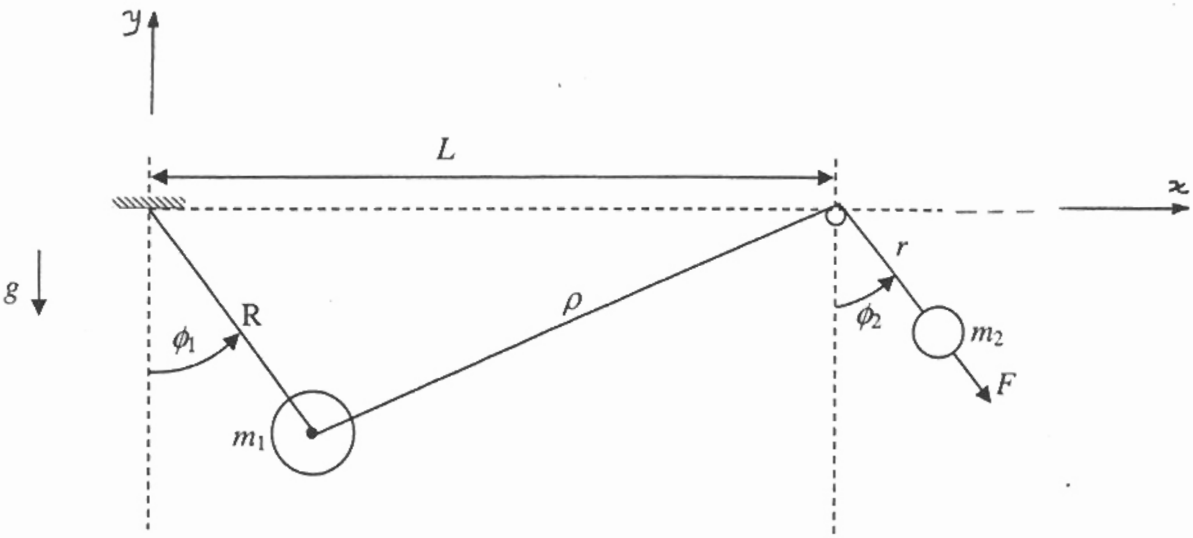
2.032 Dynamics

Fall 2004

Quiz No. 2

Problem 1

(a)



Constraints :

$$x_1^2 + y_1^2 = R^2$$

$$\rho + r = L \longrightarrow \sqrt{(L-x_1)^2 + y_1^2} + \sqrt{(x_2-L)^2 + y_2^2} = L$$

where $\begin{cases} x_1 = R\sin\varphi_1 \\ y_1 = -R\cos\varphi_1 \end{cases}$ are the coordinates of m_1, and $\begin{cases} x_2 = L + r\sin\varphi_2 \\ y_2 = -r\cos\varphi_2 \end{cases}$ are the coordinates of m_2.

Velocity and time do not appear explicitly in the constraints so both are holonomic scleronomic constraints.

Constraint forces do not do work and are not active forces.

Active forces are gravity and the force $\underline{F}$.

Gravity is potential.

Force $\underline{F}$:

$$\underline{F} = F\left[\frac{x-L}{\sqrt{(x-L)^2+y^2}}\,\underline{i} + \frac{y}{\sqrt{(x-L)^2+y^2}}\,\underline{j}\right]$$

Problem 1

(a) $$\frac{\partial F_x}{\partial y} = F\,\frac{-y(x-L)}{\sqrt{(x-L)^2+y^2}^{\,3}} \qquad\qquad \frac{\partial F_y}{\partial x} = \frac{-y(x-L)}{\sqrt{(x-L)^2+y^2}^{\,3}}$$

$$\longrightarrow \quad \frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} \quad \longrightarrow \quad \text{force } \underline{F} \text{ is potential.}$$

$$\underline{F} = -\nabla V \quad \longrightarrow \quad V = -F\sqrt{(x-L)^2+y^2} = -Fr$$

$$\# \text{DOF} = \underbrace{2\times 2}_{\text{point mass in a plane}} - \underbrace{2}_{\text{Constraints}} = \boxed{2}$$

(b) It is convenient to use three generalized coordinates and introduce one Lagrange multiplier:

$$q_1 = \varphi_1 , \quad q_2 = \varphi_2 , \quad q_3 = r \qquad\qquad \text{Lagrange multiplier } \lambda_1$$

$$T = \tfrac{1}{2} m_1 R^2 \dot\varphi_1^2 + \tfrac{1}{2} m_2 \left(\dot r^2 + r^2 \dot\varphi_2^2\right)$$

$$V = -m_1 g R \cos\varphi_1 - m_2 g r \cos\varphi_2 - Fr$$

Constraint : $$r + \rho = L \quad \longrightarrow \quad \dot r + \frac{\partial \rho}{\partial \varphi_1}\dot\varphi_1 = 0 , \qquad \rho = \sqrt{(L - R\sin\varphi_1)^2 + R^2\cos^2\varphi_1}$$

$$\longrightarrow \quad a_{11} = \frac{\partial \rho}{\partial \varphi_1} , \quad a_{12} = 0 , \quad a_{13} = 1$$

$L = T - V$ All active forces are potential. $\longrightarrow$ $Q_j = 0$

Equations of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi_1}\right) - \frac{\partial L}{\partial \varphi_1} = 0 + \lambda_1 a_{11} \quad \longrightarrow \quad m_1 R^2 \ddot\varphi_1 + m_1 g R \sin\varphi_1 = \lambda_1 \frac{\partial \rho}{\partial \varphi_1} \qquad (1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi_2}\right) - \frac{\partial L}{\partial \varphi_2} = 0 + \lambda_1 a_{12} \quad \longrightarrow \quad m_2 r^2 \ddot\varphi_2 + 2 m_2 r \dot r \dot\varphi_2 + m_2 g r \sin\varphi_2 = 0 \qquad (2)$$

Problem 1

(b)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) - \frac{\partial L}{\partial r} = 0 + \lambda_1 a_{13} \longrightarrow m_2\ddot r - m_2 r\dot\varphi_2^2 - m_2 g\cos\varphi_2 - F = \lambda_1 \qquad (3)$$

Constraint : $\dot r + \frac{\partial \rho}{\partial \varphi_1}\dot\varphi_1 = 0 \qquad (r = L-\rho) \qquad (4)$

$$(4) \longrightarrow \ddot r = -\frac{\partial \rho}{\partial \varphi_1}\ddot\varphi_1 - \frac{\partial^2 \rho}{\partial \varphi_1^2}\dot\varphi_1^2 \qquad (5)$$

(1), (3), (4), (5) $\Longrightarrow$

$$m_1 R^2\ddot\varphi_1 + m_1 g R\sin\varphi_1 = \frac{\partial \rho}{\partial \varphi_1}\left[m_2\left(-\frac{\partial \rho}{\partial \varphi_1}\ddot\varphi_1 - \frac{\partial^2\rho}{\partial\varphi_1^2}\dot\varphi_1^2\right) - m_2(L-\rho)\dot\varphi_2^2 - m_2 g\cos\varphi_2 - F\right]$$

(2), (4) $\Longrightarrow$

$$m_2(L-\rho)^2\ddot\varphi_2 + 2m_2(L-\rho)\left(-\frac{\partial\rho}{\partial\varphi_1}\dot\varphi_1\right)\dot\varphi_2 + m_2 g(L-\rho)\sin\varphi_2 = 0$$

$\therefore$ Equations of motion in terms of φ_1 and φ_2 :

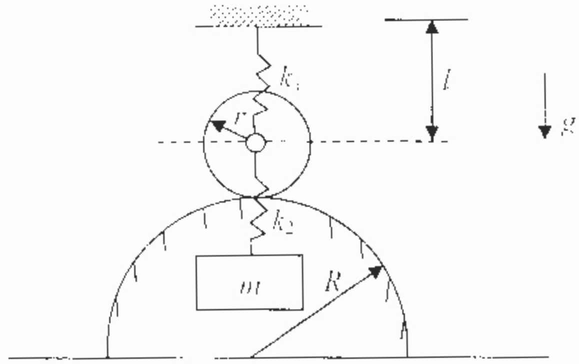
$$\begin{cases} \left[m_1R^2 + m_2\left(\frac{\partial\rho}{\partial\varphi_1}\right)^2\right]\ddot\varphi_1 + m_2\frac{\partial\rho}{\partial\varphi_1}\frac{\partial^2\rho}{\partial\varphi_1^2}\dot\varphi_1^2 + m_2(L-\rho)\frac{\partial\rho}{\partial\varphi_1}\dot\varphi_2^2 + m_1 g R\sin\varphi_1 + m_2 g\frac{\partial\rho}{\partial\varphi_1}\cos\varphi_2 = -F\frac{\partial\rho}{\partial\varphi_1} \\ (L-\rho)\ddot\varphi_2 - 2\frac{\partial\rho}{\partial\varphi_1}\dot\varphi_1\dot\varphi_2 + g\sin\varphi_2 = 0 \end{cases}$$

where $\rho = f(\varphi_1) = \sqrt{(L-R\sin\varphi_1)^2 + R^2\cos^2\varphi_1} = \sqrt{L^2+R^2-2LR\sin\varphi_1}$

$$\frac{\partial\rho}{\partial\varphi_1} = \frac{-LR\cos\varphi_1}{\sqrt{L^2+R^2-2LR\sin\varphi_1}}$$

Problem 2

- Active forces are potential (gravity, spring).
- Constraint forces do not work (no slip).



$\Longrightarrow$ System is conservative.

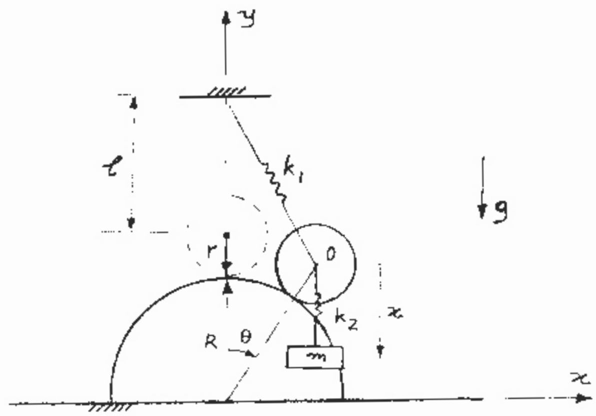
$\longrightarrow$ Dirichlet's theorem applies.

$\# DOF = 2\times 3 - 2 - 2 = 2 \qquad q_1 = x, \quad q_2 = \theta$

$x_0 = (R+r)\sin\theta$

$y_0 = (R+r)\cos\theta$

$y_m = (R+r)\cos\theta - x$



$$V = V^{disk} + V^{block} + V^{Spring\,2} + V^{Spring\,1}$$

$$V = Mg y_0 + mg y_m + \frac{1}{2}k_2(x-\ell_0)^2 + \frac{1}{2}k_1\left[\sqrt{x_0^2 + (R+r+\ell-y_0)^2} - \ell_0\right]^2$$

$$V = Mg(R+r)\cos\theta + mg\left[(R+r)\cos\theta - x\right] + \frac{1}{2}k_2(x-\ell_0)^2 + \frac{1}{2}k_1\left[\underbrace{\sqrt{(R+r)^2\sin^2\theta + \left[\ell + (R+r)(1-\cos\theta)\right]^2}}_{\sqrt{f(\theta)}} - \ell_0\right]^2$$

$$\frac{\partial V}{\partial x} = -mg + k_2(x-\ell_0)$$

$$\frac{\partial V}{\partial \theta} = -g(R+r)\sin\theta\,(M+m) + k_1\left(\sqrt{f} - \ell_0\right)\frac{f'}{2\sqrt{f}} = -g(R+r)(M+m)\sin\theta + \frac{k_1 f'}{2}\left(1 - \frac{\ell_0}{\sqrt{f}}\right)$$

where $\quad f' = 2(R+r)(\ell+R+r)\sin\theta$

$$\begin{cases} \dfrac{\partial V}{\partial x} = 0 \\[2mm] \dfrac{\partial V}{\partial \theta} = 0 \end{cases} \Longrightarrow \begin{cases} x = \ell_0 + \dfrac{mg}{k_2} \\[2mm] \theta = 0 \end{cases}$$ is the equilibrium shown in the figure.

Problem 2

For stability:

$$\frac{\partial^2 V}{\partial x^2} = k_2$$

$$\frac{\partial^2 V}{\partial \theta \partial x} = 0$$

$$\frac{\partial^2 V}{\partial \theta^2} = -g(R+r)(M+m)\cos\theta + \frac{k_1 f''}{2}\left(1-\frac{\ell_0}{\sqrt{f}}\right) + \frac{1}{4}k_1\ell_0\frac{f'^2}{\sqrt{f}^3}$$

$$f'' = 2(R+r)(\ell+R+r)\cos\theta$$

$$f|_{\theta=0} = \ell^2 \qquad f'|_{\theta=0} = 0 \qquad f''|_{\theta=0} = 2(R+r)(\ell+R+r)$$

$$\longrightarrow \quad \frac{\partial^2 V}{\partial \theta^2}\Big|_{\theta=0} = -g(R+r)(M+m) + k_1(R+r)(\ell+R+r)\left(1-\frac{\ell_0}{\ell}\right)$$

$$= (R+r)\left[k_1(\ell+R+r)\left(\frac{\ell-\ell_0}{\ell}\right) - g(M+m)\right]$$

Hessian of V at equilibrium $\left(\theta=0,\ x=\ell_0+\frac{mg}{k_2}\right)$:

$$D^2V = \begin{pmatrix} k_2 & 0 \\ 0 & (R+r)\left[k_1(\ell+R+r)\frac{\ell-\ell_0}{\ell} - g(M+m)\right] \end{pmatrix}$$

D^2V is positive definite if and only if $(k_2>0,\ R+r>0)$:

$$k_1(\ell+R+r)\frac{\ell-\ell_0}{\ell} - g(M+m) > 0 \longrightarrow \ell^2 + \left[R+r-\ell_0-\frac{g}{k_1}(M+m)\right]\ell - (R+r)\ell_0 > 0$$

$$\Longrightarrow \quad \ell > \frac{1}{2}\left\{\frac{g}{k_1}(M+m) - R - r + \ell_0 + \sqrt{\left[R+r-\ell_0-\frac{g}{k_1}(M+m)\right]^2 + 4(R+r)\ell_0}\right\}$$