10: Cornu's spiral, a plot of the Fresnel integrals

Figure by MIT OpenCourseWare.

In more complex propagation problems, the parabolic approximation can simplify the numerical task in that an elliptic boundary value problem involving an infinite domain is reduced to an initial boundary value problem. One can use Crank-Nicholson scheme to march in "time", i.e., x'.

Homework Find by the parabolic approximation the transition solution along the edge of the reflection zone.

8 Rayleigh surface waves

Refs. Graff, Achenbach, Fung

In a homogeneous elastic half plane, in addition to P, SV and SH waves, another wave which is trapped along the surface of a half plane can also be present. Because most of the action is near the surface, this *surface wave* is of special importance to seismic effects on the ground surface.

Let us start from the governing equations again

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{1}{c_L^2}\frac{\partial^2 \phi}{\partial t^2},e \tag{8.1}$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} = \frac{1}{c_T^2}\frac{\partial^2 H_z}{\partial t^2} \tag{8.2}$$

We now seek waves propagating along the x direction

$$\phi = \Re\left(f(y)\ ^{i\xi x - i\omega t}\right), \quad H_z = \Re\left(h(y)\ ^{i\xi x - i\omega t}\right) \tag{8.3}$$

Then $f(y), h(y)$ must satisfy

$$\frac{d^2 fe}{dy^2} + \left(\omega^2/c_L^2 - \xi^2\right) fe = 0,e \qquad \frac{d^2 he}{dy^2} + \left(\omega^2/c_T^2 - \xi^2\right) he = 0,e \tag{8.4}$$

To have surface waves we insist that

$$\overline{\alpha} = \sqrt{\xi^2 - \omega^2/c_L^2},e \quad \overline{\beta} = \sqrt{\xi^2 - \omega^2/c_T^2} \tag{8.5}$$

be real and positive. Keeping only the solutions which are bounded for $ye\sim \infty$, we get

$$\phi = Ae^{-\overline{\alpha}y\ \ i(\xi x - \omega t)}, \quad H_z = Be^{-\overline{\beta}y\ \ i(\xi x - \omega t)}. \tag{8.6}$$

The expressions for the displacements and stresses can be found straightforwardly.

$$u_x = \left(i\xi Ae^{-\overline{\alpha}y} - \overline{\beta}Be^{-\overline{\beta}y}\right)^{\ i(\xi x-\omega t)},e \tag{8.7}$$

$$u_y = -\left(\overline{\alpha}Ae^{-\overline{\alpha}y} + i\xi Be^{-\overline{\beta}y}\right)^{\ i(\xi x-\omega t)},e \tag{8.8}$$

$$\tau_{xx} = \mu e\left\{\left(\overline{\beta}^2 - \xi^2 - 2\overline{\alpha}^2\right) Ae^{-\overline{\alpha}y} - 2i\overline{\beta}\xi Be^{-\overline{\beta}y}\right\}^{\ i(\xi x-\omega t)},e \tag{8.9}$$

$$\tau_{yy} = \mu e\left\{\left(\overline{\beta}^2 + \xi^2\right) Ae^{-\overline{\alpha}y} + 2i\overline{\beta}\xi Be^{-\overline{\beta}y}\right\}^{\ i(\xi x-\omega t)},e \tag{8.10}$$

$$\tau_{xy} = \mu e\left\{-2i\overline{\alpha}\xi Ae^{-\overline{\alpha}y} + \left(\xi^2 + \overline{\beta}^2\right) Be^{-\overline{\beta}y}\right\}^{\ i(\xi x-\omega t)} \tag{8.11}$$

On the free surface the traction-free conditions $\tau_{yy} = \tau_{xy} = 0$ require that

$$\left(\overline{\beta}^2 + \xi^2\right) Ae + 2i\overline{\beta}\xi Be = 0,e \tag{8.12}$$

$$-2i\overline{\alpha}\xi Ae + \left(\overline{\beta}^2 + \xi^2\right) Be = 0.e \tag{8.13}$$

For nontrivial solutions of $A, Be$the coefficient determinant must vanish,

$$\left(\overline{\beta}^2 + \xi^2\right)^2 - 4\overline{\alpha}\overline{\beta}\xi^2 = 0,e \tag{8.14}$$

or

$$\left[2\xi^2 - \frac{\omega^2}{c_T^2}\right]^2 - 4\xi^2\sqrt{\xi^2 - \frac{\omega^2}{c_L^2}}\sqrt{\xi^2 - \frac{\omega^2}{c_T^2}} = 0 \tag{8.15}$$

which is the dispersion relation between frequency ω and wavenumber ξ. From either (8.12) or (8.13) we get the amplitude ratio:

$$\frac{Ae}{Be} = -\frac{2i\overline{\beta}\xi}{\overline{\beta}^2 + \xi^2} = \frac{\overline{\beta}^2 + \xi^2}{2i\overline{\alpha}\xi},e \tag{8.16}$$

In terms of the wave velocity $ce= \omega/\xi$, (8.15) becomes

$$\left(2 - \frac{c^2}{c_T^2}\right)^2 = 4\left(1 - \frac{c^2}{c_L^2}\right)^{\frac{1}{2}}\left(1 - \frac{c^2}{c_T^2}\right)^{\frac{1}{2}}.e \tag{8.17}$$

or, upon squaring both sides finally

$$\frac{c^2}{c_2^2}\left\{\left(\frac{c}{c_2}\right)^6 - 8\left(\frac{ce}{c_2}\right)^4 + \left(24 - \frac{16}{\kappa^2}\right)\left(\frac{ce}{c_2}\right)^2 - 16\left(1 - \frac{1}{\kappa^2}\right)\right\} = 0.e \tag{8.18}$$

The first solution $ce= \omega = 0$ is at best a static problem. In fact $\overline{\alpha} = \overline{\beta} = \xi$ and $Ae= -iB$, so that $u_x = u_y \equiv 0$ which is of no interest.

We need only consider the cubic equation for c^2. Note that the roots of the cubic equation depend only on Poisson's ratio, through $\kappa^2 = 2(1-\nu)/(1-2\nu)$. There can be three real roots for c or ω, or one real root and two complex-conjugate roots. We rule out the latter because the complex roots imply either temporal damping or instability; neither of which is a propagating wave. When all three roots are real we must pick the one so that both $\overline{\alpha}$ and $\overline{\beta}$ are real.

For $c = 0$, the factor in curly brackets is

$$\{.\} = -16\left(1 - \frac{c_T^2}{c_L^2}\right) < 0$$

For $c = c_T$ the same factor is equal to unity and hence positive. There must be a solution for c such that $0 < c < c_T$. Furthermore, we cannot have roots in the range $c/c_T > 1$. If so,

$$\overline{\beta}^2 = \xi^2\left(1 - \frac{c^2}{c_T^2}\right) < 0$$

which is not a surface wave. Thus the surface wave, if it exists, is slower than the shear wave.

Numerical studies for the entire range of Poisson's ratio $(0 < \nu < 0.5)$ have shown that there are one real and two complex conjugate roots if $\nu > 0.263\ldots$ and three real roots if $\nu < 0.263\ldots$. But there is only one real root that gives the surface wave velocity c_R. A graph of c_R for all values of Poisson's ratio, due to Knopoff , is shown in Fig. **??**. A curve-fitted expression for the Rayleigh wave velocity is

$$c_R/c_2 = (0\cdot 87 + 1\cdot 12\nu)/(1+\nu).e \tag{8.19}$$

For rocks, $\lambda = \mu$ and $\nu = \frac{1}{4}$, the roots are

$$(c/c_2)^2 = 4, 2 + 2/\sqrt{3}, 2 - 2/\sqrt{3}.e \tag{8.20}$$

The only acceptable root for Rayleigh wave speed c_R is

$$(c_R/c_T)^2 = (2 - 2/\sqrt{3})^{\frac{1}{2}} = 0\cdot 9194 \tag{8.21}$$

or

$$c_R = 0\cdot 9194 c_T.e \tag{8.22}$$

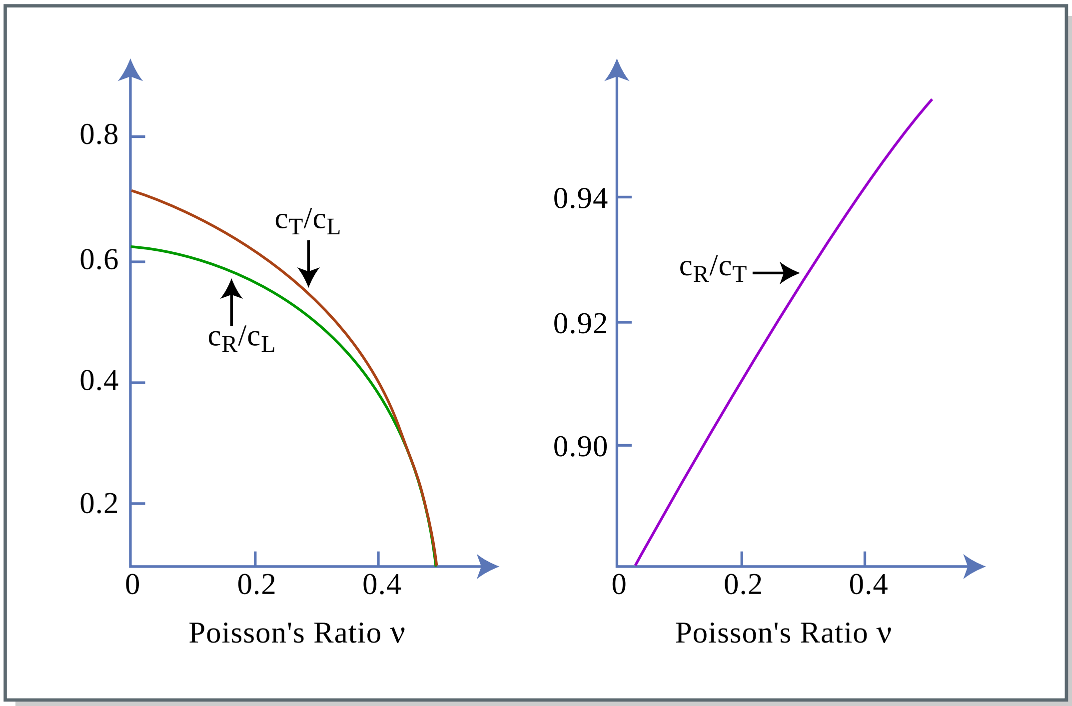

Figure 11: The velocity of Rayleigh surface waves c_R
Figure by MIT OpenCourseWare.

The particle displacement of a particle on the free surface is, from (??) and (??)

$$u_x = iA\left(\xi - \frac{\overline{\beta}^2+\xi^2}{2\xi}\right) e^{i(\xi x-\omega t)} \tag{8.23}$$

$$u_y = A\left(-\overline{\alpha} + \frac{\overline{\beta}^2+\xi^2}{2\xi}\right) e^{i(\xi x-\omega t)} \tag{8.24}$$

Note that

$$a = A\left[\xi - \frac{\overline{\beta}^2+\xi^2}{2\xi}\right] = A\left[\xi + \frac{k_T^2}{2\xi}\right] > 0$$

$$b = A\left[-\overline{\alpha} + \frac{\overline{\beta}^2+\xi^2}{2\xi}\right] = A\left[\frac{(\overline{\alpha}-\overline{\beta})^2+k_L^2}{2\overline{\beta}}\right] > 0$$

hence

$$u_x = a\sin(\omega t - \xi x), \quad u_y = b\cos(\omega t - \xi x)$$

and

$$\frac{u_x^2}{a^2} + \frac{u_y^2}{b^2} = 1 \tag{8.25}$$

The particle trajectory is an ellipse. In complex form we have

$$\frac{u_x}{a} + i\frac{u_y}{b} = \exp\{i(\omega t - \xi x - \pi/2)\} \tag{8.26}$$

Hence as t increases, a particle at $(x, 0)$ traces the ellipse in the counter-clockwise direction. See figure (8).

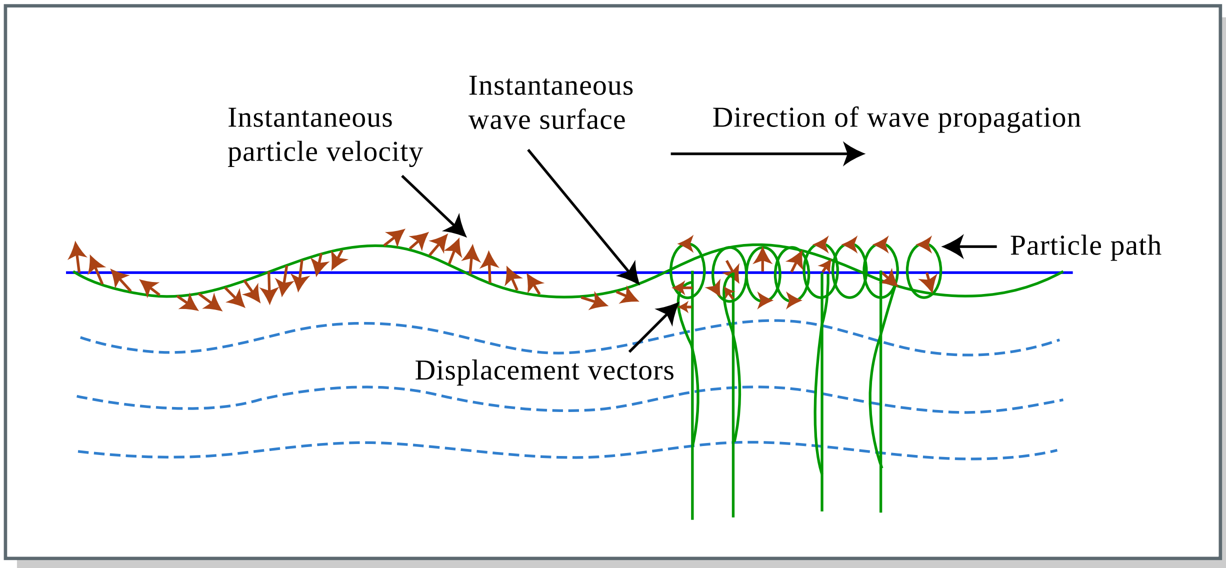

Figure 12: Displacement of particles on the ground surface in Rayleigh surface wave
Figure by MIT OpenCourseWare.

9 Elastic waves due to a load traveling on the ground surface

Refs: Fung: *Foundations of Solid Mechanics*

Cole and Huth: (1956, Elastic half space ; *J Appl Mech*25, 433-436.)

Mei, Si & Chen , (1985, Poro-elastic half space, *Wave Motion*, 7, 129-141.).

In this section the ye**axis is positive if pointing upwards.**

Let the traction on the ground surface be :

$$\tau_{yy} = -P(x+Ut).e \quad \tau_{xy} = 0,e \text{ on } \; ye = 0 \tag{9.1}$$

Let us make a (Galilean) transformation to a coordinate system moving to the left at the speed of U, so that the load appears stationary, Then, by the chain rule, derivatives are changed according

$$\frac{\partial}{\partial xe} \to \frac{\partial}{\partial xe}, e\frac{\partial}{\partial ye} \to \frac{\partial}{\partial ye}, e \; \frac{\partial}{\partial te} \to \frac{\partial}{\partial te} + U\frac{\partial}{\partial xe} \tag{9.2}$$

In the moving coordinates, the wave equations are changed to

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = \frac{1}{c_L^2}\left(\frac{\partial}{\partial te} + Ue\frac{\partial}{\partial xe}\right)^2 \Phi, \text{ e} \tag{9.3}$$

$$\frac{\partial^2 He}{\partial x^2} + \frac{\partial^2 He}{\partial y^2} = \frac{1}{c_T^2}\left(\frac{\partial}{\partial te} + U\frac{\partial}{\partial xe}\right)^2 H \text{ e} \tag{9.4}$$

where we have abbreviated H_z simply by H.

In the steady state limit they become

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = \frac{U^2}{c_L^2} \frac{\partial^2 \Phi}{\partial x^2} \tag{9.5}$$

$$\frac{\partial^2 He}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = \frac{U^2}{c_T^2} \frac{\partial^2 He}{\partial x^2} \tag{9.6}$$

Introducing the Mach numbers:

$$M_1 = \frac{U}{c_L}, \quad M_2 = \frac{U}{c_T} \tag{9.7}$$

then (9.5) amd (9.6) become

$$\left(1 - M_1^2\right) \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \tag{9.8}$$

$$\left(1 - M_2^2\right) \frac{\partial^2 He}{\partial x^2} + \frac{\partial^2 He}{\partial y^2} = 0 \tag{9.9}$$

The stress components can be derived straightforwardly in terms of the potentials

$$\begin{aligned}
\tau_{xx} &= (\lambda + 2\mu)\nabla^2 \Phi - 2\mu e\left(\frac{\partial^2 \Phi}{\partial y^2} - \frac{\partial^2 He}{\partial x \partial ye}\right) \\
&= (\lambda + 2\mu)\frac{U^2}{c_L^2} \frac{\partial^2 \Phi}{\partial x^2} - 2\mu(M_1^2 - 1)\frac{\partial^2 \Phi}{\partial x^2} + 2\mu e\frac{\partial^2 He}{\partial x \partial ye}
\end{aligned}$$

Using the fact that $\lambda + 2\mu)/\mu = c_L^2/c_T^2$, we further get

$$\tau_{xx} = \mu \left[\left(M_2^2 - 2M_1^2 + 2\right) \frac{\partial^2 \Phi}{\partial x^2} + 2\frac{\partial^2 He}{\partial x \partial ye}\right] \tag{9.10}$$

Similarly we find

$$\tau_{yy} = \mu \left[(M_2^2 - 2)\frac{\partial^2 \Phi}{\partial x^2} - 2\frac{\partial^2 He}{\partial x \partial ye}\right] \tag{9.11}$$

$$\tau_{xy} = \mu \left[2\frac{\partial^2 \Phi}{\partial x \partial ye} + (M_2^2 - 2)\frac{\partial^2 He}{\partial x^2}\right] \tag{9.12}$$

We now examine the special pressure distribution, as shown in figure (9):

$$p(x, 0) = P_o \mathcal{P}(x) \equiv P_0 \begin{cases} ^{-\alpha_1 x}, & x > 0,e \\ ^{\alpha_2 x}, & x < 0 \end{cases} \tag{9.13}$$

Thus the traction boundary conditions (9.3) on the ground surface become

$$(M_2^2 - 2)\frac{\partial^2 \Phi}{\partial x^2} - 2\frac{\partial^2 H}{\partial x \partial y} = -\frac{P_o}{\mu e}\mathcal{P}(x), \quad y = 0; \tag{9.14}$$

$$2\frac{\partial^2 \Phi}{\partial x \partial ye} + (M_2^2 - 2)\frac{\partial^2 He}{\partial x^2} = 0, \quad y = 0.e \tag{9.15}$$

Three cases will be distinguished:

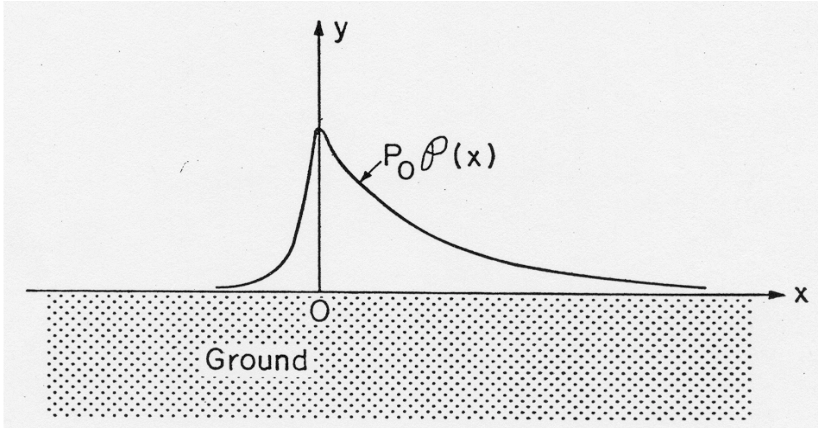

Figure 13: A moving pressure distribution on an elastic half space. Shown in a moving coordinate system, the pressure appears stationary.

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9.1 Supersonic: $M_1 > M_2 > e1$

Let us apply the exponential Fourier transform defined by

$$f(\lambda) = \int_{-\infty}^{\infty} F(x)\ {}^{-i\lambda x} dx, e$$

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\lambda)\ {}^{i\lambda x} d\lambda. e \tag{9.16}$$

From the governing wave equations we get

$$\frac{d^2\phi}{\partial y^2} + \bar{\beta}_1^2 \lambda^2 \phi = 0, e\ \frac{d^2 h}{\partial y^2} + \bar{\beta}_2^2 \lambda^2 h e= 0; \quad y < 0. e \tag{9.17}$$

where

$$\bar{\beta}_j^2 = M_j^2 - 1, \quad j = e1, 2. e \tag{9.18}$$

The general solutions of the Fourier Transforms are

$$\phi = A(\lambda)\ {}^{i\bar{\beta}_1 y} + B(\lambda)\ {}^{-i\bar{\beta}_1 y},$$

$$h e= C(\lambda)\ {}^{i\bar{\beta}_2 y} + D(\lambda)\ {}^{i\bar{\beta}_2 y}$$

so that

$$\Phi(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[A(\lambda)\ {}^{i\lambda(x+\bar{\beta}_1 y)} + B(\lambda)\ {}^{i\lambda(x-\bar{\beta}_1 y)} \right] d\lambda. e \tag{9.19}$$

$$H(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[C(\lambda)\ {}^{i\lambda(x+\bar{\beta}_2 y)} + D(\lambda)\ {}^{i\lambda(x-\bar{\beta}_2 y)} \right] d\lambda. e \tag{9.20}$$

In order that waves below the ground surface trail behind the surface load, we discard the second term in each integral. Thus

$$\phi = A(\lambda)\ \ {}^{i\bar{\beta}_1 y}, \quad h = eC(\lambda)\ \ {}^{i\bar{\beta}_2 y} \tag{9.21}$$

Now the boundary conditions require

$$2i\lambda\frac{d\phi}{dye} - \lambda^2\bar{\beta}_2^2 he = 0 \tag{9.22}$$

and

$$-\lambda^2\bar{\beta}_2^2\phi - 2i\lambda\frac{dhe}{dye} = \frac{iP_o}{\mu e}\left(\frac{1}{\lambda - i\alpha_1} - \frac{1}{\lambda + i\alpha_2}\right) \tag{9.23}$$

Use has been made of the result

$$\begin{aligned}\int_{-\infty}^{\infty}\ {}^{-i\lambda x}\mathcal{P}(x)dxe &= \int_{-\infty}^{0}\ {}^{-i\lambda x}\ \ {}^{\alpha_2 x}dxe + \int_{0}^{\infty}\ {}^{-i\lambda x}\ \ {}^{-\alpha_1 x}dxe \\ &= \frac{1}{ie}\left(\frac{1}{\lambda - i\alpha_1} - \frac{1}{\lambda + i\alpha_2}\right)\end{aligned} \tag{9.24}$$

It follows that

$$-2\lambda^2\bar{\beta}_1 Ae - \bar{\beta}_2^2\lambda^2 Ce = 0 \tag{9.25}$$

and

$$-\beta_2^2\lambda^2 Ae + 2\lambda^2\bar{\beta}_2 Ce = \frac{iP_o}{\mu e}\left(\frac{1}{\lambda - i\alpha_1} - \frac{1}{\lambda + i\alpha_2}\right) \tag{9.26}$$

The last two equations can be solved to give

$$Ae = \frac{iP_o}{\mu e}\left(\frac{1}{\lambda - i\alpha_1} + \frac{1}{\lambda + i\alpha_2}\right)\frac{-\bar{\beta}_2^2}{\lambda^2(\bar{\beta}_2^4 + 4\bar{\beta}_1\bar{\beta}_2)} \tag{9.27}$$

$$Ce == \frac{iP_o}{\mu e}\left(\frac{1}{\lambda - i\alpha_1} + \frac{1}{\lambda + i\alpha_2}\right)\frac{2\bar{\beta}_1}{\lambda^2(\bar{\beta}_2^4 + 4\bar{\beta}_1\bar{\beta}_2)} \tag{9.28}$$

The inverse transforms of ϕ and he are:

$$\begin{aligned}\begin{bmatrix}\Phi \\ -H\end{bmatrix} &= \frac{iP_0}{2\pi\mu e}\begin{bmatrix}k_1 \\ k_2\end{bmatrix}\int_{-\infty}^{\infty}\left(\frac{1}{\lambda - i\alpha_1} - \frac{1}{\lambda + i\alpha_2}\right) \\ &\cdot\left[\exp i\lambda\left(xe + \begin{bmatrix}\bar{\beta}_1 \\ \bar{\beta}_2\end{bmatrix} y\right)\right]\frac{d\lambda}{\lambda^2}\end{aligned} \tag{9.29}$$

where

$$k_1 = \frac{-(M_2^2-2)}{(M_1^2-2)^2+4\overline{\beta}_1\overline{\beta}_2}, e$$
$$k_2 = \frac{-2\overline{\beta}_1}{(M_2^2-2)^2+4\overline{\beta}_1\overline{\beta}_2}. e \tag{9.30}$$

Using (9.10), (9.11) and (9.12), we get the stress components. For example

$$\frac{\tau_{xy}}{P_o} = \frac{2\bar{\beta}_1 k_1}{2\pi ie}\int_{-\infty}^{\infty}\left(\frac{1}{\lambda-i\alpha_1}-\frac{1}{\lambda+i\alpha_2}\right)^{i\lambda\xi_1} d\lambda \tag{9.31}$$
$$- \frac{(M_2^2-2)k_2}{2\pi ie}\int_{-\infty}^{\infty}\left(\frac{1}{\lambda-i\alpha_1}-\frac{1}{\lambda+i\alpha_2}\right)^{i\lambda\xi_2} d\lambda \tag{9.32}$$

where

$$\xi_1 = x+\overline{\beta}_1|y|, e \qquad \xi_2 = x+\overline{\beta}_2|y| \tag{9.33}$$

In view of (9.24), the inverse transform is immediate,

$$\frac{\tau_{xy}}{P_o} = 2\bar{\beta}_1 k_1 \mathcal{P}(\xi_1) - (M_2^2-2)k_2\mathcal{P}(\xi_2) \tag{9.34}$$

As a check, the shear stress on the ground surface $y = 0$ is

$$\frac{\tau_{xy}}{P_o} = \left[2\bar{\beta}_1 k_1 - (M_2^2-2)k_2\right]\mathcal{P}(x) = 0 \tag{9.35}$$

in view of (9.30).

It can be shown that

$$\tilde{\tau}_{xx} = \frac{\tau_{xx}^0}{P_0} = \left(M_2^2 - M_1^2 + 2\right)k_1\mathcal{P}(\xi_1) - 2\overline{\beta}_2 k_2 \mathcal{P}(\xi_2), e$$
$$\tilde{\tau}_{yy} = \text{(to be worked out)} \tag{9.36}$$

Note that the disturbances in the half space indeed trail behind the surface pressure. The P and SV waves are concentrated respectively along the characteristics $x+\overline{\beta}_1|y| =$ constant and $x+\overline{\beta}_2|y|$ =constant.

Homework: Verify the above results by the method of characteristics.

9.2 Subsonic case, $1 > M_1 > M_2$

In this case (9.8) and (9.9) are elliptic. Let

$$\beta_1^2 = 1 - M_1^2, e \qquad \beta_2 = 1 - M_2^2,$$

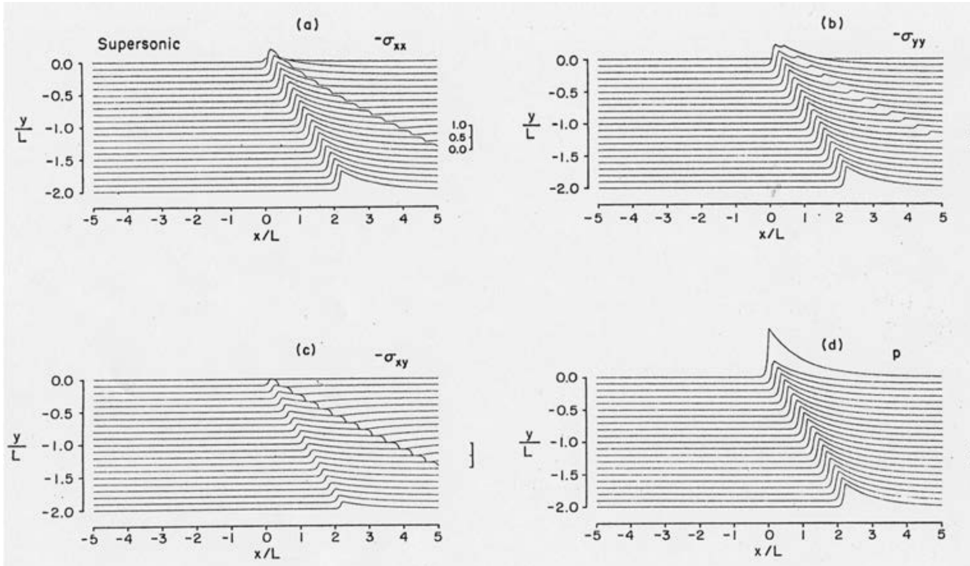

Figure 14: Stress variations in the ground under supersonic load on the surface.

$$
\begin{aligned}
k_3 &= \frac{-M_2^2+2}{(M_2^2-2)^2 - 4\beta_1\beta_2}, e \\
k_4 &= \frac{-2\beta_1}{(M_2^2-2)^2 - 4\beta_1\beta_2}. e
\end{aligned} \tag{9.37}
$$

The formal solutions for Φ and Heare

$$
\begin{aligned}
\Phi &= \frac{iP_0k_3}{2\pi\mu e}\int_{-\infty}^{\infty}\left(\frac{1}{\lambda - i\alpha_1} - \frac{1}{\lambda + i\alpha_2}\right) \\
&\quad \times {}^{|\lambda|\beta_1 y}\ {}^{i\lambda x}\frac{d\lambda}{\lambda^2}, e \\
-He &= \frac{P_0k_4}{2\pi\mu e}\int_{-\infty}^{\infty}\left(\frac{1}{\lambda - i\alpha_1} - \frac{1}{\lambda + i\alpha_2}\right) \\
&\quad \times {}^{|\lambda|\beta_2 y}\ {}^{i\lambda x}\frac{\operatorname{sgn}\lambda}{\lambda^2}d\lambda.\ \mathrm{e}
\end{aligned} \tag{9.38}
$$

By using (9.10), (9.11) and (9.12), the stress components can be expressed as Fourier integrals, which can be evaluated in terms of the exponential integral defined by

$$
\begin{aligned}
E_1(z) &= \int_x^{\infty} \frac{{}^{-\tau}}{\tau} d\tau \\
&= -\gamma - \ln z e - \sum_{n=1}^{\infty}(-1)^n \frac{z^n}{nn!}.\ \mathrm{e}
\end{aligned} \tag{9.39}
$$

Let

$$z_1 - x + i\beta_1 y,e \qquad z_2 = x + i\beta_2 ye \tag{9.40}$$

and

$$G(z) = \ {}^{-\alpha_1 z} E_1(-\alpha_1 z) - \ {}^{\alpha_2 z} E_1(\alpha_2 z).e \tag{9.41}$$

Then the stress components are

$$\begin{aligned}
\tilde{\tau}_{xx} = \frac{\tau^o_{xx}}{P_0} &= -\left(M_2^2 - 2M_1^2 + 2\right)\frac{k_3}{\pi}\Im G(z_1) - \frac{2\beta_2 k_4}{\pi}\Im G(z_2), \\
\tilde{\tau}_{yy} = \frac{\tau^o_{yy}}{P_0} &= -\left(M_2^2 - 2\right)\frac{k_3}{\pi}\Im G(z_1) + \frac{2\beta_2 k_4}{\pi}\Im G(z_2) \\
\tilde{\tau}_{xy} = \frac{\tau^o_{xy}}{P_0} &= \frac{2\beta_1 k_3}{\pi}\Re\left[G(z_2) - G(z_1)\right]
\end{aligned} \tag{9.42}$$

Note that

$$\lim_{y\uparrow 0^-} E_1(-\alpha_1 z_1) = \begin{cases} E_1(\alpha|x|) & \text{if } x < 0,e \\ -Ei(\alpha x) - i\pi & \text{if } x > 0, \text{e} \end{cases}$$

$$\lim_{y\uparrow 0^-} E_1(\alpha z_1) = \begin{cases} -Ei(\alpha|x|) + \pi ie & \text{if } x < 0, \text{e} \\ E_1(\alpha x) & \text{if } x > 0 \end{cases} \tag{9.43}$$

where

$$Ei(x) = -\text{PV}\int_x^\infty \frac{{}^{-\tau}}{\tau} d\tau \tag{9.44}$$

with the integral being a principal value.

From the definitions k_3 and k_4 become infinite when their denominators vanish; This occurs when the external load travels at the speed of the Rayleigh surface wave and indicates resonance. The unbounded resonance need not be a threat in practice because the model of steady two-dimensional line load is an idealization not usually realized.

9.3 Transonic case, $M_1 > 1 > M_2$

The scalar potentials are

$$\begin{aligned}
\Phi &= \frac{P_0}{2\pi\mu e}\int_{-\infty}^{\infty} d\lambda \ {}^{i\lambda x} A(\lambda) \ {}^{|\lambda|\beta_1 y}, \\
-He &= \frac{P_0}{2\pi\mu e}\int_{-\infty}^{\infty} d\lambda \ {}^{i\lambda x} B(\lambda) \ {}^{i\lambda\overline{\beta}_2 y}
\end{aligned} \tag{9.45}$$

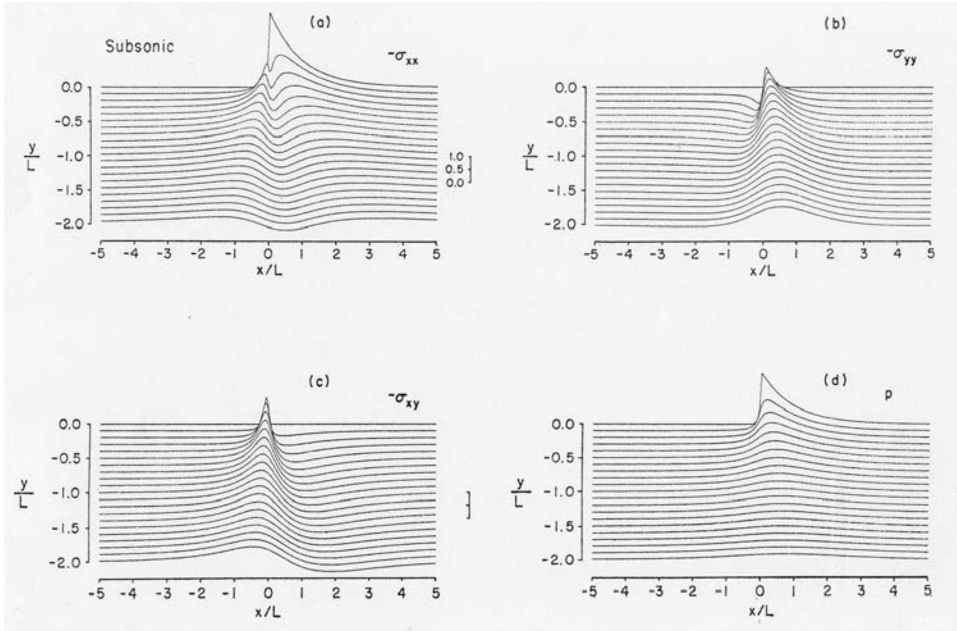

Figure 15: Stress variations in the ground under subsonic load on the surface.

where

$$\begin{aligned} A(\lambda) &= -\left(M_2^2-2\right)\left(k_5+\frac{i|\lambda|}{\lambda}k_6\right)\frac{ie}{\lambda^2}\left(\frac{1}{\lambda-i\alpha_1}-\frac{1}{\lambda+i\alpha_2}\right),e \\ B(\alpha) &= 2i\beta_1\left(\frac{|\lambda|}{\lambda}k_5+ik_6\right)\frac{ie}{\lambda^2}\left(\frac{1}{\lambda-i\alpha_1}-\frac{1}{\lambda+i\alpha_2}\right),e \end{aligned} \tag{9.46}$$

$$k_5 = \frac{-\left(M_2^2-2\right)^2}{\left(M_2^2-2\right)^4+16\beta_1^2\overline{\beta}_2^2},e$$

$$k_6 = \frac{-4\beta_1\overline{\beta}_2}{\left(M_2^2-2\right)^4+16\beta_1^2\overline{\beta}_2^2}.e$$

In terms of $z_1 = x + i\beta_1 y$and $\xi_2 = x + \overline{\beta}|y|$ all the integrals in (3.18) can again be evaluated. The results involve the following functions:

$$\begin{aligned} H(\xi) &= {}^{\alpha_2\xi}E_1\left(\alpha_2\xi\right) + {}^{-\alpha_1\xi}Ei\left(\alpha_1\xi\right),e \\ H\overset{*}{e}(\xi) &= {}^{\alpha_1|\xi|}E_1\left(\alpha_1|\xi|\right) + {}^{-\alpha_2|\xi|}Ei\left(\alpha_2|\xi|\right) \\ &\equiv H(|\xi|).e \end{aligned} \tag{9.47}$$

The stresses are

$$\begin{aligned}\tilde{\tau}_{xx} &= \frac{\tau^0_{xx}}{P_0} = -\left(M_2^2 - 2M_1^2 + 2\right)\left(M_2^2 - 2\right)\frac{1}{\pi}\Im\left\{\left(k_5 - ik_6\right)G\left(z_1\right)\right\} \\ &- \frac{4\beta_1\overline{\beta}_2}{\pi}\begin{cases} k_5 H\left(\xi_2\right) + \pi k_6 e^{-\alpha_1\xi_2}, & \xi_2 > 0, \\ -k_5 H^*\left(\xi_2\right) + \pi k_6 e^{-\alpha_2|\xi_2|}, & \xi_2 < 0, \end{cases}\end{aligned} \tag{9.48}$$

$$\begin{aligned}\tilde{\tau}_{yy} &= \frac{\tau^0_{yy}}{P_0} = -\left(M_2^2 - 2\right)^2\frac{1}{\pi}\Im\left\{\left(k_5 - ik_6\right)G\left(z_1\right)\right\} \\ &+ \frac{4\beta_1\overline{\beta}_2}{\pi}\begin{cases} k_5 H\left(\xi_2\right) + \pi k_6 e^{-\alpha_1\xi_2}, & \xi_2 > 0, \\ -k_5 H^*\left(\xi_2\right) + \pi k_6 e^{-\alpha_2|\xi_2|}, & \xi_2 < 0 \end{cases}\end{aligned} \tag{9.49}$$

$$\begin{aligned}\tilde{\tau}_{xy} &= \frac{\tau^0_{xy}}{P_0} = -\left(M_2^2 - 2\right)\frac{2\beta_1}{\pi}\Re\left\{\left(k_5 - ik_6\right)G\left(z_1\right)\right\} \\ &- \left(M_2^2 - 2\right)\frac{2\beta_1}{\pi}\begin{cases} k_5 H\left(\xi_2\right) + \pi k_6 e^{-\alpha_1\xi_2}, & \xi_2 > 0, \\ -k_5 H^*\left(\xi_2\right) + \pi k_6 e^{-\alpha_2|\xi_2|}, & \xi_2 < 0 \end{cases}\end{aligned} \tag{9.50}$$

A Partial wave expansion

A useful result in wave theory is the expansion of the plane wave in a Fourier series of the polar angle θ. In polar coordinates the spatial factor of a plane wave of unit amplitude is

$$e^{ikx} = e^{ikr\cos\theta}.$$

Consider the following product of exponential functions

$$\begin{aligned} e^{zt/2}e^{-z/2t} &= \left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{zt}{2}\right)^n\right]\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-z}{2t}\right)^n\right] \\ &= \sum_{-\infty}^{\infty} t^n\left[\frac{(z/2)^n}{n!} - \frac{(z/2)^{n+2}}{1!(n+1)!} + \frac{(z/2)^{n+4}}{2!(n+2)!} + \cdots + (-1)^r\frac{(z/2)^{n+2r}}{r!(n+r)!} + \cdots\right]. \end{aligned}$$

The coefficient of t^n is nothing but $J_n(z)$, hence

$$\exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{-\infty}^{\infty} t^n J_n(z).$$

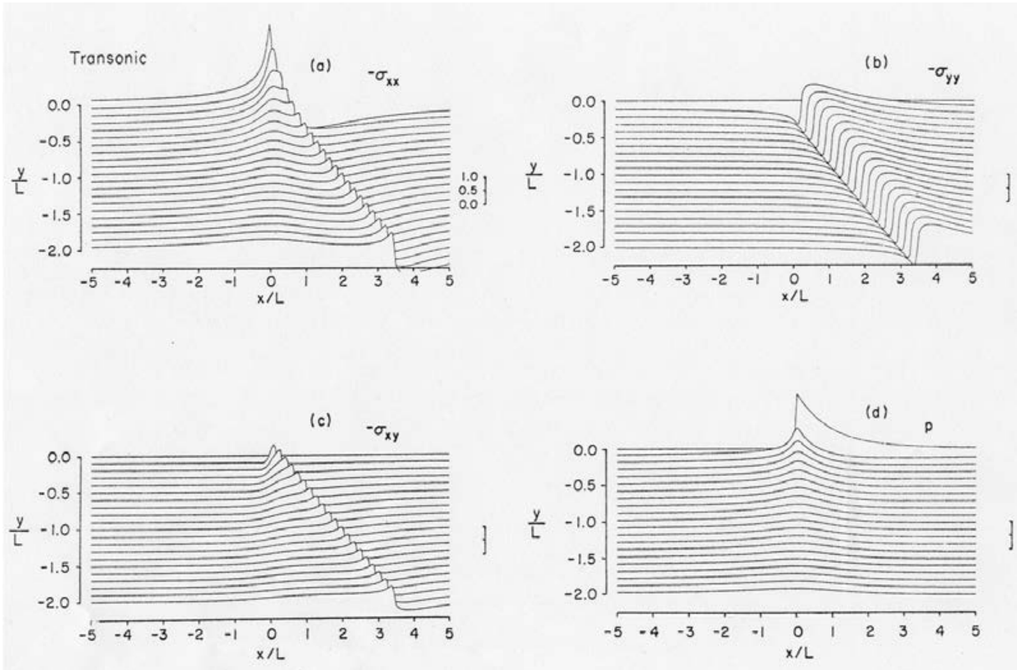

Figure 16: Stress variations in the ground under transonic load on the surface.

Now we set

$$te= ie^{i\theta} \quad ze= kr.e$$

The plane wave then becomes

$$^{ikx} = \sum_{N=-\infty}^{\infty} {}^{in(\theta+\pi/2} J_n(z).e$$

Using the fact that $J_{-n} = (-1)^n J_n$, we finally get

$$^{ikx} = {}^{ikr\cos\theta} = \sum_{n=0}^{\infty} \epsilon_n i^n J_n(kr) \cos n\theta, e \tag{A.1}$$

where ϵ_n is the Jacobi symbol. The above result may be viewed as the Fourier expansion of the plane wave with Bessel functions being the expansion coefficients. In wave propagation theories, each term in the series represents a distinct angular variation and is called a *partial wave.*

Using the orthogonality of $\cos n\theta$, we may evaluate the Fourier coefficient

$$J_n(kr) = \frac{2}{\epsilon_n i^n \pi} \int_0^{\pi} {}^{ikr\cos\theta} \cos n\theta d\theta, \tag{A.2}$$

which is one of a host of integral representations of Bessel functions.

B Approximate evaluation of an integral

Consider the integral

$$\int_0^{2\pi} d\theta[1 + \cos(\theta - \theta_o)] \quad {}^{ikr(1-\cos(\theta-\theta_o))}$$

For large kr the stationary phase points are found from

$$\frac{\partial}{\partial\theta}[1 - \cos(\theta - \theta_o)] = \sin(\theta - \theta_o) = 0$$

or $\theta = \theta_o, \theta_o + \pi$ within the range $[0, 2\pi]$. Near the first stationary point the integrand is dominated by

$$2\mathcal{A}(\theta_o) \quad {}^{ikt(\theta-\theta_o)^2/2}.$$

When the limits are approximated by $(-\infty, \infty)$, the integral can be evaluated to give

$$\mathcal{A}(\theta_o) \int_{-\infty}^{\infty} \quad {}^{ikr\theta^2/2} d\theta = \sqrt{\frac{2\pi}{kre}} \quad {}^{i\pi/4} \mathcal{A}(\theta_o)$$

Near the second stationary point the integral vanishes since $1+\cos(\theta-\theta_o) == 1-1 = 0$. Hence the result (6.28) follows.