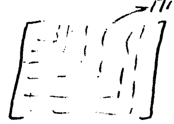
Stability of equilibria

For systems with Conservative forces $\frac{\partial V(q_0)}{\partial q} = 0$ (1)

$\left.\frac{\partial^2 V}{\partial q^2}\right|_{q_0}$ pos. definite



A_i

$\det(A_i) > 0$

$i = 1, \dots, n$

Stability & Small Oscillations in general holonomic Systems

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \underline{Q} \qquad q = (q_1, \dots, q_N)$$

$$L = T - V \qquad \underline{Q}(q, \dot{q}, t)$$

Kinetic Energy:

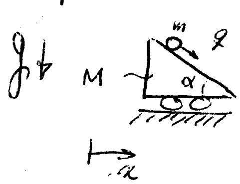
$$T = \frac{1}{2} \sum_{i=1}^{n} m_i |\dot{\underline{r}}_i|^2 ; \quad \underline{r}_i = r_i(q_1, \dots, q_N, t)$$

$$\Rightarrow T = \frac{1}{2} \sum_{i=1}^{n} m_i \left[\sum_{j=1}^{N} \left(\frac{\partial r_i}{\partial q_j} \dot{q}_j + \frac{\partial r_i}{\partial t} \right) \right] \cdot \left[\qquad \right]$$

$$T = \underbrace{\frac{1}{2} \sum_{i,j=1}^{N} m_{ij}(q;t)\, \dot{q}_i \dot{q}_j}_{T_2} + \underbrace{\sum_{i=1}^{N} b_i(q,t) \dot{q}_i}_{T_1} + \underbrace{c(q,t)}_{T_0}$$

Natural Mechanical System: By definition is one for which $T_1 = T_0 = 0$

Example (1)



$$T = \frac{1}{2} m \left[(\dot{x} + \dot{q}\cos\alpha)^2 + (\dot{q}\sin\alpha)^2 \right] + \frac{1}{2} M \dot{x}^2$$

Gen. Coordinates (x, q) $\Rightarrow$ Natural System

(2) Same Problem with the Constraint

$\dot{x} = V(t) =$ prescribed

(Integrable non holonomic Constraint) $\Big\}$ $\Rightarrow$ gen. Coordinate q

$$T = \frac{1}{2} m \left[(V + \dot{q}\cos\alpha)^2 + (\dot{q}\sin\alpha)^2 \right] + \frac{1}{2} M V^2$$

non-natural System

(3) if $q = (\hat{q}, \varphi)$, $\frac{\partial L}{\partial \varphi} = 0$; $Q_\varphi = 0 \longrightarrow \varphi$ is a cyclic (ignorable) Coordinate

$\Rightarrow \frac{d}{dt}\underbrace{\left(\frac{\partial L}{\partial \dot\varphi}\right)}_{P_p\text{: gen. momentum associated with } \varphi} = 0 \quad \Rightarrow P_p = \frac{\partial L}{\partial \dot\varphi} \Rightarrow \dot\varphi = F(p, \hat{q}, \dot{\hat{q}})$

$\Downarrow$

reduced set of eqns for $\hat{q}$

$\Rightarrow$ kinetic energy in reduced coordinates (Assume unreduced system natural)

$$T = \frac{1}{2}\sum_{i,j=1}^{N-1} m_{ij}\dot{\hat{q}}_i\dot{\hat{q}}_j + \frac{1}{2}\sum_{i=1}^{N-1} m_{iN}\dot{\hat{q}}_i\underbrace{\dot\varphi}_{F(\varphi,\hat{q},\dot{\hat{q}})} + \frac{1}{2}m_{NN}\underbrace{\dot\varphi^2}_{F(\varphi,\hat{q},\dot{\hat{q}})}$$

$\Rightarrow T_1$ & T_2-Type term

$\Rightarrow$ reduced system for $\hat{q}$ is non-natural

For general non-natural system:

$$L = T - V = T_2 + T_1 + T_0 - V$$

let: $T_2 = \frac{1}{2}\dot{q}^T \underline{\underline{M}}(q,t)\dot{q}$, $\underline{\underline{M}} = [m_{ij}]_{i,j=1}^{n}$

$T_1 = \underline{b}\cdot\dot{q} = \underline{b}^T(q,t)\dot{q}$; $\underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix}$

$V = V(q,t)$

$Q = Q(q, \dot{q}, t)$

(Assuming $\frac{\partial T_0}{\partial q} = 0$)

$\Rightarrow$ Lagrangian eq. of motion $\frac{d}{dt}(\underline{M}\dot{q} + \underline{b}) - \underbrace{\frac{1}{2}\dot{q}^T\frac{\partial \underline{M}}{\partial q}\dot{q}}_{\text{Nonlinear term}} - \frac{\partial b^T}{\partial q}\dot{q} + \frac{\partial V}{\partial q} = \underline{Q}$

Assume $\frac{\partial \underline{\underline{M}}}{\partial t} = 0$, $\frac{\partial b}{\partial t} = 0$

$$\Rightarrow \underline{M}\ddot{q} + \frac{1}{2}\dot{q}^T\frac{\partial \underline{M}}{\partial q}\dot{q} + \left[\frac{\partial b}{\partial q} - \frac{\partial b^T}{\partial q}\right]\dot{q} + \frac{\partial V}{\partial q} - \frac{\partial T_0}{\partial q} = \underline{Q}(q,\dot{q}) \qquad (*)$$

we can assume $\frac{\partial T_0}{\partial t} \neq 0$!!

equilibria: $q = q_0 \Rightarrow \dot{q} = 0$, $\ddot{q} = 0$

$\left.\frac{\partial V}{\partial q}\right|_{q_0} = Q(q_0, 0, t)$ defines equilibria

to understand the stability of q_0, linearize $(*)$ about q_0

$$\Rightarrow \underline{M}(q_0)\ddot{q} + \left[\frac{\partial b}{\partial q} - \frac{\partial b^T}{\partial q}\right]\Big|_{q=q_0}\dot{q} + \cancel{\left.\frac{\partial V}{\partial q}\right|_{q_0}} + \left.\frac{\partial^2 V}{\partial q^2}\right|_{q=q_0}(q-q_0) + O(2)$$

$$- \cancel{Q(q_0, 0, t)} + \left.\frac{\partial Q}{\partial q}\right|_{q=q_0}(q-q_0)$$

$$+ \left.\frac{\partial Q}{\partial \dot{q}}\right|_{q_0, q}\dot{q} + O(2) = 0$$

let (1) $\underline{\underline{M}} = \underline{\underline{M}}(q_0)$: mass matrix

Symmetric ($\underline{\underline{M}} = \underline{\underline{M}}^T$)

and positive-definite ($\underline{x}^T \underline{\underline{M}} \, \underline{x} > 0$)

$\underline{x} \neq 0$

(2) $\underline{\underline{G}} = \left[\frac{\partial b}{\partial q} - \frac{\partial b^T}{\partial q} \right] \Big|_{q=0}$ gyroscopic matrix

only presents for non-natural system

$\underline{\underline{G}}^T = -\underline{\underline{G}}$ skew-symmetric

(3) $\underline{\underline{K}} = \frac{\partial^2 V}{\partial q^2} \Big|_{q=q_0}$ Stiffness matrix $\underline{\underline{K}} = \underline{\underline{K}}^T \Rightarrow K$ is symmetric

if q is stable eq. $\Rightarrow K$ = positive definite

(4) $\underline{\underline{B}} = -\frac{\partial Q}{\partial q} \Big|_{q=q_0}$

$\underline{\underline{C}} = -\frac{\partial Q}{\partial \dot{q}} \Big|_{q=q_0}$

$\Rightarrow$ Linearized eq. of motion

with $\underline{x} = q - q_0$

$$\underline{\underline{M}} \cdot \ddot{\underline{x}} + (\underline{\underline{G}} + \underline{\underline{C}}) \dot{\underline{x}} + (\underline{\underline{K}} + \underline{\underline{B}}) \underline{x} = \underline{0}$$