Today's plan

- 1^{st} order system response: review
  - the role of zeros

- 2^{nd} order system response:
  - example: DC motor with inductance
  - response classifications:
    - overdamped
    - underdamped
    - undamped

- Linearization
  - from pendulum equation to the harmonic oscillator

Review: step response of 1st order systems

Step response in the s–domain

$$\frac{a}{s(s+a)};$$

in the time domain

$$\left(1 - e^{-at}\right) u(t);$$

time constant

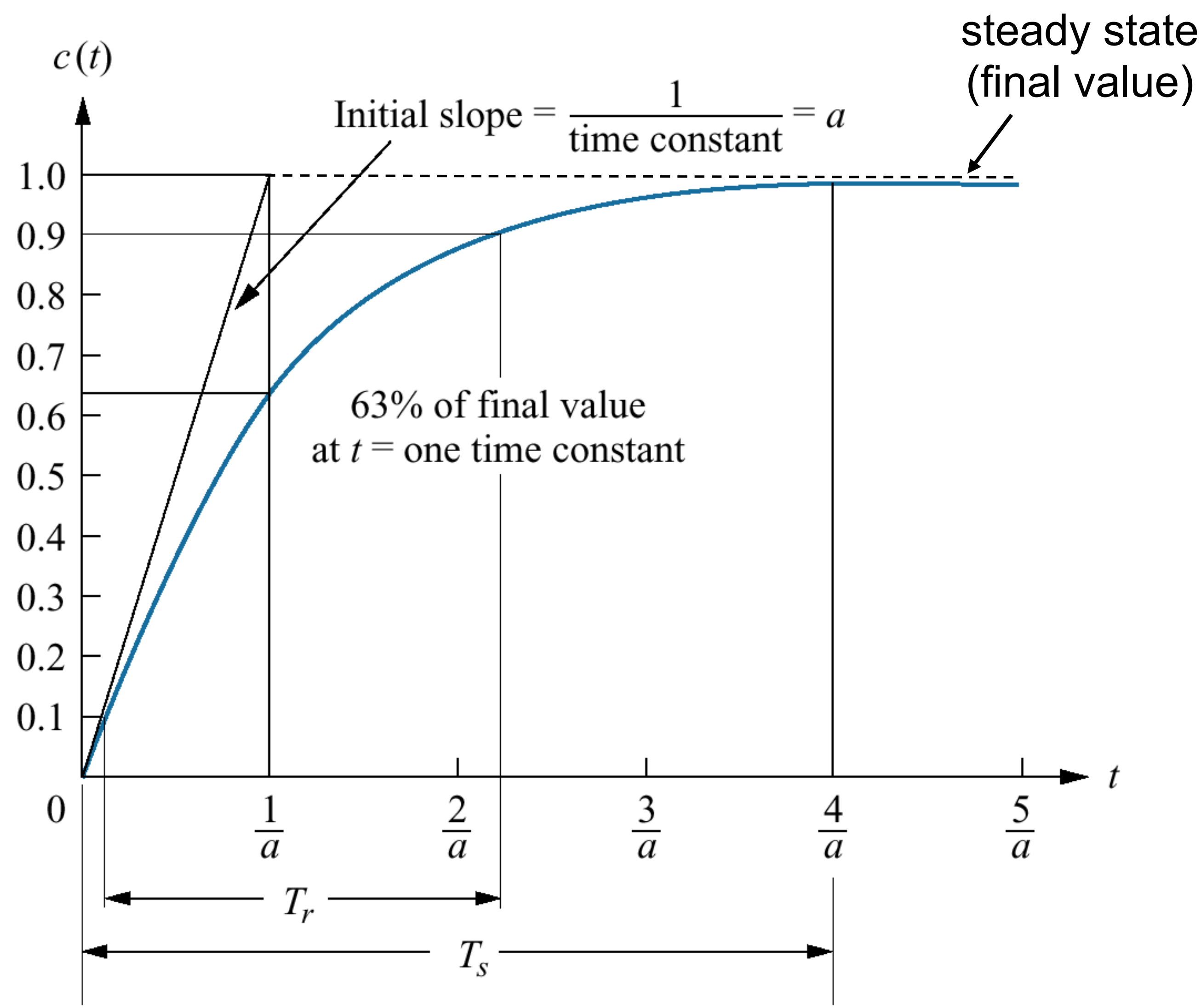
$$\tau = \frac{1}{a};$$

rise time (10%→90%)

$$T_r = \frac{2.2}{a};$$

settling time (98%)

$$T_s = \frac{4}{a}.$$



Nise Figure 4.3

Steady-state

- Note that as t→∞, the exponential decays away, and the step response tends to 1, i.e. the value of the driving force.
- More generally, if we consider a flywheel-like viscously damped system with equation of motion expressed as 1st order linear time-invariant ODE

$$J\dot{\omega}(t) + b\omega(t) = f(t)$$ and step excitation $$f(t) = F_0\,\mathrm{step}(t)$$

[where f(t) and F0 have units of torque,]

the step response is $$\omega(t) = \frac{F_0}{b}\left(1 - \mathrm{e}^{-t/\tau}\right), \quad t > 0 \qquad \left(\tau = \frac{J}{b}\right)$$

the angular velocity as t→∞ tends to $$\omega_\infty = \frac{F_0}{b}$$

- This long-term value is known as the system's **steady state**
- In systems where the output physically represents a velocity, the steady state is also known as **terminal velocity**.

Steady state in the Laplace domain: the final value thm.

- It turns out that we can predict the steady state of a system directly in the Laplace domain by using the following property known, for obvious reasons, as the **final value theorem**:

$$\lim_{t\to\infty} g(t) = \lim_{s\to 0} sG(s).$$

which holds generally if g(t) and G(s) form a Laplace transform pair.

- In the case of the flywheel, the transfer function is $\dfrac{\Omega(s)}{F(s)} = \dfrac{1}{Js+b}$

For the step response we have $F(s) = \dfrac{F_0}{s} \Rightarrow \Omega(s) = \dfrac{F_0}{s\,(Js+b)}$

It is easy to verify that $\lim_{s\to 0} s\Omega(s) = \dfrac{F_0}{b}$

in agreement with the result for the steady state of this system in the previous page

A word on zeros

Transfer function
with a zero

$$H(s) = \frac{s+z}{D(s)} = s \cdot \frac{1}{D(s)} + z \cdot \frac{1}{D(s)}$$

Derivative
operator

Amplification
("gain")

Example: compare the step response of the two systems below:

$$H_0(s) = \frac{1}{s+p}$$

$$H(s) = \frac{s+z}{s+p}$$

Step response without zero *vs* with zero

$U(s) = \dfrac{1}{s}$ → $H_0(s) = \dfrac{1}{s+p}$ → $F_0(s)$

$$F_0(s) = \frac{1}{s}\,\frac{1}{s+p} = \frac{1}{p}\left(\frac{1}{s} - \frac{1}{s+p}\right)$$

$$\Rightarrow f_0(t) = \frac{1}{p}\left(1 - \mathrm{e}^{-pt}\right), \qquad t > 0.$$

Please verify yourselves this partial-fraction expansion!

$U(s) = \dfrac{1}{s}$ → $H(s) = \dfrac{s+z}{s+p}$ → $F(s)$

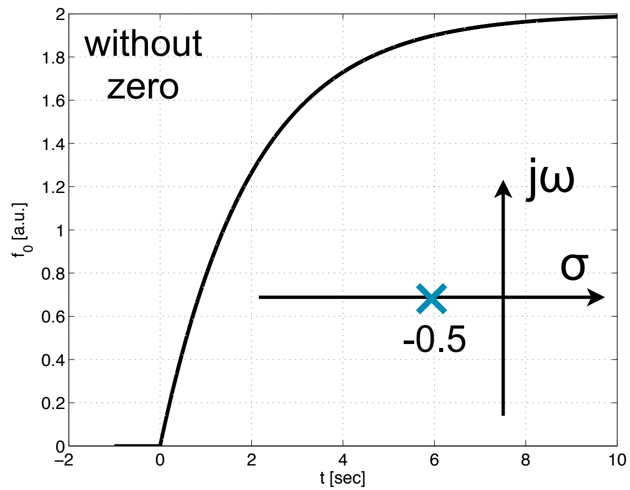
$$F(s) = sF_0(s) + zF_0(s)$$

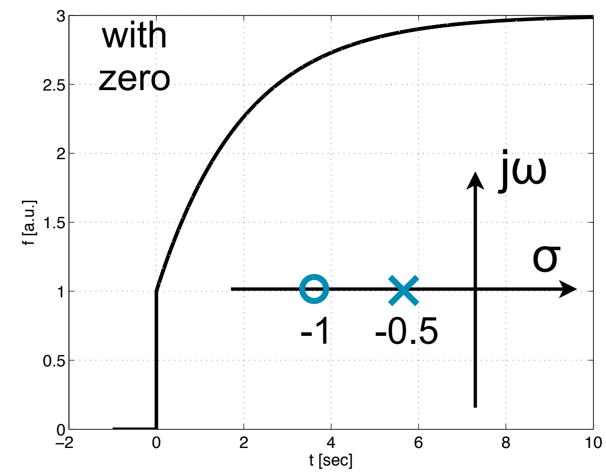
$$\Rightarrow f(t) = \frac{\mathrm{d}}{\mathrm{d}t} f_0(t) + z f_0(t) = \mathrm{e}^{-pt} + \frac{z}{p}\left(1 - \mathrm{e}^{-pt}\right), \quad t > 0.$$

<u>Example:</u>

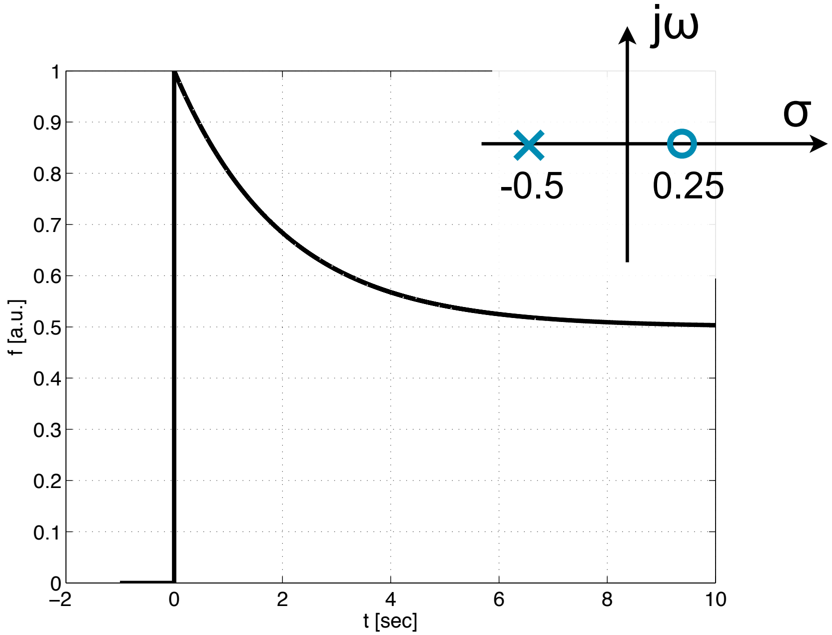
$p = 0.5;$

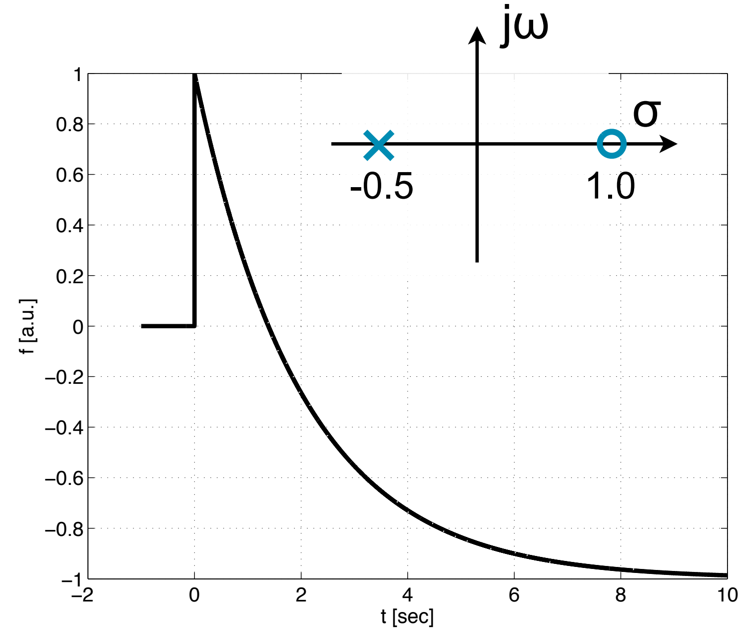
$z = 1.0$





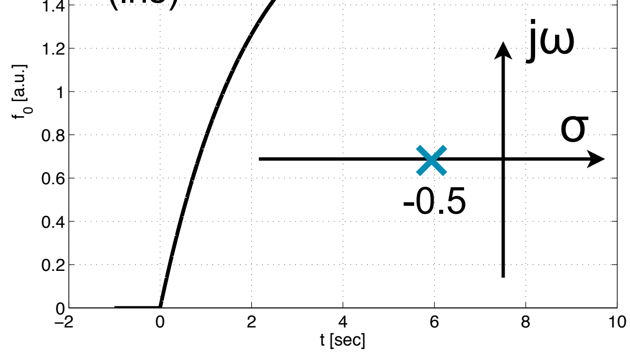
Zero on the right-hand side?

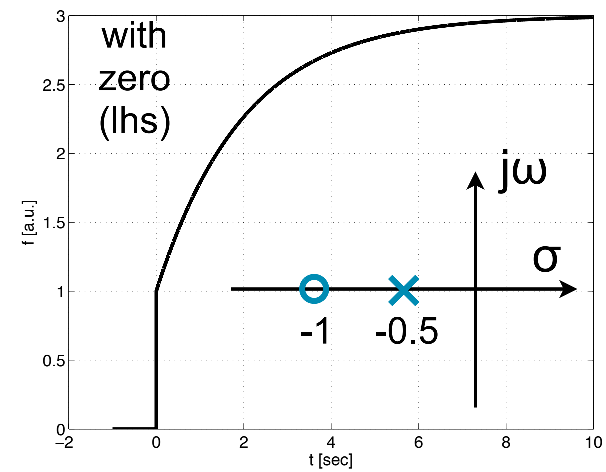




Example:

$p = 0.5$





The general 2nd order system

We can write the transfer function of the general 2^{nd}–order system with *unit* steady state response as follows:

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \qquad \text{where}$$

- ω_n is the system's *natural frequency*, and
- ζ is the system's *damping ratio*.

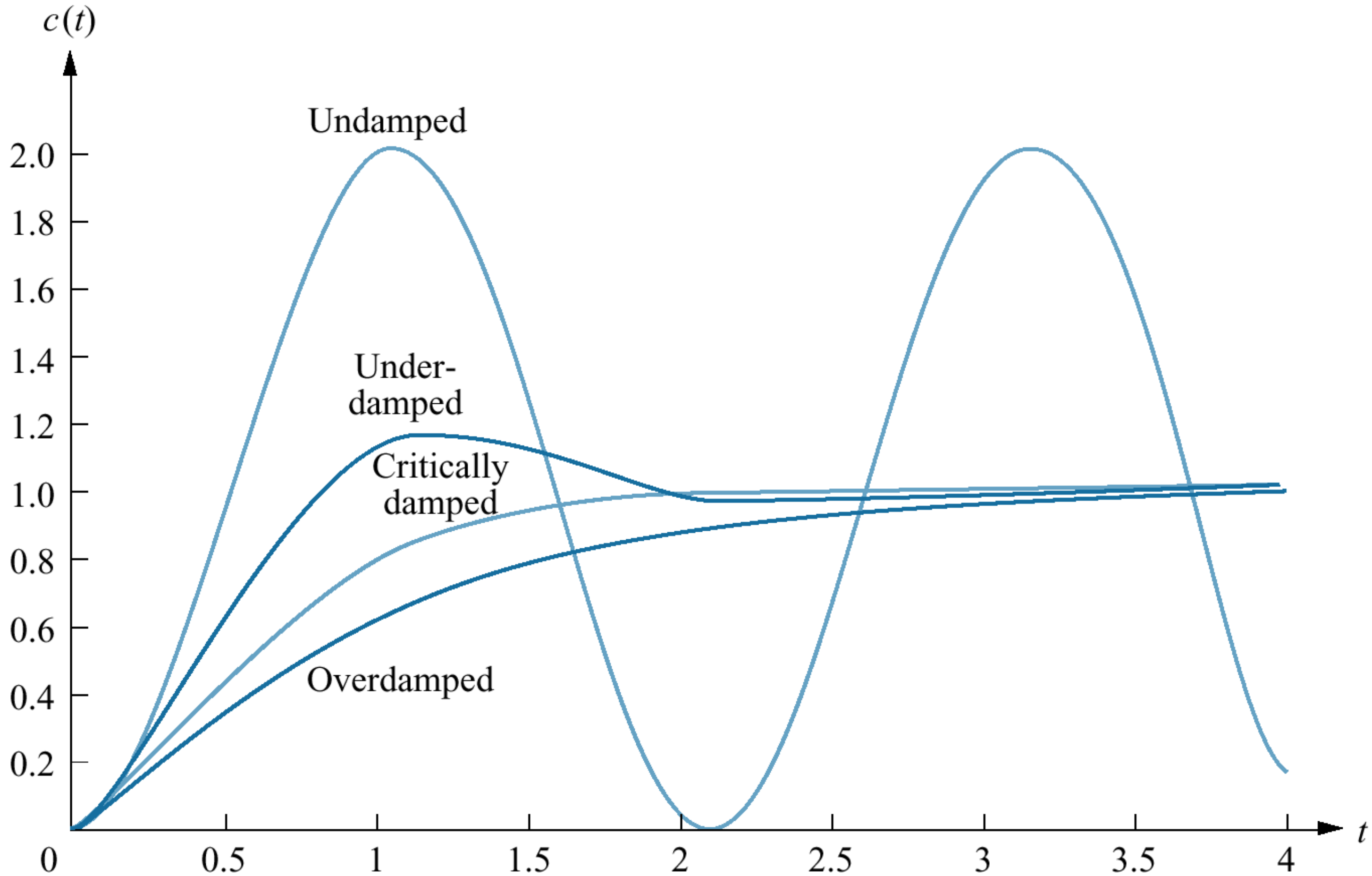
The natural frequency indicates the oscillation frequency of the undamped ("natural") system, *i.e.* the system with energy storage elements only and without any dissipative elements. The damping ratio denotes the relative contribution to the system dynamics by energy storage elements and dissipative elements. Recall,

$$\zeta \equiv \frac{1}{2\pi} \frac{\text{Undamped ("natural") period}}{\text{Time constant of exponential decay}}.$$

Depending on the damping ratio ζ, the system response is

- *undamped* if $\zeta = 0$;
- *underdamped* if $0 < \zeta < 1$;
- *critically damped* if $\zeta = 1$;
- *overdamped* if $\zeta > 1$.

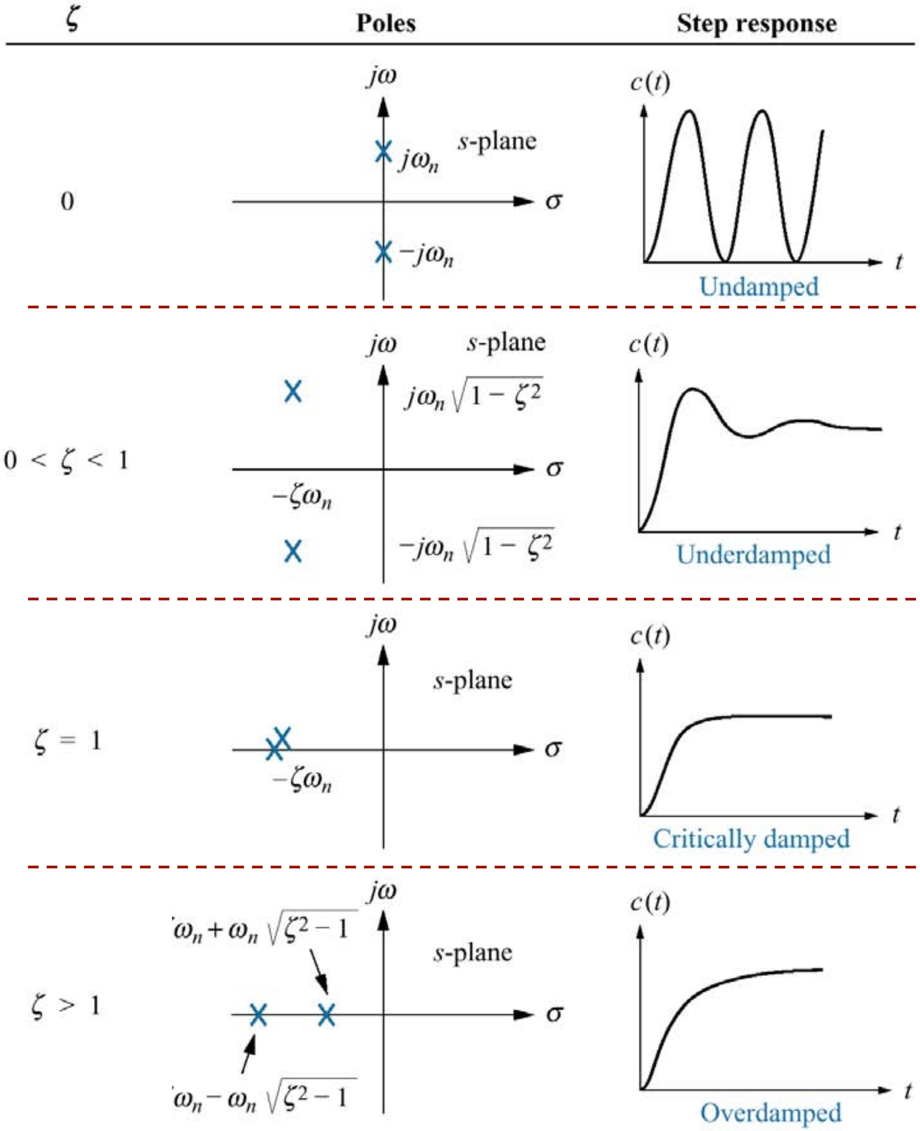
The general 2nd order system



Nise Figure 4.10

The general 2nd order system

$$\frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$



$$\left[1 - \cos\left(\omega_n t\right)\right] u(t)$$

ω_n, *natural* oscillation frequency

$$\left[1 - Ae^{-\sigma_d t} \cos\left(\omega_d t - \phi\right)\right] u(t)$$

$\sigma_d = \zeta\omega_n$, decay time constant;

$\omega_d = \omega_n\sqrt{1-\zeta^2}$, *damped* oscillation frequency

$$\left[1 + K_1 e^{-\sigma_c t} + K_2 t e^{-\sigma_c t}\right] u(t)$$

σ_c, double pole

$$\left(\text{Transfer function} \quad \frac{\omega_n^2}{(s+\omega_n)^2} \equiv \frac{\omega_n^2}{(s+\sigma_c)^2}\right)$$

$$\left[1 + K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}\right] u(t)$$

$\sigma_{1,2} = \omega_n\left(1 \pm \sqrt{\zeta^2 - 1}\right)$, real poles.

Nise Figure 4.11

The underdamped 2nd order system

$$\frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}, \quad 0 < \zeta < 1$$

The step response's Laplace transform is

$$\frac{1}{s} \times \frac{\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

We find

$$K_1 = \frac{1}{\omega_n^2}, \qquad K_2 = -\frac{1}{\omega_n^2}, \qquad K_3 = \frac{2\zeta}{\omega_n}$$

Substituting and applying the same method of completing squares that we did in the numerical example of the DC motor's angular velocity response, we can rewrite the laplace transform of the step response as

$$\frac{1}{s} - \frac{(s + \zeta\omega_n) + \dfrac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^2\left(1 - \zeta^2\right)}.$$

Using the frequency shifting property of Laplace transforms we finally obtain the step response in the time domain as

$$1 - \mathrm{e}^{-\zeta\omega_n t}\left[\cos\left(\omega_n\sqrt{1-\zeta^2}t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\right].$$

The underdamped 2nd order system

$$\frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}, \quad 0 < \zeta < 1$$

Finally, using some additional trigonometry and the definitions

$$\sigma_d = \zeta\omega_n, \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}, \qquad \tan\phi = \frac{\zeta}{\sqrt{1-\zeta^2}}$$

we can rewrite the step response as

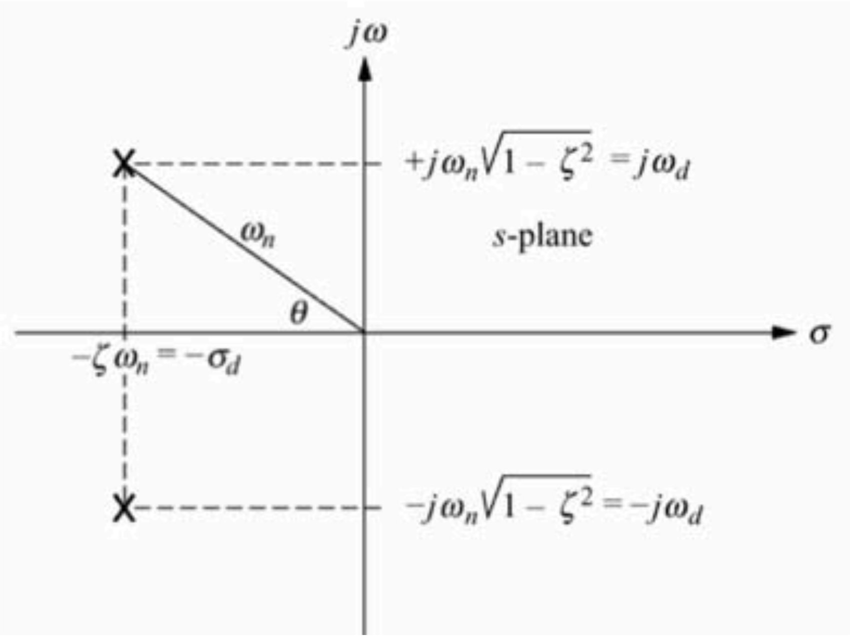
$$1 \quad - \quad \frac{1}{\sqrt{1-\zeta^2}} \quad \times \quad e^{-\sigma_d t} \quad \times \quad \cos(\omega_d t - \phi)$$

The definitions above can be re–written

$$\zeta = \frac{\sigma_d}{\omega_n},$$

$$\sqrt{1-\zeta^2} = \frac{\omega_d}{\omega_n},$$

$$\Rightarrow \tan\theta = \frac{\omega_d}{\sigma_d} = \frac{\sqrt{1-\zeta^2}}{\zeta}.$$



Nise Figure 4.10

The underdamped 2nd order system

$$\frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}, \qquad 0 < \zeta < 1$$

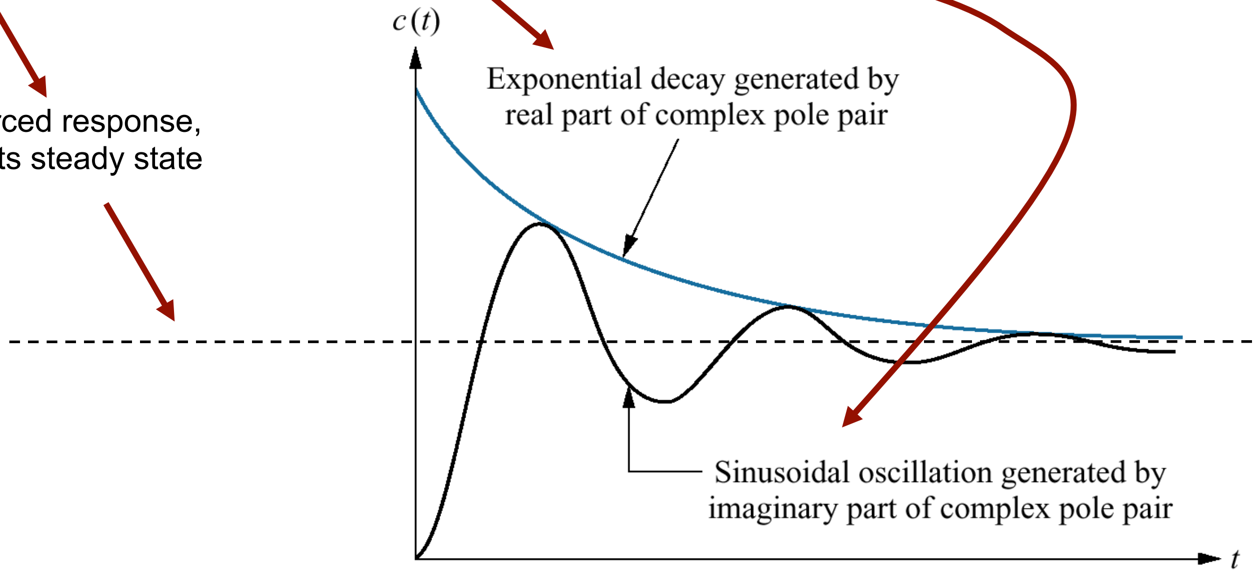
Finally, using some additional trigonometry and the definitions

$$\sigma_d = \zeta\omega_n, \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}, \qquad \tan\phi = \frac{\zeta}{\sqrt{1-\zeta^2}}$$

we can rewrite the step response as

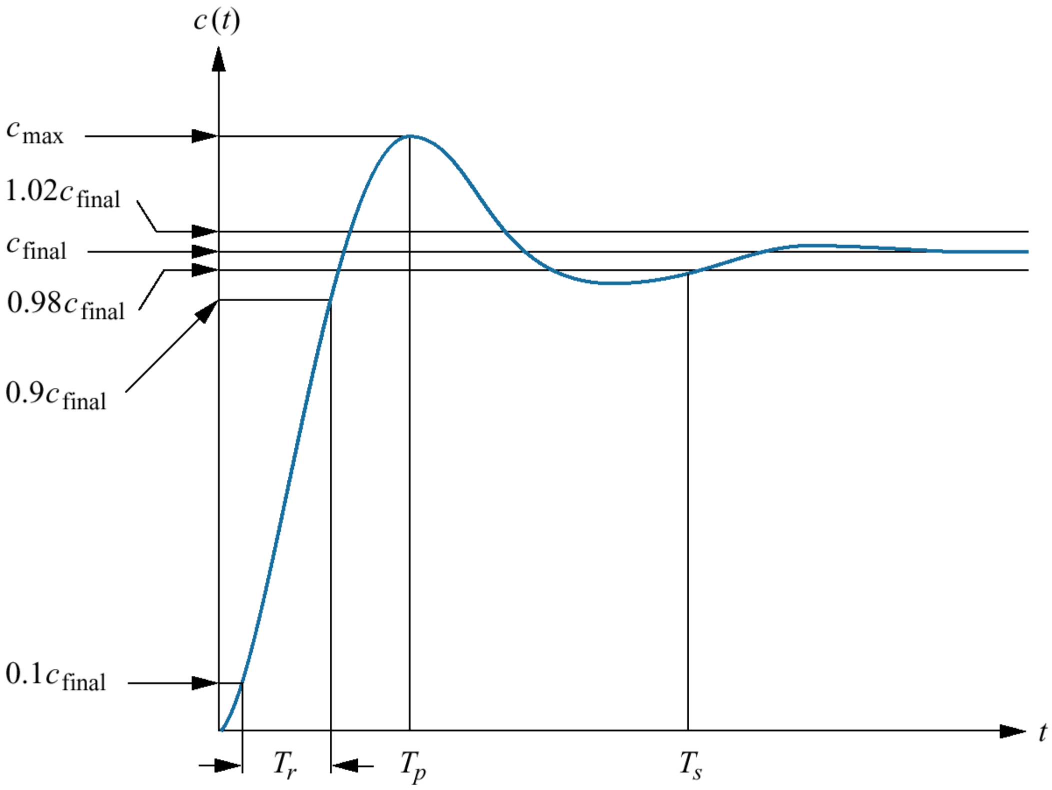
$$1 \quad - \quad \frac{1}{\sqrt{1-\zeta^2}} \quad \times \quad \mathrm{e}^{-\sigma_d t} \quad \times \quad \cos\left(\omega_d t - \phi\right)$$

forced response,
sets steady state



Nise Figure 4.10

Transients in the underdamped 2nd order system



Nise Figure 4.14

Peak time

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}.$$

Percent overshoot (%OS)

$$\%OS = \exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right) \times 100$$

$$\Leftrightarrow \zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

Settling time
(to within ±2% of steady state)

$$T_s = -\frac{\ln\left(0.02\sqrt{1-\zeta^2}\right)}{\zeta\omega_n} \approx \frac{4}{\zeta\omega_n}.$$

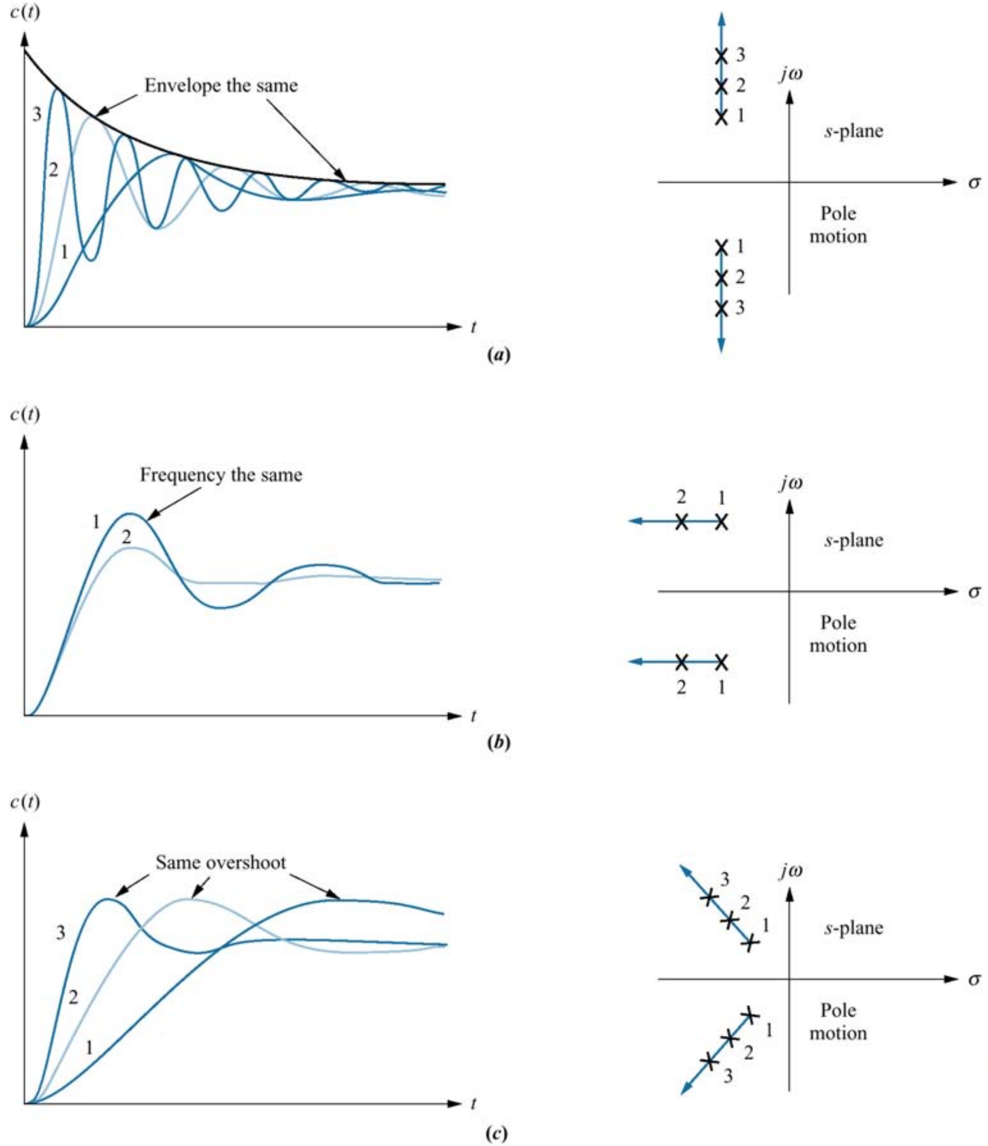
(approximation valid for
$0 < \zeta < 0.9$.)

Transient qualities from pole location in the *s*-plane

Same T_s.

Same ω_d.

Same %OS.



Nise Figure 4.19

Recall

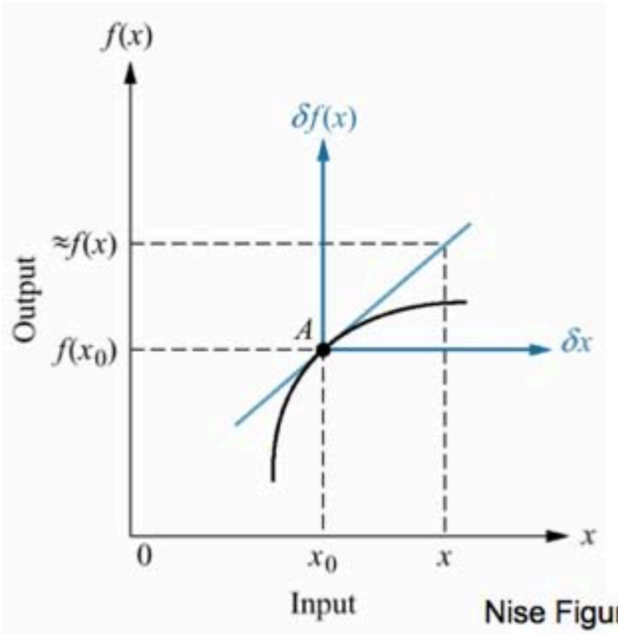
$$\zeta = \frac{\sigma_d}{\omega_n},$$

$$\sqrt{1-\zeta^2} = \frac{\omega_d}{\omega_n},$$

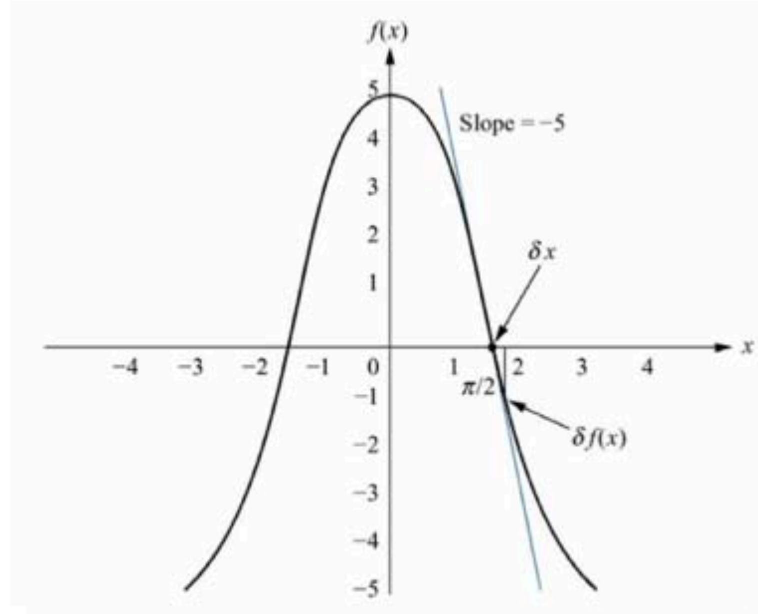
$$\Rightarrow \tan\theta = \frac{\omega_d}{\sigma_d} = \frac{\sqrt{1-\zeta^2}}{\zeta}.$$

Linearization

- This technique can be used to approximate a non-linear system



Nise Figure 2.47



Nise Figure 2.48

$$f(x) \approx f(x_0) + m_a\,(x - x_0)$$

where

$$m_a = \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_0}.$$

First order Taylor expansion

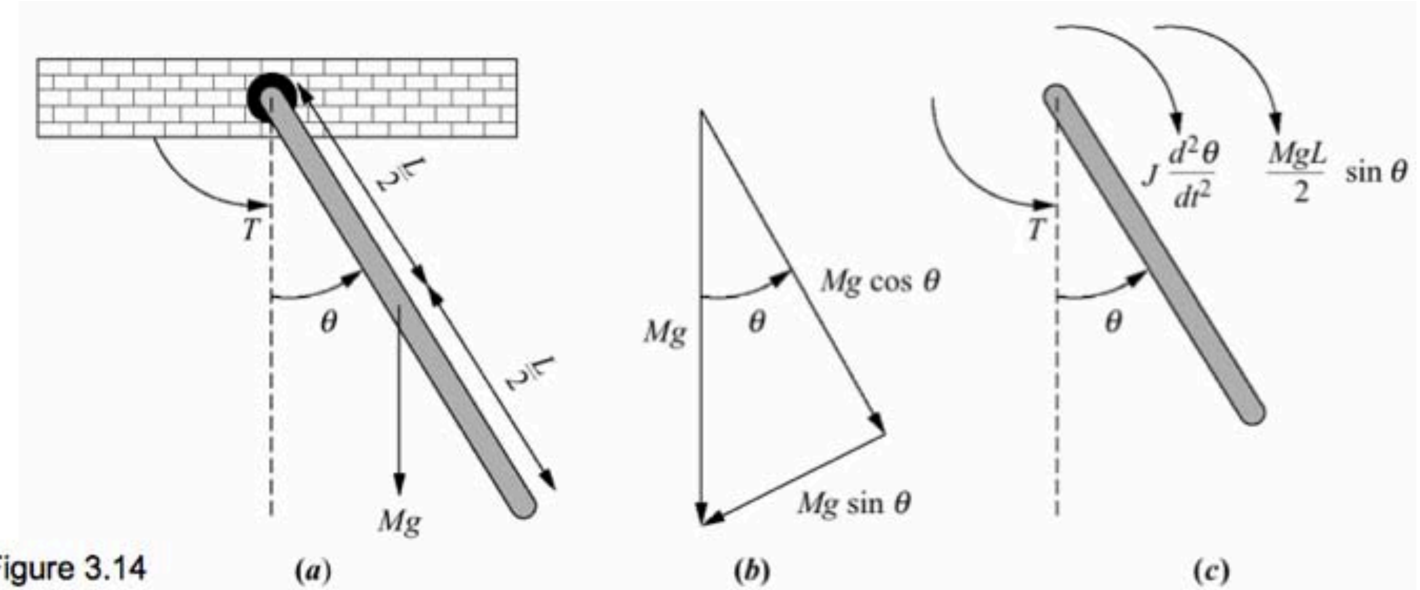
Example

Linearize $f(x) = 5\cos x$ near $x = \pi/2$.

Answer: We have $f(\pi/2) = 0$, $m_a = -5$, so

$$f(x) \approx -5\left(x - \frac{\pi}{2}\right) \qquad (x \approx \pi/2)$$

Linearizing systems: the pendulum



Nise Figure 3.14

The equation of motion is found as

$$J\ddot{\theta} + \frac{MgL}{2}\sin\theta = T. \qquad \text{(We \textbf{cannot} Laplace transform!)}$$

For small angles $\theta \approx 0$, we have

$$\sin\theta \approx \left.\frac{\mathrm{d}\sin\theta}{\mathrm{d}\theta}\right|_{\theta=0} \times \theta = \cos\theta|_{\theta=0} \times \theta = 1 \times \theta = \theta.$$

Therefore, the **linearized** equation of motion is

$$J\ddot{\theta} + \frac{MgL}{2}\theta = T \Rightarrow Js^2\Theta(s) + \frac{MgL}{2}\Theta(s) = T(s).$$